Schrödinger Equation and Free Particle Wave Function

Karl De Paepe

Abstract

Using the wave function of a free particle we obtain a solution of the Schrödinger equation for a class of potentials.

1 Time dependent accelerating frame of reference

Consider an accelerating frame of reference $\mathcal{F}'$ with coordinates $x', t'$ and an inertial frame of reference $\mathcal{F}$ with coordinates $x, t$. The coordinates of the frames being related by

$$x' = x - f(t) \quad t' = t$$

(1)

Since $dx' = dx$ and position probabilities are the same for $\mathcal{F}'$ and $\mathcal{F}$ we have for the wave function $\psi(x, t)$ with respect to $\mathcal{F}$ and corresponding wave function $\psi'(x', t')$ with respect to $\mathcal{F}'$ that [1]

$$|\psi'(x', t')|^2 = |\psi(x, t)|^2$$

(2)

Consequently there is a real valued function $\beta(x, t)$ such that

$$\psi'(x', t') = e^{-\frac{i}{\hbar}\beta(x, t)}\psi(x, t)$$

(3)

With respect to $\mathcal{F}$ let the wave function $\psi(x, t)$ satisfies the Schrödinger equation

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2}(x, t) = i\hbar \frac{\partial \psi}{\partial t}(x, t)$$

(4)

With respect to $\mathcal{F}'$ we have an additional force $m\ddot{f}(t)x + V_0(t')$. The wave function $\psi'(x', t')$ then satisfies the Schrödinger equation

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi'}{\partial x'^2}(x', t') + \left( m\ddot{f}(t')x' + V_0(t') \right) \psi'(x', t') = i\hbar \frac{\partial \psi'}{\partial t'}(x', t')$$

(5)

Now

$$\frac{\partial}{\partial x'} = \frac{\partial}{\partial x} \quad \frac{\partial}{\partial t'} = \dot{f} \frac{\partial}{\partial x} + \frac{\partial}{\partial t}$$

(6)

and on substituting (3) in (5) and using (4) and (6) gives

$$\left[ \frac{i\hbar}{2m} \frac{\partial^2 \beta}{\partial x'^2} + \frac{1}{2m} \left( \frac{\partial \beta}{\partial x'} \right)^2 + m\ddot{f}(x - f) + V_0 - \dot{f} \frac{\partial \beta}{\partial x} - \frac{\partial \beta}{\partial t} \right]\psi + \frac{i\hbar}{m} \left[ \frac{\partial \beta}{\partial x} - m\ddot{f} \right] \frac{\partial \psi}{\partial x} = 0$$

(7)

We have

$$\beta(x, t) = m\ddot{f}(t)x + \int_0^t \left[ V_0(s) - m\ddot{f}(s)\ddot{f}(s) - \frac{1}{2}m\ddot{f}(s)^2 \right] ds + C$$

(8)

is the unique solution of (7) satisfying the initial condition [2]

$$\beta(x, 0) = m\ddot{f}(0)x + C$$

(9)
2 Space and time dependent velocity

Let $v_\epsilon(x, t)$ be a smooth function in variables $\epsilon, x, t$. Require $v_\epsilon(x, 0) = 0$. Define $X_\epsilon(u; t)$ to be the curve $x(t)$ such that
\[
\frac{dx}{dt} = v_\epsilon(x, t)
\]
and $x(0) = u$. Require that the curves are defined for all $t$ and the curves do not intersect. We then have a frame of reference $F_\epsilon$ with coordinates $x_\epsilon, t_\epsilon$ such that
\[
x_\epsilon = X_\epsilon(x; t) \quad t_\epsilon = t
\]
Let $\psi(x, t)$ satisfy (4). Let $V_\epsilon(x_\epsilon, t_\epsilon)$ be the potential in these coordinates. We have
\[
\frac{1}{m} \frac{\partial V_\epsilon}{\partial x}(x_\epsilon, t_\epsilon) = v(x, t) \frac{\partial v}{\partial x}(x, t) + \frac{\partial v}{\partial t}(x, t)
\]
Let $\psi_\epsilon(x_\epsilon, t_\epsilon)$ be the wave function satisfying the Schrödinger equation in $x_\epsilon, t_\epsilon$ coordinates and $\psi_\epsilon(x, 0) = \psi(x, 0)$. Let $B(x_0; \epsilon)$ be the set of points $x_0 - \epsilon < x < x_0 + \epsilon$. Choose $v_\epsilon(x, t)$ so that for $u \in B(x_0; \epsilon)$
\[
X_\epsilon(u; t) = X_0(x_0; t) + u - x_0
\]
Let $\hat{F}$ be a frame of reference with coordinates $\hat{x}, \hat{t}$ related to coordinates $x, t$ of $F$ by
\[
\hat{x} = x - X_0(x_0; t) \quad \hat{t} = t
\]
The potential in these coordinates is $m\hat{X}_0(\hat{x}_0 : \hat{t})\hat{x} + V_0(\hat{t})$. Let $\hat{\psi}(\hat{x}, \hat{t})$ be the wave function satisfying the Schrödinger equation with this potential and $\hat{\psi}(x_0, 0) = \psi(x, 0)$. We have by (8) a $\tilde{\beta}(x, t)$ such that
\[
\frac{\tilde{\psi}(\hat{x}, \hat{t})}{\psi(x, t)} = e^{-\frac{i}{\hbar}\tilde{\beta}(x, t)} \quad \frac{\partial \tilde{\beta}}{\partial x}(x, t) = m\hat{X}_0(x_0; t)
\]
and hence for points $(X_\epsilon(u; t), t)$ where $u \in B(x_0; \epsilon)$ we have
\[
\frac{\psi_\epsilon(x_\epsilon, t_\epsilon)}{\psi(x, t)} = e^{-\frac{i}{\hbar}\tilde{\beta}(x, t)} \quad \frac{\partial \tilde{\beta}}{\partial x}(x, t) = m\hat{X}_0(x_0; t)
\]
Define coordinates $x' = x_0, t' = t_0$. Let $\psi'(x', t') = \psi_\epsilon(x_\epsilon, t_\epsilon)$. Now $x_0$ is arbitrary and let $\beta(x, t)$ be the limit of $\tilde{\beta}(x, t)$ as $\epsilon \to 0$ so we get
\[
\frac{\partial \beta}{\partial x}(x, t) = mv_0(x, t)
\]
Require $v(x, t) \to 0$ as $v \to -\infty$. We then have $\beta(x, t) \to 0$ as $x \to -\infty$ hence by (17)
\[
\beta(x, t) = \int_{-\infty}^{x} v_0(u, t) du
\]
Consequently
\[
\psi'(x', t') = e^{-\frac{i}{\hbar}\int_{-\infty}^{x_0} v_0(u, t) du} \psi(x, t)
\]

References
