Equiprobability for any non null natural integer of having either an odd or even number of prime factor(s) counted with multiplicity.

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Abstract. Redefining the set of all non null natural integers $\mathbb{N}^*$ as the union of infinitely many disjoint sets, we prove the equiprobability for any integer of each said set to have either an odd or even number of prime factor(s) counted with multiplicity. The thus established equiprobability on $\mathbb{N}^*$ allows us to use the standard normal distribution to establish that $\lim\limits_{N\to+\infty} \dfrac{L(N)}{\sqrt{N}} = 0$, $L(N)$ the summatory Liouville function. Recalling the Dirichlet series for the Liouville function we deduce that $\frac{\zeta(2s)}{\zeta(s)}$, $s = \sigma + it$, is analytic for $\sigma > \frac{1}{2}$, $\zeta(s)$ the Riemann zeta function. Consequently the veracity of the Riemann hypothesis is being established.

Introduction

On the topic of the probability of the parity of the number of prime factor(s) counted with multiplicity, we have not come across any article nor book that deals directly with the matter in a fashion that is similar to that of the present article.

Indeed by introducing a novel approach, we have been able to prove the equiprobability for any non null natural integer of having either an odd or even number of prime factor(s) counted with multiplicity. The equiprobability of which does in turn have remarkable implications.

Lemma 1. *Considering an infinite number of probability spaces defined by :* $\forall i \in \mathbb{N}$ *let* $\langle \Omega_i = \omega_i \cup \overline{\omega_i}, \mathcal{F}_i = \{\emptyset, \omega_i, \overline{\omega_i}, \Omega_i\}, P_i : \mathcal{F}_i \to [0,1]\rangle$, *with* $P_i(\omega_i) = a, a \in [0,1]$, $P_i(\overline{\omega_i}) = 1 - a$, ω_i *and* $\overline{\omega_i}$ *being both non-empty countable sets while one or both being possibly infinite, be the probability space uniquely indexed by* $i \in \mathbb{N}$.

If $\Omega_U = \bigcup_{i\in\mathbb{N}} \Omega_i$ *and* $\forall i, j \in \mathbb{N}, i \neq j, \Omega_i \cap \Omega_j = \emptyset$ *and* $\forall i \in \mathbb{N}$, $P_i(\omega_i) = a$, *then on the probability space* $\langle \Omega_U = \bigcup_{i\in\mathbb{N}} \omega_i \cup \bigcup_{i\in\mathbb{N}} \overline{\omega_i}, \mathcal{F}_U = \{\emptyset, \bigcup_{i\in\mathbb{N}} \omega_i, \bigcup_{i\in\mathbb{N}} \overline{\omega_i}, \Omega_U\}, P_U : \mathcal{F}_U \to [0,1]\rangle$ *we have :*

$$P_U\left(\bigcup_{i\in\mathbb{N}} \omega_i\right) = a$$

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Proof

Let $t \in \Omega_U$ be an outcome of Ω_U then by definition : $t \in \bigcup_{i\in\mathbb{N}} \omega_i \Leftrightarrow \exists$ a unique $j \in \mathbb{N}$ such that $t \in \omega_j$.

That is to say that for a given outcome $t \in \Omega_U$, the event $\bigcup_{i\in\mathbb{N}} \omega_i \in \mathcal{F}_U$ has occurred *iff* for that same outcome $t \in \Omega_U$, $\exists$ a unique $j \in \mathbb{N}$ such that the event $\omega_j \in \mathcal{F}_j$ has occurred. Given that $\forall i \in \mathbb{N}, P_i(\omega_i) = a$, therefore $P_U(\bigcup_{i\in\mathbb{N}} \omega_i) = P_j(\omega_j) = a$; then by considering the complementary event of $\bigcup_{i\in\mathbb{N}} \omega_i \in \mathcal{F}_U$: $P_U(\bigcup_{i\in\mathbb{N}} \overline{\omega_i}) = P_j(\overline{\omega_j}) = 1 - a$ and $P_U(\Omega_U) = P_U(\bigcup_{i\in\mathbb{N}} \omega_i) + P_U(\bigcup_{i\in\mathbb{N}} \overline{\omega_i}) = 1$.

Lemma 2. *Considering a finite number of probability spaces defined by : $\forall i, V \in \mathbb{N}$, $\forall i \in [0, V]$, let $\langle \Omega_i = \omega_i \cup \overline{\omega_i}, \mathcal{F}_i = \{\emptyset, \omega_i, \overline{\omega_i}, \Omega_i\}, P_i : \mathcal{F}_i \to [0, 1]\rangle$, with $P_i(\omega_i) = a, a \in [0, 1]$, $P_i(\overline{\omega_i}) = 1 - a$, ω_i and $\overline{\omega_i}$ being both non-empty countable sets while one or both being possibly infinite, be the probability space uniquely indexed by $i \in [0, V]$.*

If $\Omega_V = \bigcup_{i\in[0,V]} \Omega_i$ and $\forall i, j \in [0, V], i \neq j, \Omega_i \cap \Omega_j = \emptyset$ and $\forall i \in [0, V]$, $P_i(\omega_i) = a$, then on the probability space $\langle \Omega_V = \bigcup_{i\in[0,V]} \omega_i \cup \bigcup_{i\in[0,V]} \overline{\omega_i}, \mathcal{F}_V = \{\emptyset, \bigcup_{i\in[0,V]} \omega_i, \bigcup_{i\in[0,V]} \overline{\omega_i}, \Omega_V\}, P_V : \mathcal{F}_V \to [0, 1]\rangle$ we have :

$$P_V(\textstyle\bigcup_{i\in[0,V]} \omega_i) = a$$

Proof

Let $t \in \Omega_V$ be an outcome of Ω_V then by definition : $t \in \bigcup_{i\in[0,V]} \omega_i \Leftrightarrow \exists$ a unique $j \in [0, V]$ such that $t \in \omega_j$.

That is to say that for a given outcome $t \in \Omega_V$, the event $\bigcup_{i\in[0,V]} \omega_i \in \mathcal{F}_V$ has occurred *iff* for that same outcome $t \in \Omega_V$, $\exists$ a unique $j \in [0, V]$ such that the event $\omega_j \in \mathcal{F}_j$ has occurred. Given that $\forall i \in [0, V], P_i(\omega_i) = a$, therefore $P_V(\bigcup_{i\in[0,V]} \omega_i) = P_j(\omega_j) = a$; then by considering the complementary event of $\bigcup_{i\in[0,V]} \omega_i \in \mathcal{F}_V$: $P_V(\bigcup_{i\in[0,V]} \overline{\omega_i}) = P_j(\overline{\omega_j}) = 1 - a$ and $P_V(\Omega_V) = P_V(\bigcup_{i\in[0,V]} \omega_i) + P_V(\bigcup_{i\in[0,V]} \overline{\omega_i}) = 1$.

Lemma 3. *For any integer n drawn randomly from $\mathbb{N}^*$, it is equiprobable that either $n \in \{2k + 1 : k \in \mathbb{N}\}$ or $n \in \{2k : k \in \mathbb{N}^*\}$. That is, considering the probability space $\langle \Omega_{N^*} = \mathbb{N}^*, \mathcal{F}_{N^*} = \{\emptyset, \{2k + 1 : k \in \mathbb{N}\}, \{2k : k \in \mathbb{N}^*\}, \Omega_{N^*}\}, P_{N^*} : \mathcal{F}_{N^*} \to [0, 1]\rangle$, then :*

$$P_{N^*}(\{2k + 1 : k \in \mathbb{N}\}) = P_{N^*}(\{2k : k \in \mathbb{N}^*\}) = \tfrac{1}{2}.$$

Proof

Let us consider the random experiment consisting in drawing randomly any integer n from $\mathbb{N}^*$ in order to note as the outcome whether $n \in \{2k + 1 : k \in \mathbb{N}\}$ or $n \in \{2k : k \in \mathbb{N}^*\}$. The probability space associated with the latter random experiment is : $\langle \Omega_{N^*} = \mathbb{N}^*, \mathcal{F}_{N^*} = \{\emptyset, \{2k + 1 : k \in \mathbb{N}\}, \{2k : k \in \mathbb{N}^*\}, \Omega_{N^*}\}, P_{N^*} : \mathcal{F}_{N^*} \to [0, 1]\rangle$.

$\forall l \in \mathbb{N}$ let us now consider the random experiment consisting in drawing randomly any integer n from the set $\{2l+1, 2l+2\}$ in order to note as the outcome whether $n \in \{2k+1 : k \in \mathbb{N}\}$ or $n \in \{2k : k \in \mathbb{N}^*\}$. The probability space associated with the latter random experiment is : $\langle \Omega_l = \{2l+1, 2l+2\}, \mathcal{F}_l = \{\emptyset, \{2l+1\}, \{2l+2\}, \Omega_l\}, P_l : \mathcal{F}_l \to [0,1]\rangle$. Given that the set $\{2l+1, 2l+2\}$ has only 2 elements that are equally likely to be picked, it is therefore obvious that to draw randomly any integer n from the set $\{2l+1, 2l+2\}$ then : $P_l(\{2l+1\}) = P_l(\{2l+2\}) = \frac{1}{2}$ and $P_l(\Omega_l) = P_l(\{2l+1\} \cup \{2l+2\}) = \frac{1}{2} + \frac{1}{2} = 1$.

Noting that $\mathbb{N}^* = \bigcup_{l \in \mathbb{N}} \Omega_l$ and $\forall l, l' \in \mathbb{N}, l \neq l', \Omega_l \cap \Omega_{l'} = \emptyset$ and $\forall l \in \mathbb{N}$, $P_l(\{2l+1\}) = \frac{1}{2}$, by applying **Lemma 1** we have : $P_{N^*}(\{2k+1 : k \in \mathbb{N}\}) = P_l(\{2l+1\}) = \frac{1}{2}$, $P_{N^*}(\{2k : k \in \mathbb{N}^*\}) = P_l(\{2l+2\}) = \frac{1}{2}$ and $P_{N^*}(\Omega_{N^*}) = P_{N^*}(\{2k+1 : k \in \mathbb{N}\} \cup \{2k : k \in \mathbb{N}^*\}) = \frac{1}{2} + \frac{1}{2} = 1$.

LEMMA 4. *For any integer n drawn randomly from $\mathbb{N}$, it is equiprobable that either $n \in \{2k+1 : k \in \mathbb{N}\}$ or $n \in \{2k : k \in \mathbb{N}\}$. That is, considering the probability space $\langle \Omega_N = \mathbb{N}, \mathcal{F}_N = \{\emptyset, \{2k+1 : k \in \mathbb{N}\}, \{2k : k \in \mathbb{N}\}, \Omega_N\}, P_N : \mathcal{F}_N \to [0,1]\rangle$, then :*

$$P_N(\{2k+1 : k \in \mathbb{N}\}) = P_N(\{2k : k \in \mathbb{N}\}) = \tfrac{1}{2}.$$

Proof

Let us consider the random experiment consisting in drawing randomly any integer n from $\mathbb{N}$ in order to note as the outcome whether $n \in \{2k+1 : k \in \mathbb{N}\}$ or $n \in \{2k : k \in \mathbb{N}\}$. The probability space associated with the latter random experiment is : $\langle \Omega_N = \mathbb{N}, \mathcal{F}_N = \{\emptyset, \{2k+1 : k \in \mathbb{N}\}, \{2k : k \in \mathbb{N}\}, \Omega_N\}, P_N : \mathcal{F}_N \to [0,1]\rangle$.

$\forall m \in \mathbb{N}$ let us now consider the random experiment consisting in drawing randomly any integer n from the set $\{2m, 2m+1\}$ in order to note as the outcome whether $n \in \{2k+1 : k \in \mathbb{N}\}$ or $n \in \{2k : k \in \mathbb{N}\}$. The probability space associated with the latter random experiment is : $\langle \Omega_m = \{2m, 2m+1\}, \mathcal{F}_m = \{\emptyset, \{2m\}, \{2m+1\}, \Omega_m\}, P_m : \mathcal{F}_m \to [0,1]\rangle$. Given that the set $\{2m, 2m+1\}$ has only 2 elements that are equally likely to be picked, it is therefore obvious that to draw randomly any integer n from the set $\{2m, 2m+1\}$ then : $P_m(\{2m\}) = P_m(\{2m+1\}) = \frac{1}{2}$ and $P_m(\Omega_m) = P_m(\{2m\} \cup \{2m+1\}) = \frac{1}{2} + \frac{1}{2} = 1$.

Noting that $\mathbb{N} = \bigcup_{m \in \mathbb{N}} \Omega_m$ and $\forall m, m' \in \mathbb{N}, m \neq m', \Omega_m \cap \Omega_{m'} = \emptyset$ and $\forall m \in \mathbb{N}$, $P_m(\{2m+1\}) = \frac{1}{2}$, by applying **Lemma 1** we have : $P_N(\{2k+1 : k \in \mathbb{N}\}) = P_m(\{2m+1\}) = \frac{1}{2}$, $P_N(\{2k : k \in \mathbb{N}\}) = P_m(\{2m\}) = \frac{1}{2}$ and $P_N(\Omega_N) = P_N(\{2k+1 : k \in \mathbb{N}\} \cup \{2k : k \in \mathbb{N}\}) = \frac{1}{2} + \frac{1}{2} = 1$.

1/ $\mathbb{N}^*$ as infinitely many complementary disjoint sets

$\forall n \in \mathbb{N}^* \setminus \{1\}$, by the unique prime factorization theorem, there exists a unique sequence $(p_1, p_2, \ldots, p_i, \ldots, p_m)$, $p_1 < p_2 < \ldots < p_i < \ldots < p_m$, $m \in \mathbb{N}^*$,

$p_i \in \mathbb{P}$, the set of all the prime numbers and a unique sequence $(f_1, f_2, \ldots, f_i, \ldots, f_m)$, $f_i \in \mathbb{N}^*$ such that $n = p_1^{f_1} \times p_2^{f_2} \times \ldots \times p_i^{f_i} \times \ldots \times p_m^{f_m}$ (the latter notation will be used for the entirety of the article).

Let us note that for the entirety of the article the adjectives odd or even are employed in the classical definition of parity i.e. a natural integer is "odd" shall it be a member of $\{2k+1 : k \in \mathbb{N}\}$ and "even" shall it be a member of $\{2k : k \in \mathbb{N}\}$. Thus 1 is the smallest odd number while 0 is the smallest even number. Additionally let us note by $\mathbb{N}_1$ the set of all the odd natural integers such that $\mathbb{N}_1 = \{2k+1 : k \in \mathbb{N}\}$, by $\mathbb{N}_2$ the set of all the even natural integers such that $\mathbb{N}_2 = \{2k : k \in \mathbb{N}\}$ and by $\mathbb{N}_2^*$ the set of all the non null even natural integers such that $\mathbb{N}_2^* = \{2k : k \in \mathbb{N}^*\}$.

Let us note for the entirety of the article by F, $F \in \mathbb{N}$, the number of prime factor(s) counted with multiplicity of $n \in \mathbb{N}^*$ and by F', $F' \in \{odd, even\}$, the parity of F.

Additionally let us note for the entirety of the article by f'_i, $i \in [1, m]$ the parity of each f_i.

Let $\mathbb{N}^*$ be as such :

$$\mathbb{N}^* = \mathbb{A}_1 \cup \mathbb{A}_2 \cup \bigcup_{m \in \mathbb{N}^*, m \geq 2} \mathbb{B}_m$$

where :

$\mathbb{A}_1 = \{2^k : k \in \mathbb{N}\}$;

$\mathbb{A}_2 = \bigcup_{p \in \mathbb{P} \setminus \{2\}} \{p^k : k \in \mathbb{N}^*\}$, $\mathbb{P} \setminus \{2\}$ denoting the set of all the prime numbers excluding $\{2\}$;

$\forall m \in \mathbb{N}^*, m \geq 2, \mathbb{B}_m = \bigcup_{p_i \in \mathbb{P}, p_1 < p_2 < \ldots < p_i < \ldots < p_m} (\bigcup_{k_m \in \mathbb{N}_1 \cup k_m \in \mathbb{N}_2^*} (\ldots (\bigcup_{k_i \in \mathbb{N}_1 \cup k_i \in \mathbb{N}_2^*} (\ldots (\bigcup_{k_2 \in \mathbb{N}_1 \cup k_2 \in \mathbb{N}_2^*} \{(p_1^{k_1} \times p_2^{k_2} \times \ldots \times p_i^{k_i} \times \ldots \times p_m^{k_m}) : k_1 \in \mathbb{N}^*\}) \ldots)) \ldots))$, thus $\forall m, m' \in \mathbb{N}, m, m' \geq 2, m \neq m', \mathbb{B}_m \cap \mathbb{B}_{m'} = \emptyset$.

Let us note for the entirety of the article that for infinitely many given sets S_i, $i \in \mathbb{N}^*$, we will be using the expression "$\bigcup_{i \in \mathbb{N}_1 \cup i \in \mathbb{N}_2^*} S_i$" in order to mean "$\bigcup_{i \in \mathbb{N}_1} S_i \cup \bigcup_{i \in \mathbb{N}_2^*} S_i$" i.e. $\bigcup_{i \in \mathbb{N}_1 \cup i \in \mathbb{N}_2^*} S_i = \bigcup_{i \in \mathbb{N}_1} S_i \cup \bigcup_{i \in \mathbb{N}_2^*} S_i$. Thus for infinitely many given sets $S_{i,j}$, $i, j \in \mathbb{N}^*$, we have : $\bigcup_{j \in \mathbb{N}_1 \cup j \in \mathbb{N}_2^*} (\bigcup_{i \in \mathbb{N}_1 \cup i \in \mathbb{N}_2^*} S_{i,j}) = \bigcup_{j \in \mathbb{N}_1 \cup j \in \mathbb{N}_2^*} (\bigcup_{i \in \mathbb{N}_1} S_{i,j} \cup \bigcup_{i \in \mathbb{N}_2^*} S_{i,j}) = \bigcup_{j \in \mathbb{N}_1} (\bigcup_{i \in \mathbb{N}_1} S_{i,j} \cup \bigcup_{i \in \mathbb{N}_2^*} S_{i,j}) \cup \bigcup_{j \in \mathbb{N}_2^*} (\bigcup_{i \in \mathbb{N}_1} S_{i,j} \cup \bigcup_{i \in \mathbb{N}_2^*} S_{i,j}) = \bigcup_{j \in \mathbb{N}_1} (\bigcup_{i \in \mathbb{N}_1} S_{i,j}) \cup \bigcup_{j \in \mathbb{N}_1} (\bigcup_{i \in \mathbb{N}_2^*} S_{i,j}) \cup \bigcup_{j \in \mathbb{N}_2^*} (\bigcup_{i \in \mathbb{N}_1} S_{i,j}) \cup \bigcup_{j \in \mathbb{N}_2^*} (\bigcup_{i \in \mathbb{N}_2^*} S_{i,j})$.

Additionally, for the entirety of the article we will be using the expression "$\bigcup_{p_i \in \mathbb{P}, p_1 < p_2 < \ldots < p_i < \ldots < p_m}$" in order to mean "$\bigcup_{\forall i \in [1,m], p_i \in \mathbb{P}, p_1 < p_2 < \ldots < p_i < \ldots < p_m}$"; the latter is to say : "the union for all m prime numbers such that $p_1 < p_2 < \ldots < p_i < \ldots < p_m$". For instance for infinitely many given sets S_{p_1,p_2}, $p_1, p_2 \in \mathbb{P}$ and for $m = 2$ we have $\bigcup_{p_i \in \mathbb{P}, p_1 < p_2} S_{p_1,p_2} = \bigcup_{p_1, p_2 \in \mathbb{P}, p_1 < p_2} S_{p_1,p_2}$ which is the union of all the sets S_{p_1,p_2} for all couples of prime numbers $p_1, p_2 \in \mathbb{P}$, $p_1 < p_2$.

Thus :

$\mathbb{A}_1 \cap \mathbb{A}_2 = \emptyset$;

$\forall m \in \mathbb{N}^*, m \geq 2, \mathbb{A}_1 \cap \mathbb{B}_m = \emptyset$;

$\forall m \in \mathbb{N}^*, m \geq 2, \mathbb{A}_2 \cap \mathbb{B}_m = \emptyset$;

$\forall m, m' \in \mathbb{N}^*, m, m' \geq 2, m \neq m', \mathbb{B}_m \cap \mathbb{B}_{m'} = \emptyset$;

$\forall p_1, p_2, \ldots, p_i, \ldots, p_m \in \mathbb{P}, p_1 < p_2 < \ldots < p_i < \ldots < p_m, \forall p'_1, p'_2, \ldots, p'_i, \ldots, p'_m \in \mathbb{P}, p'_1 < p'_2 < \ldots < p'_i < \ldots < p'_m$, $p_1 \times p_2 \times \ldots \times p_i \times \ldots \times p_m \neq p'_1 \times p'_2 \times \ldots \times p'_i \times \ldots \times p'_m$, $\bigcup_{k_m \in \mathbb{N}_1 \cup k_m \in \mathbb{N}_2^*}(\cdots(\bigcup_{k_i \in \mathbb{N}_1 \cup k_i \in \mathbb{N}_2^*}(\cdots(\bigcup_{k_2 \in \mathbb{N}_1 \cup k_2 \in \mathbb{N}_2^*}\{(p_1^{k_1} \times p_2^{k_2} \times \ldots \times p_i^{k_i} \times \ldots \times p_m^{k_m}) : k_1 \in \mathbb{N}^*\})\ldots))\ldots) \cap \bigcup_{k_m \in \mathbb{N}_1 \cup k_m \in \mathbb{N}_2^*}(\cdots(\bigcup_{k_i \in \mathbb{N}_1 \cup k_i \in \mathbb{N}_2^*}(\cdots(\bigcup_{k_2 \in \mathbb{N}_1 \cup k_2 \in \mathbb{N}_2^*}\{(p_1'^{k_1} \times p_2'^{k_2} \times \ldots \times p_i'^{k_i} \times \ldots \times p_m'^{k_m}) : k_1 \in \mathbb{N}^*\})\ldots))\ldots) = \emptyset$.

Let us remark that $1 = 2^0 \in \mathbb{A}_1$.

Let us note that $\forall m \in \mathbb{N}^*$, $m \geq 2$, $\forall p_1, p_2, \ldots, p_i, \ldots, p_m \in \mathbb{P}$, $p_1 < p_2 < \ldots < p_i < \ldots < p_m, \forall k_2, \ldots, k_i, \ldots, k_m, \forall i \in [2, m], k_i \in \mathbb{N}_1 \cup k_i \in \mathbb{N}_2^*$, each of the sets $\{(p_1^{k_1} \times p_2^{k_2} \times \ldots \times p_i^{k_i} \times \ldots \times p_m^{k_m}) : k_1 \in \mathbb{N}^*\}$, $\mathbb{A}_1$ and $\{p^k : k \in \mathbb{N}^*\} \subset \mathbb{A}_2$, $\forall p \in \mathbb{P} \setminus \{2\}$, can be considered in a strictly increasing order that is being conferred by the original strictly increasing order of $\mathbb{N}$ and $\mathbb{N}^*$.

2/ n belongs to $\mathbb{A}_1$ or n belongs to $\mathbb{A}_2$

a. n belongs to $\mathbb{A}_1$

THEOREM 1. *To draw randomly any integer n from $\mathbb{A}_1$ then the probability that F being odd is equal to the probability that F being even which is $\frac{1}{2}$. That is, considering the probability space $\langle \Omega_{A_1} = \{n \in \mathbb{A}_1 : F' \in \{odd\}\} \cup \{n \in \mathbb{A}_1 : F' \in \{even\}\}, \mathcal{F}_{A_1} = \{\emptyset, \{n \in \mathbb{A}_1 : F' \in \{odd\}\}, \{n \in \mathbb{A}_1 : F' \in \{even\}\}, \Omega_{A_1}\}, P_{A_1} : \mathcal{F}_{A_1} \to [0,1]\rangle$ then :*

$$P_{A_1}(\{n \in \mathbb{A}_1 : F' \in \{odd\}\}) = P_{A_1}(\{n \in \mathbb{A}_1 : F' \in \{even\}\}) = \tfrac{1}{2}$$

Proof

Since $n \in \mathbb{A}_1, \mathbb{A}_1 = \{2^k : k \in \mathbb{N}\}$, it comes that the parity of F is given by the parity of f_1 – let us note the special case of $f_1 = 0$ where $n = 1$ has 0 prime factor; 0 being considered as even, which is consistent with the Liouville function as $\lambda(1) = 1$.

By the unique prime factorization theorem, it is clear that to any integer $n \in \mathbb{A}_1, n = 2^{f_1}$, corresponds the unique integer $f_1 \in \mathbb{N}$, and vice versa to any integer $f_1 \in \mathbb{N}$, corresponds the unique integer $n \in \mathbb{A}_1, n = 2^{f_1}$. That is to say that be a function $f_{A_1} : \mathbb{A}_1 \to \mathbb{N}$ such that for any $n \in \mathbb{A}_1, n = 2^{f_1}$, we have $f_{A_1}(n) = f_1, f_1 \in \mathbb{N}$, then f_{A_1} is a bijective function from $\mathbb{A}_1$ toward $\mathbb{N}$, whose inverse function is $f_{A_1}^{-1} : \mathbb{N} \to \mathbb{A}_1$, such that for any integer $f_1, f_1 \in \mathbb{N}$, we have $f_{A_1}^{-1}(f_1) = 2^{f_1} = n, n \in \mathbb{A}_1$.

Thus : $\mathbb{A}_1 = \{2^k : k \in \mathbb{N}\} = \{f_{A_1}^{-1}(k) : k \in \mathbb{N}\}$.

Thus each integer n in $\mathbb{A}_1$ necessarily has only but one corresponding integer f_1 that determines the parity of the number of prime factor(s) counted with multiplicity of n, F.

Let us now consider the random experiment consisting in drawing randomly any integer n from $\mathbb{A}_1$ in order to note as the outcome the parity f_1' of the corresponding exponent f_1 given by $f_{T_1}(n)$. The probability space associated with the latter random experiment is $\langle \Omega_{T_1} = \{k' : k' \in \{odd, even\}\}, \mathcal{F}_{T_1} = 2^{\Omega_{T_1}}, P_{T_1} : \mathcal{F}_{T_1} \to [0,1]\rangle$.

Let us note by $e_{f_1} \in \mathcal{F}_{T_1}$ the event that f_1' is odd and $\overline{e_{f_1}} \in \mathcal{F}_{T_1}$ the event that f_1' is even.

Given **Lemma 4**, it is equiprobable for any integer drawn randomly from $\mathbb{N}$ to be either odd or even, therefore $P_{T_1}(e_{f_1}) = P_{T_1}(\overline{e_{f_1}}) = \frac{1}{2}$.

It is clear that the total number of possible outcomes k', $card(\Omega_{T_1})$ is equal to 2; thus $P_{T_1}(\Omega_{T_1}) = P_{T_1}(e_{f_1} \cup \overline{e_{f_1}}) = \frac{1}{2} + \frac{1}{2} = 1$ indeed.

F is odd iff f_1' is odd and F is even iff f_1' is even. Let us note by $A_F \in \mathcal{F}_{T_1}, A_F = e_{f_1}$ the event that F is odd and $\overline{A_F} \in \mathcal{F}_{T_1}, \overline{A_F} = \overline{e_{f_1}}$ the event that F is even. It comes that $P_{T_1}(A_F) = P_{T_1}(\overline{A_F}) = \frac{1}{2}$ with $P_{T_1}(\Omega_{T_1}) = P_{T_1}(A_F) + P_{T_1}(\overline{A_F}) = 1$ indeed.

By definition $A_F \in \mathcal{F}_{T_1}$ is equivalent to $\{n \in \mathbb{A}_1 : F' \in \{odd\}\} \in \mathcal{F}_{A_1}$ and $\overline{A_F} \in \mathcal{F}_{T_1}$ is equivalent to $\{n \in \mathbb{A}_1 : F' \in \{even\}\} \in \mathcal{F}_{A_1}$, therefore **Theorem 1** is established.

b. n belongs to $\mathbb{A}_2$

THEOREM 2. *To draw randomly any integer n from $\mathbb{A}_2$ then the probability that F being odd is equal to the probability that F being even which is $\frac{1}{2}$. That is, considering the probability space $\langle \Omega_{A_2} = \{n \in \mathbb{A}_2 : F' \in \{odd\}\} \cup \{n \in \mathbb{A}_2 : F' \in \{even\}\}, \mathcal{F}_{A_2} = \{\emptyset, \{n \in \mathbb{A}_2 : F' \in \{odd\}\}, \{n \in \mathbb{A}_2 : F' \in \{even\}\}, \Omega_{A_2}\}, P_{A_2} : \mathcal{F}_{A_2} \to [0,1]\rangle$ then :*

$$P_{A_2}(\{n \in \mathbb{A}_2 : F' \in \{odd\}\}) = P_{A_2}(\{n \in \mathbb{A}_2 : F' \in \{even\}\}) = \frac{1}{2}$$

Proof

Let p be any given prime number in $\mathbb{P} \setminus \{2\}$.

When considering $n \in \{p^k : k \in \mathbb{N}^*\}$, $n = p^{f_1}$, it comes that the parity of F is given by the parity of f_1.

By the unique prime factorization theorem, it is clear that to any integer $n \in \{p^k : k \in \mathbb{N}^*\}, n = p^{f_1}$, corresponds the unique integer $f_1 \in \mathbb{N}^*$, and vice versa to any integer $f_1 \in \mathbb{N}^*$, corresponds the unique integer $n \in \{p^k : k \in \mathbb{N}^*\}, n = p^{f_1}$. That is to say that be a function $f_{T_2} : \{p^k : k \in \mathbb{N}^*\} \to \mathbb{N}^*$ such that for any $n \in \{p^k : k \in \mathbb{N}^*\}, n = p^{f_1}$, we have $f_{T_2}(n) = f_1, f_1 \in \mathbb{N}^*$, then f_{T_2} is a bijective function from $\{p^k : k \in \mathbb{N}^*\}$ toward $\mathbb{N}^*$, whose inverse function is $f_{T_2}^{-1} : \mathbb{N}^* \to \{p^k : k \in \mathbb{N}^*\}$, such that for any integer $f_1, f_1 \in \mathbb{N}^*$, we have $f_{T_2}^{-1}(f_1) = p^{f_1} = n, n \in \{p^k : k \in \mathbb{N}^*\}$.

Thus : $\{p^k : k \in \mathbb{N}^*\} = \{f_{T_2}^{-1}(k) : k \in \mathbb{N}^*\}\}$.

Thus each integer n in $\{p^k : k \in \mathbb{N}^*\}$ necessarily has only but one corresponding integer f_1 that determines the parity of the number of prime factor(s) counted with multiplicity of n, F.

Let us now consider the random experiment consisting in drawing randomly any integer n from $\{p^k : k \in \mathbb{N}^*\}$ in order to note as the outcome the parity f_1' of the corresponding exponent f_1 given by $f_{T_2}(n)$. The probability space associated with the latter random experiment is $\langle \Omega_{T_2} = \{k' : k' \in \{odd, even\}\}, \mathcal{F}_{T_2} = 2^{\Omega_{T_2}}, P_{T_2} : \mathcal{F}_{T_2} \to [0,1]\rangle$.

Let us note by $e_{f_1} \in \mathcal{F}_{T_2}$ the event that f_1' is odd and $\overline{e_{f_1}} \in \mathcal{F}_{T_2}$ the event that f_1' is even.

Given **Lemma 3**, it is equiprobable for any integer drawn randomly from $\mathbb{N}^*$ to be either odd or even, therefore $P_{T_2}(e_{f_1}) = P_{T_2}(\overline{e_{f_1}}) = \frac{1}{2}$.

It is clear that the total number of possible outcomes k', $card(\Omega_{T_2})$ is equal to 2; thus $P_{T_2}(\Omega_{T_2}) = P_{T_2}(e_{f_1} \cup \overline{e_{f_1}}) = \frac{1}{2} + \frac{1}{2} = 1$ indeed.

F is odd iff f_1' is odd and F is even iff f_1' is even. Let us note by $E_{T_2} \in \mathcal{F}_{T_2}, E_{T_2} = e_{f_1}$ the event that F is odd and $\overline{E_{T_2}} \in \mathcal{F}_{T_2}, \overline{E_{T_2}} = \overline{e_{f_1}}$ the event that F is even. It comes that $P_{T_2}(E_{T_2}) = P_{T_2}(e_{f_1}) = \frac{1}{2}$, $P_{T_2}(\overline{E_{T_2}}) = P_{T_2}(\overline{e_{f_1}}) = \frac{1}{2}$ and $P_{T_2}(\Omega_{T_2}) = P_{T_2}(E_{T_2}) + P_{T_2}(\overline{E_{T_2}}) = 1$ indeed.

* * *

Since $\mathbb{A}_2 = \bigcup_{p \in \mathbb{P} \setminus \{2\}} \{q^k : k \in \mathbb{N}^*\}$ and $\forall p, p' \in \mathbb{P} \setminus \{2\}$, $p \neq p'$, $\{p^k : k \in \mathbb{N}^*\} \cap \{p'^k : k \in \mathbb{N}^*\} = \emptyset$ and $\forall p \in \mathbb{P} \setminus \{2\}$, $P_{T_2}(E_{T_2}) = \frac{1}{2}$, by applying **Lemma 1** we can deduce that : $P_{A_2}(\{n \in \mathbb{A}_2 : F' \in \{odd\}\}) = P_{T_2}(E_{T_2}) = \frac{1}{2}$, $P_{A_2}(\{n \in \mathbb{A}_2 : F' \in \{even\}\}) = P_{T_2}(\overline{E_{T_2}}) = \frac{1}{2}$ and $P_{A_2}(\Omega_{A_2}) = P_{A_2}(\{n \in \mathbb{A}_2 : F' \in \{odd\}\}) + P_{A_2}(\{n \in \mathbb{A}_2 : F' \in \{even\}\}) = \frac{1}{2} + \frac{1}{2} = 1$ which establishes **Theorem 2**.

3/ n belongs to $\mathbb{B}_2$

LEMMA 5. *$\forall p_1, p_2 \in \mathbb{P}$, $p_1 < p_2$, to draw randomly any integer n from $\bigcup_{k_2 \in \mathbb{N}_1 \cup k_2 \in \mathbb{N}_2^*} \{p_1^{k_1} \times p_2^{k_2} : k_1 \in \mathbb{N}^*\}$ then the probability that F being odd is equal to the probability that F being even which is $\frac{1}{2}$. That is, considering the probability space $\langle \Omega_{L_5} = \bigcup_{k_2 \in \mathbb{N}_1 \cup k_2 \in \mathbb{N}_2^*} \{p_1^{k_1} \times p_2^{k_2} : k_1 \in \mathbb{N}^*\}, \mathcal{F}_{L_5} = \{\emptyset, \{n \in \Omega_{L_5} : F' \in \{odd\}\}, \{n \in \Omega_{L_5} : F' \in \{even\}\}, \Omega_{L_5}\}, P_{L_5} : \mathcal{F}_{L_5} \to [0,1]\rangle$ then :*

$$P_{L_5}(\{n \in \Omega_{L_5} : F' \in \{odd\}\}) = P_{L_5}(\{n \in \Omega_{L_5} : F' \in \{even\}\}) = \frac{1}{2}$$

Proof

Let $p_1, p_2, p_1 < p_2$ be any 2 given prime numbers in $\mathbb{P}$ and let f_2 be any given integer in $\mathbb{N}^*$.

When considering $n \in \{p_1^{k_1} \times p_2^{f_2} : k_1 \in \mathbb{N}^*\}$, $n = p_1^{f_1} \times p_2^{f_2}$, it comes that the parity of F is given by the compounding by additivity of the parity of f_1 and the parity of f_2.

* * *

Given that p_2 and f_2 being given and fixed, by the unique prime factorization theorem, it is clear that to any integer $n \in \{p_1^{k_1} \times p_2^{f_2} : k_1 \in \mathbb{N}^*\}, n = p_1^{f_1} \times p_2^{f_2}$, corresponds the unique integer $f_1 \in \mathbb{N}^*$, and vice versa to any integer $f_1 \in \mathbb{N}^*$, corresponds the unique integer $n \in \{p_1^{k_1} \times p_2^{f_2} : k_1 \in \mathbb{N}^*\}, n = p_1^{f_1} \times p_2^{f_2}$. That is to say that be a function $f_{T_3} : \{p_1^{k_1} \times p_2^{f_2} : k_1 \in \mathbb{N}^*\} \to \mathbb{N}^*$ such that for any $n \in \{p_1^{k_1} \times p_2^{f_2} : k_1 \in \mathbb{N}^*\}, n = p_1^{f_1} \times p_2^{f_2}$, we have $f_{T_3}(n) = f_1, f_1 \in \mathbb{N}^*$, then f_{T_3} is a bijective function from $\{p_1^{k_1} \times p_2^{f_2} : k_1 \in \mathbb{N}^*\}$ toward $\mathbb{N}^*$, whose inverse function is $f_{T_3}^{-1} : \mathbb{N}^* \to \{p_1^{k_1} \times p_2^{f_2} : k_1 \in \mathbb{N}^*\}$, such that for any integer $f_1, f_1 \in \mathbb{N}^*$, we have $f_{T_3}^{-1}(f_1) = p_1^{f_1} \times p_2^{f_2} = n, n \in \{p_1^{k_1} \times p_2^{f_2} : k_1 \in \mathbb{N}^*\}$.

Thus : $\{p_1^{k_1} \times p_2^{f_2} : k_1 \in \mathbb{N}^*\} = \{f_{T_3}^{-1}(k_1) : k_1 \in \mathbb{N}^*\}$.

Thus each integer n in $\{p_1^{k_1} \times p_2^{f_2} : k_1 \in \mathbb{N}^*\}$ necessarily has only but one corresponding integer f_1 that determines the parity of the number of prime factor(s) counted with multiplicity of n, F (the parity f_2' being given and fixed).

Let us now consider the random experiment consisting in drawing randomly any integer n from $\{p_1^{k_1} \times p_2^{f_2} : k_1 \in \mathbb{N}^*\}$ in order to note as the outcome the parity f_1' of the corresponding exponent f_1 given by $f_{T_3}(n)$. The probability space associated with the latter random experiment is $\langle \Omega_{T_3} = \{k_1' : k_1' \in \{odd, even\}\}, \mathcal{F}_{T_3} = 2^{\Omega_{T_3}}, P_{T_3} : \mathcal{F}_{T_3} \to [0,1]\rangle$.

Let us note by $e_{f_1} \in \mathcal{F}_{T_3}$ the event that f_1' is odd and $\overline{e_{f_1}} \in \mathcal{F}_{T_3}$ the event that f_1' is even.

Given **Lemma 3**, it is equiprobable for any integer drawn randomly from $\mathbb{N}^*$ to be either odd or even, therefore $P_{T_3}(e_{f_1}) = P_{T_3}(\overline{e_{f_1}}) = \frac{1}{2}$.

It is clear that the total number of possible outcomes k_1', $card(\Omega_{T_3})$ is equal to 2; thus $P_{T_3}(\Omega_{T_3}) = P_{T_3}(e_{f_1} \cup \overline{e_{f_1}}) = \frac{1}{2} + \frac{1}{2} = 1$ indeed.

* * *

For any given $f_2 \in \mathbb{N}_1$, let us now consider the following probability space :

$\langle \Omega'_{T_3} = \{p_1^{k_1} \times p_2^{f_2} : k_1 \in \mathbb{N}^*\}, \mathcal{F}'_{T_3} = \{\emptyset, \{n \in \Omega'_{T_3} : F' \in \{odd\}\}, \{n \in \Omega'_{T_3} : F' \in \{even\}\}, \Omega'_{T_3}\}, P'_{T_3} : \mathcal{F}'_{T_3} \to [0,1]\rangle$.

$f_2 \in \mathbb{N}_1$, f_2 being given and fixed, F is odd iff f_1 is even and F is even iff f_1 is odd. Therefore we have : $P'_{T_3}(\{n \in \Omega'_{T_3} : F' \in \{odd\}\}) = P_{T_3}(\overline{e_{f_1}}) = \frac{1}{2}$, $P'_{T_3}(\{n \in \Omega'_{T_3} : F' \in \{even\}\}) = P_{T_3}(e_{f_1}) = \frac{1}{2}$ and $P'_{T_3}(\Omega'_{T_3}) = \frac{1}{2} + \frac{1}{2} = 1$.

Then let us consider the following probability space :

$\langle \Omega'_3 = \bigcup_{f_2 \in \mathbb{N}_1} \{p_1^{k_1} \times p_2^{f_2} : k_1 \in \mathbb{N}^*\}, \mathcal{F}'_3 = \{\emptyset, \{n \in \Omega'_3 : F' \in \{odd\}\}, \{n \in \Omega'_3 : F' \in \{even\}\}, \Omega'_3\}, P'_3 : \mathcal{F}'_3 \to [0,1] \rangle$.

Since $\Omega'_3 = \bigcup_{f_2 \in \mathbb{N}_1} \{p_1^{k_1} \times p_2^{f_2} : k_1 \in \mathbb{N}^*\}$ and $\forall f_2, f''_2 \in \mathbb{N}_1,\ f_2 \neq f''_2$, $\{p_1^{k_1} \times p_2^{f_2} : k_1 \in \mathbb{N}^*\} \cap \{p_1^{k_1} \times p_2^{f''_2} : k_1 \in \mathbb{N}^*\} = \emptyset$ and $\forall f_2 \in \mathbb{N}_1,\ P'_{T_3}(\{n \in \Omega'_{T_3} : F' \in \{odd\}\}) = \frac{1}{2}$, by applying **Lemma 1** we can deduce that : $P'_3(\{n \in \Omega'_3 : F' \in \{odd\}\}) = P'_{T_3}(\{n \in \Omega'_{T_3} : F' \in \{odd\}\}) = \frac{1}{2}$, $P'_3(\{n \in \Omega'_3 : F' \in \{even\}\}) = P'_{T_3}(\{n \in \Omega'_{T_3} : F' \in \{even\}\}) = \frac{1}{2}$ and $P'_3(\Omega'_3) = P'_3(\{n \in \Omega'_3 : F' \in \{odd\}\}) + P'_3(\{n \in \Omega'_3 : F' \in \{even\}\}) = \frac{1}{2} + \frac{1}{2} = 1$.

* * *

For any given $f_2 \in \mathbb{N}_2^*$, let us now consider the following probability space :

$\langle \Omega''_{T_3} = \{p_1^{k_1} \times p_2^{f_2} : k_1 \in \mathbb{N}^*\}, \mathcal{F}''_{T_3} = \{\emptyset, \{n \in \Omega''_{T_3} : F' \in \{odd\}\}, \{n \in \Omega''_{T_3} : F' \in \{even\}\}, \Omega''_{T_3}\}, P''_{T_3} : \mathcal{F}''_{T_3} \to [0,1] \rangle$.

$f_2 \in \mathbb{N}_2^*$, f_2 being given and fixed, F is odd iff f_1 is odd and F is even iff f_1 is even. Therefore we have : $P''_{T_3}(\{n \in \Omega''_{T_3} : F' \in \{odd\}\}) = P_{T_3}(e_{f_1}) = \frac{1}{2}$, $P''_{T_3}(\{n \in \Omega''_{T_3} : F' \in \{even\}\}) = P_{T_3}(\overline{e_{f_1}}) = \frac{1}{2}$ and $P''_{T_3}(\Omega''_{T_3}) = \frac{1}{2} + \frac{1}{2} = 1$.

* * *

Then let us consider the following probability space :

$\langle \Omega''_3 = \bigcup_{f_2 \in \mathbb{N}_2^*} \{p_1^{k_1} \times p_2^{f_2} : k_1 \in \mathbb{N}^*\}, \mathcal{F}''_3 = \{\emptyset, \{n \in \Omega''_3 : F' \in \{odd\}\}, \{n \in \Omega''_3 : F' \in \{even\}\}, \Omega''_3\}, P''_3 : \mathcal{F}''_3 \to [0,1] \rangle$.

Since $\Omega''_3 = \bigcup_{f_2 \in \mathbb{N}_2^*} \{p_1^{k_1} \times p_2^{f_2} : k_1 \in \mathbb{N}^*\}$ and $\forall f_2, f''_2 \in \mathbb{N}_2^*,\ f_2 \neq f''_2$, $\{p_1^{k_1} \times p_2^{f_2} : k_1 \in \mathbb{N}^*\} \cap \{p_1^{k_1} \times p_2^{f''_2} : k_1 \in \mathbb{N}^*\} = \emptyset$ and $\forall f_2 \in \mathbb{N}_2^*,\ P''_{T_3}(\{n \in \Omega''_{T_3} : F' \in \{odd\}\}) = \frac{1}{2}$, by applying **Lemma 1** we can deduce that : $P''_3(\{n \in \Omega''_3 : F' \in \{odd\}\}) = P''_{T_3}(\{n \in \Omega''_{T_3} : F' \in \{odd\}\}) = \frac{1}{2}$, $P''_3(\{n \in \Omega''_3 : F' \in \{even\}\}) = P''_{T_3}(\{n \in \Omega''_{T_3} : F' \in \{even\}\}) = \frac{1}{2}$ and $P''_3(\Omega''_3) = P''_3(\{n \in \Omega''_3 : F' \in \{odd\}\}) + P''_3(\{n \in \Omega''_3 : F' \in \{even\}\}) = \frac{1}{2} + \frac{1}{2} = 1$.

* * *

Then we can consider the following probability space :

$\langle \Omega_3 = \bigcup_{f_2 \in \mathbb{N}_1} \{p_1^{k_1} \times p_2^{f_2} : k_1 \in \mathbb{N}^*\} \cup \bigcup_{f_2 \in \mathbb{N}_2^*} \{p_1^{k_1} \times p_2^{f_2} : k_1 \in \mathbb{N}^*\}, \mathcal{F}_3 = \{\emptyset, \{n \in \Omega_3 : F' \in \{odd\}\}, \{n \in \Omega_3 : F' \in \{even\}\}, \Omega_3\}, P_3 : \mathcal{F}_3 \to [0,1] \rangle$.

Since $\Omega_3 = \Omega'_3 \cup \Omega''_3$ and $\Omega'_3 \cap \Omega''_3 = \emptyset$ and $P'_3(\{n \in \Omega'_3 : F' \in \{odd\}\}) = P''_3(\{n \in \Omega''_3 : F' \in \{odd\}\}) = \frac{1}{2}$, by applying **Lemma 2** we can deduce that

$P_3(\{n \in \Omega_3 : F' \in \{odd\}\}) = P'_3(\{n \in \Omega'_3 : F' \in \{odd\}\}) = P''_3(\{n \in \Omega''_3 : F' \in \{odd\}\}) = \frac{1}{2}$, $P_3(\{n \in \Omega_3 : F' \in \{even\}\}) = P'_3(\{n \in \Omega'_3 : F' \in \{even\}\}) = P''_3(\{n \in \Omega''_3 : F' \in \{even\}\}) = \frac{1}{2}$ and $P_3(\Omega_3) = P_3(\{n \in \Omega_3 : F' \in \{odd\}\}) + P_3(\{n \in \Omega_3 : F' \in \{even\}\}) = \frac{1}{2} + \frac{1}{2} = 1$.

* * *

Having shown for any 2 given prime numbers $p_1, p_2 \in \mathbb{P}$, $p_1 < p_2$, that $P_3(\{n \in \Omega_3 : F' \in \{odd\}\}) = P_3(\{n \in \Omega_3 : F' \in \{even\}\}) = \frac{1}{2}$ and given that by definition $\bigcup_{k_2 \in \mathbb{N}_1 \cup k_2 \in \mathbb{N}^*_2} \{p_1^{k_1} \times p_2^{k_2} : k_1 \in \mathbb{N}^*\} = \bigcup_{f_2 \in \mathbb{N}_1} \{p_1^{k_1} \times p_2^{f_2} : k_1 \in \mathbb{N}^*\} \cup \bigcup_{f_2 \in \mathbb{N}^*_2} \{p_1^{k_1} \times p_2^{f_2} : k_1 \in \mathbb{N}^*\}$ we can immediately deduce that : $P_{L_5}(\{n \in \Omega_{L_5} : F' \in \{odd\}\}) = P_3(\{n \in \Omega_3 : F' \in \{odd\}\}) = \frac{1}{2}$, $P_{L_5}(\{n \in \Omega_{L_5} : F' \in \{even\}\}) = P_3(\{n \in \Omega_3 : F' \in \{even\}\}) = \frac{1}{2}$ and $P_{L_5}(\Omega_{L_5}) = P_{L_5}(\{n \in \Omega_{L_5} : F' \in \{odd\}\}) + P_{L_5}(\{n \in \Omega_{L_5} : F' \in \{even\}\}) = \frac{1}{2} + \frac{1}{2} = 1$; which establishes **Lemma 5**.

Theorem 3. *To draw randomly any integer n from $\mathbb{B}_2$ then the probability that F being odd is equal to the probability that F being even which is $\frac{1}{2}$. That is, considering the probability space $\langle \Omega_{B_2} = \{n \in \mathbb{B}_2 : F' \in \{odd\}\} \cup \{n \in \mathbb{B}_2 : F' \in \{even\}\}, \mathcal{F}_{B_2} = \{\emptyset, \{n \in \mathbb{B}_2 : F' \in \{odd\}\}, \{n \in \mathbb{B}_2 : F' \in \{even\}\}, \Omega_{B_2}\}, P_{B_2} : \mathcal{F}_{B_2} \to [0,1]\rangle$ then :*

$$P_{B_2}(\{n \in \mathbb{B}_2 : F' \in \{odd\}\}) = P_{B_2}(\{n \in \mathbb{B}_2 : F' \in \{even\}\}) = \frac{1}{2}$$

Proof

Let us consider the probability space :

$\langle \Omega_{B_2} = \{n \in \mathbb{B}_2 : F' \in \{odd\}\} \cup \{n \in \mathbb{B}_2 : F' \in \{even\}\}, \mathcal{F}_{B_2} = \{\emptyset, \{n \in \mathbb{B}_2 : F' \in \{odd\}\}, \{n \in \mathbb{B}_2 : F' \in \{even\}\}, \Omega_{B_2}\}, P_{B_2} : \mathcal{F}_{B_2} \to [0,1]\rangle$.

Since $\mathbb{B}_2 = \bigcup_{p_1, p_2 \in \mathbb{P}, p_1 < p_2} (\bigcup_{k_2 \in \mathbb{N}_1 \cup k_2 \in \mathbb{N}^*_2} \{p_1^{k_1} \times p_2^{k_2} : k_1 \in \mathbb{N}^*\})$ and $\forall p_1, p_2 \in \mathbb{P}, p_1 < p_2, \forall p'_1, p'_2 \in \mathbb{P}, p'_1 < p'_2$, $p_1 \times p_2 \neq p'_1 \times p'_2$, $\bigcup_{k_2 \in \mathbb{N}_1 \cup k_2 \in \mathbb{N}^*_2} \{p_1^{k_1} \times p_2^{k_2} : k_1 \in \mathbb{N}^*\} \cap \bigcup_{k_2 \in \mathbb{N}_1 \cup k_2 \in \mathbb{N}^*_2} \{{p'_1}^{k_1} \times {p'_2}^{k_2} : k_1 \in \mathbb{N}^*\} = \emptyset$ and given **Lemma 5**, $\forall p_1, p_2 \in \mathbb{P}, p_1 < p_2$, $P_{L_5}(\{n \in \Omega_{L_5} : F' \in \{odd\}\}) = \frac{1}{2}$, by applying **Lemma 1** we can deduce that $P_{B_2}(\{n \in \mathbb{B}_2 : F' \in \{odd\}\}) = P_{L_5}(\{n \in \Omega_{L_5} : F' \in \{odd\}\}) = \frac{1}{2}$, $P_{B_2}(\{n \in \mathbb{B}_2 : F' \in \{even\}\}) = P_{L_5}(\{n \in \Omega_{L_5} : F' \in \{even\}\}) = \frac{1}{2}$ and $P_{B_2}(\Omega_{B_2}) = P_{B_2}(\{n \in \mathbb{B}_2 : F' \in \{odd\}\}) + P_{B_2}(\{n \in \mathbb{B}_2 : F' \in \{even\}\}) = \frac{1}{2} + \frac{1}{2} = 1$; which establishes **Theorem 3**.

4/ n belongs to $\mathbb{B}_m, m \in \mathbb{N}^*, m \geq 2$

Lemma 6. *$\forall m \in \mathbb{N}^*$, $m \geq 2$, $\forall p_1, p_2, \ldots, p_i, \ldots, p_m \in \mathbb{P}$, $p_1 < p_2 < \ldots < p_i < \ldots < p_m$ to draw randomly any integer n from*

$\bigcup_{k_m \in \mathbb{N}_1 \cup k_m \in \mathbb{N}_2^*}(\ldots(\bigcup_{k_i \in \mathbb{N}_1 \cup k_i \in \mathbb{N}_2^*}(\ldots(\bigcup_{k_2 \in \mathbb{N}_1 \cup k_2 \in \mathbb{N}_2^*}\{p_1^{k_1} \times p_2^{k_2} \times \ldots \times p_i^{k_i} \cdots \times p_m^{k_m} : k_1 \in \mathbb{N}^*\})\ldots))\ldots)$ *then the probability that F being odd is equal to the probability that F being even which is $\frac{1}{2}$. That is, considering the probability space* $\langle \Omega_{L_6} = \bigcup_{k_m \in \mathbb{N}_1 \cup k_m \in \mathbb{N}_2^*}(\ldots(\bigcup_{k_i \in \mathbb{N}_1 \cup k_i \in \mathbb{N}_2^*}(\ldots(\bigcup_{k_2 \in \mathbb{N}_1 \cup k_2 \in \mathbb{N}_2^*}\{p_1^{k_1} \times p_2^{k_2} \times \ldots \times p_i^{k_i} \cdots \times p_m^{k_m} : k_1 \in \mathbb{N}^*\})\ldots))\ldots), \mathcal{F}_{L_6} = \{\emptyset, \{n \in \Omega_{L_6} : F' \in \{odd\}\}, \{n \in \Omega_{L_6} : F' \in \{even\}\}, \Omega_{L_6}\}, P_{L_6} : \mathcal{F}_{L_6} \to [0,1]\rangle$ *then :*

$$P_{L_6}(\{n \in \Omega_{L_6} : F' \in \{odd\}\}) = P_{L_6}(\{n \in \Omega_{L_6} : F' \in \{even\}\}) = \frac{1}{2}$$

Proof

Let us prove by mathematical induction **Lemma 6**.

Lemma 5 means that **Lemma 6** is true for $m = 2$.

Let $m \in \mathbb{N}^*$, $m > 2$ be any given integer in $\mathbb{N}^*$ and let us assume that **Lemma 6** is true for m. Let us now prove that **Lemma 6** is true for $m + 1$.

* * *

Let $p_1, p_2 \ldots, p_i, \ldots, p_{m+1}, p_1 < p_2 < \ldots < p_i < \ldots < p_{m+1}$ be any $m + 1$ given prime numbers in $\mathbb{P}$ and let f_{m+1} be any given integer in $\mathbb{N}^*$.

When considering $n \in \bigcup_{k_m \in \mathbb{N}_1 \cup k_m \in \mathbb{N}_2^*}(\ldots(\bigcup_{k_i \in \mathbb{N}_1 \cup k_i \in \mathbb{N}_2^*}(\ldots(\bigcup_{k_2 \in \mathbb{N}_1 \cup k_2 \in \mathbb{N}_2^*}\{p_1^{k_1} \times p_2^{k_2} \times \ldots \times p_i^{k_i} \cdots \times p_m^{k_m} \times p_{m+1}^{f_{m+1}} : k_1 \in \mathbb{N}^*\})\ldots))\ldots), n = p_1^{f_1} \times p_2^{f_2} \times \ldots \times p_i^{f_i} \times \cdots \times p_m^{f_m} \times p_{m+1}^{f_{m+1}}$, it comes that the parity of F is given by the compounding by additivity of the parity of the sum $(f_1 + f_2 + \ldots + f_i + \ldots + f_m)$ and the parity of f_{m+1}.

* * *

Given that p_{m+1} and f_{m+1} being given and fixed, by the unique prime factorization theorem, it is clear that to any integer $n \in \bigcup_{k_m \in \mathbb{N}_1 \cup k_m \in \mathbb{N}_2^*}(\ldots(\bigcup_{k_i \in \mathbb{N}_1 \cup k_i \in \mathbb{N}_2^*}(\ldots(\bigcup_{k_2 \in \mathbb{N}_1 \cup k_2 \in \mathbb{N}_2^*}\{p_1^{k_1} \times p_2^{k_2} \times \ldots \times p_i^{k_i} \cdots \times p_m^{k_m} \times p_{m+1}^{f_{m+1}} : k_1 \in \mathbb{N}^*\})\ldots))\ldots), n = p_1^{f_1} \times p_2^{f_2} \times \ldots \times p_i^{f_i} \times \cdots \times p_m^{f_m} \times p_{m+1}^{f_{m+1}}$, corresponds the unique sequence $(f_1, f_2, \ldots, f_i, \ldots, f_m) \in \bigcup_{f_m \in \mathbb{N}_1 \cup f_m \in \mathbb{N}_2^*}(\ldots(\bigcup_{f_i \in \mathbb{N}_1 \cup f_i \in \mathbb{N}_2^*}(\ldots(\bigcup_{f_2 \in \mathbb{N}_1 \cup f_2 \in \mathbb{N}_2^*}\{(f_1, f_2, \ldots, f_i, \ldots, f_m) : f_1 \in \mathbb{N}^*\})\ldots))\ldots)$, and vice versa to any sequence $(f_1, f_2, \ldots, f_i, \ldots, f_m) \in \bigcup_{f_m \in \mathbb{N}_1 \cup f_m \in \mathbb{N}_2^*}(\ldots(\bigcup_{f_i \in \mathbb{N}_1 \cup f_i \in \mathbb{N}_2^*}(\ldots(\bigcup_{f_2 \in \mathbb{N}_1 \cup f_2 \in \mathbb{N}_2^*}\{(f_1, f_2, \ldots, f_i, \ldots, f_m) : f_1 \in \mathbb{N}^*\})\ldots))\ldots)$, corresponds the unique integer $n \in \bigcup_{k_m \in \mathbb{N}_1 \cup k_m \in \mathbb{N}_2^*}(\ldots(\bigcup_{k_i \in \mathbb{N}_1 \cup k_i \in \mathbb{N}_2^*}(\ldots(\bigcup_{k_2 \in \mathbb{N}_1 \cup k_2 \in \mathbb{N}_2^*}\{p_1^{k_1} \times p_2^{k_2} \times \ldots \times p_i^{k_i} \times \cdots \times p_m^{k_m} \times p_{m+1}^{f_{m+1}} : k_1 \in \mathbb{N}^*\})\ldots))\ldots), n = p_1^{f_1} \times p_2^{f_2} \times \ldots \times p_i^{f_i} \times \cdots \times p_m^{f_m} \times p_{m+1}^{f_{m+1}}$. That is to say that be a function $f_{T_4} : \bigcup_{k_m \in \mathbb{N}_1 \cup k_m \in \mathbb{N}_2^*}(\ldots(\bigcup_{k_i \in \mathbb{N}_1 \cup k_i \in \mathbb{N}_2^*}(\ldots(\bigcup_{k_2 \in \mathbb{N}_1 \cup k_2 \in \mathbb{N}_2^*}\{p_1^{k_1} \times p_2^{k_2} \times \ldots \times p_i^{k_i} \cdots \times p_m^{k_m} \times p_{m+1}^{f_{m+1}} : k_1 \in \mathbb{N}^*\})\ldots))\ldots) \to \bigcup_{f_m \in \mathbb{N}_1 \cup f_m \in \mathbb{N}_2^*}(\ldots(\bigcup_{f_i \in \mathbb{N}_1 \cup f_i \in \mathbb{N}_2^*}(\ldots(\bigcup_{f_2 \in \mathbb{N}_1 \cup f_2 \in \mathbb{N}_2^*}\{(f_1, f_2, \ldots, f_i, \ldots, f_m) :$

$f_1 \in \mathbb{N}^*\})\ldots))\ldots)$ such that for any $n \in \bigcup_{k_m\in\mathbb{N}_1\cup k_m\in\mathbb{N}_2^*}(\cdots(\bigcup_{k_i\in\mathbb{N}_1\cup k_i\in\mathbb{N}_2^*}(\cdots(\bigcup_{k_2\in\mathbb{N}_1\cup k_2\in\mathbb{N}_2^*}\{p_1^{k_1} \times p_2^{k_2} \times \ldots \times p_i^{k_i} \cdots \times p_m^{k_m} \times p_{m+1}^{f_{m+1}} : k_1 \in \mathbb{N}^*\})\ldots))\ldots), n = p_1^{f_1} \times p_2^{f_2} \times \ldots \times p_i^{f_i} \times \cdots \times p_m^{f_m} \times p_{m+1}^{f_{m+1}}$, we have $f_{T_4}(n) = (f_1, f_2, \ldots, f_i, \ldots, f_m) \in \bigcup_{f_m\in\mathbb{N}_1\cup f_m\in\mathbb{N}_2^*}(\cdots(\bigcup_{f_i\in\mathbb{N}_1\cup f_i\in\mathbb{N}_2^*}(\cdots(\bigcup_{f_2\in\mathbb{N}_1\cup f_2\in\mathbb{N}_2^*}\{(f_1, f_2, \ldots, f_i, \ldots, f_m) : f_1 \in \mathbb{N}^*\})\ldots))\ldots)$, then f_{T_4} is a bijective function from $\bigcup_{k_m\in\mathbb{N}_1\cup k_m\in\mathbb{N}_2^*}(\cdots(\bigcup_{k_i\in\mathbb{N}_1\cup k_i\in\mathbb{N}_2^*}(\cdots(\bigcup_{k_2\in\mathbb{N}_1\cup k_2\in\mathbb{N}_2^*}\{p_1^{k_1} \times p_2^{k_2} \times \ldots \times p_i^{k_i} \times \cdots \times p_m^{k_m} \times p_{m+1}^{f_{m+1}} : k_1 \in \mathbb{N}^*\})\ldots))\ldots)$ toward $\bigcup_{f_m\in\mathbb{N}_1\cup f_m\in\mathbb{N}_2^*}(\cdots(\bigcup_{f_i\in\mathbb{N}_1\cup f_i\in\mathbb{N}_2^*}(\cdots(\bigcup_{f_2\in\mathbb{N}_1\cup f_2\in\mathbb{N}_2^*}\{(f_1, f_2, \ldots, f_i, \ldots, f_m) : f_1 \in \mathbb{N}^*\})\ldots))\ldots)$, whose inverse function is $f_{T_4}^{-1} : \bigcup_{f_m\in\mathbb{N}_1\cup f_m\in\mathbb{N}_2^*}(\cdots(\bigcup_{f_i\in\mathbb{N}_1\cup f_i\in\mathbb{N}_2^*}(\cdots(\bigcup_{f_2\in\mathbb{N}_1\cup f_2\in\mathbb{N}_2^*}\{(f_1, f_2, \ldots, f_i, \ldots, f_m) : f_1 \in \mathbb{N}^*\})\ldots))\ldots) \to \bigcup_{k_m\in\mathbb{N}_1\cup k_m\in\mathbb{N}_2^*}(\cdots(\bigcup_{k_i\in\mathbb{N}_1\cup k_i\in\mathbb{N}_2^*}(\cdots(\bigcup_{k_2\in\mathbb{N}_1\cup k_2\in\mathbb{N}_2^*}\{p_1^{k_1}\times p_2^{k_2} \times \ldots \times p_i^{k_i} \times \cdots \times p_m^{k_m} \times p_{m+1}^{f_{m+1}} : k_1 \in \mathbb{N}^*\})\ldots))\ldots)$, such that for any sequence $(f_1, f_2, \ldots, f_i, \ldots, f_m) \in \bigcup_{f_m\in\mathbb{N}_1\cup f_m\in\mathbb{N}_2^*}(\cdots(\bigcup_{f_i\in\mathbb{N}_1\cup f_i\in\mathbb{N}_2^*}(\cdots(\bigcup_{f_2\in\mathbb{N}_1\cup f_2\in\mathbb{N}_2^*}\{(f_1, f_2, \ldots, f_i, \ldots, f_m) : f_1 \in \mathbb{N}^*\})\ldots))\ldots)$, we have $f_{T_4}^{-1}((f_1, f_2, \ldots, f_i, \ldots, f_m)) = p_1^{f_1} \times p_2^{f_2} \times \ldots \times p_i^{f_i} \times \cdots \times p_m^{f_m} \times p_{m+1}^{f_{m+1}} = n, n \in \bigcup_{k_m\in\mathbb{N}_1\cup k_m\in\mathbb{N}_2^*}(\cdots(\bigcup_{k_i\in\mathbb{N}_1\cup k_i\in\mathbb{N}_2^*}(\cdots(\bigcup_{k_2\in\mathbb{N}_1\cup k_2\in\mathbb{N}_2^*}\{p_1^{k_1} \times p_2^{k_2} \times \ldots \times p_i^{k_i} \cdots \times p_m^{k_m} \times p_{m+1}^{f_{m+1}} : k_1 \in \mathbb{N}^*\})\ldots))\ldots)$.

Thus : $\bigcup_{k_m\in\mathbb{N}_1\cup k_m\in\mathbb{N}_2^*}(\cdots(\bigcup_{k_i\in\mathbb{N}_1\cup k_i\in\mathbb{N}_2^*}(\cdots(\bigcup_{k_2\in\mathbb{N}_1\cup k_2\in\mathbb{N}_2^*}\{p_1^{k_1} \times p_2^{k_2} \times \ldots \times p_i^{k_i} \times \cdots \times p_m^{k_m} \times p_{m+1}^{f_{m+1}} : k_1 \in \mathbb{N}^*\})\ldots))\ldots) = \bigcup_{f_m\in\mathbb{N}_1\cup f_m\in\mathbb{N}_2^*}(\ldots(\bigcup_{f_i\in\mathbb{N}_1\cup f_i\in\mathbb{N}_2^*}(\cdots(\bigcup_{f_2\in\mathbb{N}_1\cup f_2\in\mathbb{N}_2^*}\{f_{T_4}^{-1}((f_1, f_2, \ldots, f_i, \ldots, f_m)) : f_1 \in \mathbb{N}^*\})\ldots))\ldots)$.

Thus each integer n in $\bigcup_{k_m\in\mathbb{N}_1\cup k_m\in\mathbb{N}_2^*}(\cdots(\bigcup_{k_i\in\mathbb{N}_1\cup k_i\in\mathbb{N}_2^*}(\cdots(\bigcup_{k_2\in\mathbb{N}_1\cup k_2\in\mathbb{N}_2^*}\{p_1^{k_1} \times p_2^{k_2} \times \ldots \times p_i^{k_i} \cdots \times p_m^{k_m} \times p_{m+1}^{f_{m+1}} : k_1 \in \mathbb{N}^*\})\ldots))\ldots)$ necessarily has only but one corresponding sequence $(f_1, f_2, \ldots, f_i, \ldots, f_m)$ that determines the parity of the number of prime factor(s) counted with multiplicity of n, F (the parity f'_{m+1} being given and fixed).

Let us now consider the random experiment consisting in drawing randomly any integer n from $\bigcup_{k_m\in\mathbb{N}_1\cup k_m\in\mathbb{N}_2^*}(\cdots(\bigcup_{k_i\in\mathbb{N}_1\cup k_i\in\mathbb{N}_2^*}(\cdots(\bigcup_{k_2\in\mathbb{N}_1\cup k_2\in\mathbb{N}_2^*}\{p_1^{k_1} \times p_2^{k_2} \times \ldots \times p_i^{k_i} \cdots \times p_m^{k_m} \times p_{m+1}^{f_{m+1}} : k_1 \in \mathbb{N}^*\})\ldots))\ldots)$ in order to note as the outcome the parity of the sum $(f_1 + f_2 + \ldots + f_i + \ldots + f_m)$ of the corresponding sequence of exponents $(f_1, f_2, \ldots, f_i, \ldots, f_m)$ given by $f_{T_n}(n)$ (let us note by S'_m the latter parity). The probability space associated with the latter random experiment is $\langle \Omega_{T_4} = \{S'_m : S'_m \in \{odd, even\}\}, \mathcal{F}_{T_4} = 2^{\Omega_{T_4}}, P_{T_4} : \mathcal{F}_{T_4} \to [0, 1]\rangle$.

Let us note by $e_{S'_m} \in \mathcal{F}_{T_4}$ the event that S'_m is odd and $\overline{e_{S'_m}} \in \mathcal{F}_{T_4}$ the event that S'_m is even.

Given that we have initially assumed that **Lemma 5** is true for m, it comes that $P_{T_4}(e_{S'_m}) = P_{T_4}(\overline{e_{S'_m}}) = \frac{1}{2}$.

It is clear that the total number of possible outcomes S'_m, $card(\Omega_{T_4})$ is equal to 2; thus $P_{T_4}(\Omega_{T_4}) = P_{T_4}(e_{S'_m} \cup \overline{e_{S'_m}}) = \frac{1}{2} + \frac{1}{2} = 1$ indeed.

* * *

For any given $f_{m+1} \in \mathbb{N}_1$, let us now consider the following probability space :

$\langle \Omega'_{T_4} = \bigcup_{k_m \in \mathbb{N}_1 \cup k_m \in \mathbb{N}_2^*}(\cdots(\bigcup_{k_i \in \mathbb{N}_1 \cup k_i \in \mathbb{N}_2^*}(\cdots(\bigcup_{k_2 \in \mathbb{N}_1 \cup k_2 \in \mathbb{N}_2^*}\{p_1^{k_1} \times p_2^{k_2} \times \ldots \times p_i^{k_i} \cdots \times p_m^{k_m} \times p_{m+1}^{f_{m+1}} : k_1 \in \mathbb{N}^*\})\ldots))\ldots), \mathcal{F}'_{T_4} = \{\emptyset, \{n \in \Omega'_{T_4} : F' \in \{odd\}\}, \{n \in \Omega'_{T_4} : F' \in \{even\}\}, \Omega'_{T_4}\}, P'_{T_4} : \mathcal{F}'_{T_4} \to [0,1]\rangle$.

$f_{m+1} \in \mathbb{N}_1$, f_{m+1} being given and fixed, F is odd iff S'_m is even and F is even iff S'_m is odd. Therefore we have : $P'_{T_4}(\{n \in \Omega'_{T_4} : F' \in \{odd\}\}) = P_{T_4}(\overline{e_{S'_m}}) = \frac{1}{2}$, $P'_{T_4}(\{n \in \Omega'_{T_4} : F' \in \{even\}\}) = P_{T_4}(e_{S'_m}) = \frac{1}{2}$ and $P'_{T_4}(\Omega'_{T_4}) = \frac{1}{2} + \frac{1}{2} = 1$.

Then let us consider the following probability space :

$\langle \Omega'_4 = \bigcup_{f_{m+1} \in \mathbb{N}_1}(\bigcup_{k_m \in \mathbb{N}_1 \cup k_m \in \mathbb{N}_2^*}(\cdots(\bigcup_{k_i \in \mathbb{N}_1 \cup k_i \in \mathbb{N}_2^*}(\cdots(\bigcup_{k_2 \in \mathbb{N}_1 \cup k_2 \in \mathbb{N}_2^*}\{p_1^{k_1} \times p_2^{k_2} \times \ldots \times p_i^{k_i} \cdots \times p_m^{k_m} \times p_{m+1}^{f_{m+1}} : k_1 \in \mathbb{N}^*\})\ldots))\ldots)), \mathcal{F}'_4 = \{\emptyset, \{n \in \Omega'_4 : F' \in \{odd\}\}, \{n \in \Omega'_4 : F' \in \{even\}\}, \Omega'_4\}, P'_4 : \mathcal{F}'_4 \to [0,1]\rangle$.

Since $\Omega'_4 = \bigcup_{f_{m+1} \in \mathbb{N}_1}(\bigcup_{k_m \in \mathbb{N}_1 \cup k_m \in \mathbb{N}_2^*}(\cdots(\bigcup_{k_i \in \mathbb{N}_1 \cup k_i \in \mathbb{N}_2^*}(\cdots(\bigcup_{k_2 \in \mathbb{N}_1 \cup k_2 \in \mathbb{N}_2^*}\{p_1^{k_1} \times p_2^{k_2} \times \ldots \times p_i^{k_i} \cdots \times p_m^{k_m} \times p_{m+1}^{f_{m+1}} : k_1 \in \mathbb{N}^*\})\ldots))\ldots))$ and $\forall f_{m+1}, f''_{m+1} \in \mathbb{N}_1$, $f_{m+1} \neq f''_{m+1}$, $\bigcup_{k_m \in \mathbb{N}_1 \cup k_m \in \mathbb{N}_2^*}(\cdots(\bigcup_{k_i \in \mathbb{N}_1 \cup k_i \in \mathbb{N}_2^*}(\cdots(\bigcup_{k_2 \in \mathbb{N}_1 \cup k_2 \in \mathbb{N}_2^*}\{p_1^{k_1} \times p_2^{k_2} \times \ldots \times p_i^{k_i} \cdots \times p_m^{k_m} \times p_{m+1}^{f_{m+1}} : k_1 \in \mathbb{N}^*\})\ldots))\ldots) \cap \bigcup_{k_m \in \mathbb{N}_1 \cup k_m \in \mathbb{N}_2^*}(\cdots(\bigcup_{k_i \in \mathbb{N}_1 \cup k_i \in \mathbb{N}_2^*}(\cdots(\bigcup_{k_2 \in \mathbb{N}_1 \cup k_2 \in \mathbb{N}_2^*}\{p_1^{k_1} \times p_2^{k_2} \times \ldots \times p_i^{k_i} \cdots \times p_m^{k_m} \times p_{m+1}^{f''_{m+1}} : k_1 \in \mathbb{N}^*\})\ldots))\ldots) = \emptyset$ and $\forall f_{m+1} \in \mathbb{N}_1$, $P'_{T_4}(\{n \in \Omega'_{T_4} : F' \in \{odd\}\}) = \frac{1}{2}$, by applying **Lemma 1** we can deduce that : $P'_4(\{n \in \Omega'_4 : F' \in \{odd\}\}) = P'_{T_4}(\{n \in \Omega'_{T_4} : F' \in \{odd\}\}) = \frac{1}{2}$, $P'_4(\{n \in \Omega'_4 : F' \in \{even\}\}) = P'_{T_4}(\{n \in \Omega'_{T_4} : F' \in \{even\}\}) = \frac{1}{2}$ and $P'_4(\Omega'_4) = P'_4(\{n \in \Omega'_4 : F' \in \{odd\}\}) + P'_4(\{n \in \Omega'_4 : F' \in \{even\}\}) = \frac{1}{2} + \frac{1}{2} = 1$.

* * *

For any given $f_{m+1} \in \mathbb{N}_2^*$, let us now consider the following probability space :

$\langle \Omega''_{T_4} = \bigcup_{k_m \in \mathbb{N}_1 \cup k_m \in \mathbb{N}_2^*}(\cdots(\bigcup_{k_i \in \mathbb{N}_1 \cup k_i \in \mathbb{N}_2^*}(\cdots(\bigcup_{k_2 \in \mathbb{N}_1 \cup k_2 \in \mathbb{N}_2^*}\{p_1^{k_1} \times p_2^{k_2} \times \ldots \times p_i^{k_i} \cdots \times p_m^{k_m} \times p_{m+1}^{f_{m+1}} : k_1 \in \mathbb{N}^*\})\ldots))\ldots), \mathcal{F}''_{T_4} = \{\emptyset, \{n \in \Omega''_{T_4} : F' \in \{odd\}\}, \{n \in \Omega''_{T_4} : F' \in \{even\}\}, \Omega''_{T_4}\}, P''_{T_4} : \mathcal{F}''_{T_4} \to [0,1]\rangle$.

$f_{m+1} \in \mathbb{N}_2^*$, f_{m+1} being given and fixed, F is odd iff S'_m is odd and F is even iff S'_m is even. Therefore we have : $P''_{T_4}(\{n \in \Omega''_{T_4} : F' \in \{odd\}\}) = P_{T_4}(e_{S'_m}) = \frac{1}{2}$, $P''_{T_4}(\{n \in \Omega''_{T_4} : F' \in \{even\}\}) = P_{T_4}(\overline{e_{S'_m}}) = \frac{1}{2}$ and $P''_{T_4}(\Omega''_{T_4}) = \frac{1}{2} + \frac{1}{2} = 1$.

* * *

Then let us consider the following probability space :

$\langle \Omega''_4 = \bigcup_{f_{m+1} \in \mathbb{N}_2^*}(\bigcup_{k_m \in \mathbb{N}_1 \cup k_m \in \mathbb{N}_2^*}(\cdots(\bigcup_{k_i \in \mathbb{N}_1 \cup k_i \in \mathbb{N}_2^*}(\cdots(\bigcup_{k_2 \in \mathbb{N}_1 \cup k_2 \in \mathbb{N}_2^*}\{ p_1^{k_1} \times p_2^{k_2} \times \ldots \times p_i^{k_i} \cdots \times p_m^{k_m} \times p_{m+1}^{f_{m+1}} : k_1 \in \mathbb{N}^*\})\ldots))\ldots)), \mathcal{F}''_4 = \{\emptyset, \{n \in \Omega''_4 : F' \in \{odd\}\}, \{n \in \Omega''_4 : F' \in \{even\}\}, \Omega''_4\}, P''_4 : \mathcal{F}''_4 \to [0,1]\rangle$.

Since $\Omega''_4 = \bigcup_{f_{m+1} \in \mathbb{N}_2^*}(\bigcup_{k_m \in \mathbb{N}_1 \cup k_m \in \mathbb{N}_2^*}(\cdots(\bigcup_{k_i \in \mathbb{N}_1 \cup k_i \in \mathbb{N}_2^*}(\cdots(\bigcup_{k_2 \in \mathbb{N}_1 \cup k_2 \in \mathbb{N}_2^*}\{ p_1^{k_1} \times p_2^{k_2} \times \ldots \times p_i^{k_i} \cdots \times p_m^{k_m} \times p_{m+1}^{f_{m+1}} : k_1 \in \mathbb{N}^*\})\ldots))\ldots))$ and $\forall f_{m+1}, f''_{m+1} \in \mathbb{N}_2^*$, $f_{m+1} \neq f''_{m+1}$, $\bigcup_{k_m \in \mathbb{N}_1 \cup k_m \in \mathbb{N}_2^*}(\cdots(\bigcup_{k_i \in \mathbb{N}_1 \cup k_i \in \mathbb{N}_2^*}(\cdots(\bigcup_{k_2 \in \mathbb{N}_1 \cup k_2 \in \mathbb{N}_2^*}\{p_1^{k_1} \times p_2^{k_2} \times \ldots \times p_i^{k_i} \cdots \times p_m^{k_m} \times p_{m+1}^{f_{m+1}} : k_1 \in \mathbb{N}^*\})\ldots))\ldots) \cap \bigcup_{k_m \in \mathbb{N}_1 \cup k_m \in \mathbb{N}_2^*}(\cdots(\bigcup_{k_i \in \mathbb{N}_1 \cup k_i \in \mathbb{N}_2^*}(\cdots(\bigcup_{k_2 \in \mathbb{N}_1 \cup k_2 \in \mathbb{N}_2^*}\{p_1^{k_1} \times p_2^{k_2} \times \ldots \times p_i^{k_i} \cdots \times p_m^{k_m} \times p_{m+1}^{f''_{m+1}} : k_1 \in \mathbb{N}^*\})\ldots))\ldots) = \emptyset$ and $\forall f_{m+1} \in \mathbb{N}_2^*$, $P''_{T_4}(\{n \in \Omega''_{T_4} : F' \in \{odd\}\}) = \frac{1}{2}$, by applying **Lemma 1** we can deduce that : $P''_4(\{n \in \Omega''_4 : F' \in \{odd\}\}) = P''_{T_4}(\{n \in \Omega''_{T_4} : F' \in \{odd\}\}) = \frac{1}{2}$, $P''_4(\{n \in \Omega''_4 : F' \in \{even\}\}) = P''_{T_4}(\{n \in \Omega''_{T_4} : F' \in \{even\}\}) = \frac{1}{2}$ and $P''_4(\Omega''_4) = P''_4(\{n \in \Omega''_4 : F' \in \{odd\}\}) + P''_4(\{n \in \Omega''_4 : F' \in \{even\}\}) = \frac{1}{2} + \frac{1}{2} = 1$.

* * *

Then we can consider the following probability space :

$\langle \Omega_4 = \bigcup_{f_{m+1} \in \mathbb{N}_1}(\bigcup_{k_m \in \mathbb{N}_1 \cup k_m \in \mathbb{N}_2^*}(\cdots(\bigcup_{k_i \in \mathbb{N}_1 \cup k_i \in \mathbb{N}_2^*}(\cdots(\bigcup_{k_2 \in \mathbb{N}_1 \cup k_2 \in \mathbb{N}_2^*}\{p_1^{k_1} \times p_2^{k_2} \times \ldots \times p_i^{k_i} \cdots \times p_m^{k_m} \times p_{m+1}^{f_{m+1}} : k_1 \in \mathbb{N}^*\})\ldots))\ldots)) \cup \bigcup_{f_{m+1} \in \mathbb{N}_2^*}(\bigcup_{k_m \in \mathbb{N}_1 \cup k_m \in \mathbb{N}_2^*}(\cdots(\bigcup_{k_i \in \mathbb{N}_1 \cup k_i \in \mathbb{N}_2^*}(\cdots(\bigcup_{k_2 \in \mathbb{N}_1 \cup k_2 \in \mathbb{N}_2^*}\{p_1^{k_1} \times p_2^{k_2} \times \ldots \times p_i^{k_i} \cdots \times p_m^{k_m} \times p_{m+1}^{f_{m+1}} : k_1 \in \mathbb{N}^*\})\ldots))\ldots)), \mathcal{F}_4 = \{\emptyset, \{n \in \Omega_4 : F' \in \{odd\}\}, \{n \in \Omega_4 : F' \in \{even\}\}, \Omega_4\}, P_4 : \mathcal{F}_4 \to [0,1]\rangle$.

Since $\Omega_4 = \Omega'_4 \cup \Omega''_4$ and $\Omega'_4 \cap \Omega''_4 = \emptyset$ and $P'_4(\{n \in \Omega'_4 : F' \in \{odd\}\}) = P''_4(\{n \in \Omega''_4 : F' \in \{odd\}\}) = \frac{1}{2}$, by applying **Lemma 2** we can deduce that $P_4(\{n \in \Omega_4 : F' \in \{odd\}\}) = P'_4(\{n \in \Omega'_4 : F' \in \{odd\}\}) = P''_4(\{n \in \Omega''_4 : F' \in \{odd\}\}) = \frac{1}{2}$, $P_4(\{n \in \Omega_4 : F' \in \{even\}\}) = P'_4(\{n \in \Omega'_4 : F' \in \{even\}\}) = P''_4(\{n \in \Omega''_4 : F' \in \{even\}\}) = \frac{1}{2}$ and $P_4(\Omega_4) = P_4(\{n \in \Omega_4 : F' \in \{odd\}\}) + P_4(\{n \in \Omega_4 : F' \in \{even\}\}) = \frac{1}{2} + \frac{1}{2} = 1$.

* * *

Having shown for any $m+1$ given prime numbers $p_1, p_2, \ldots, p_i, \ldots, p_{m+1} \in \mathbb{P}$, $p_1 < p_2 < \ldots < p_i < \ldots < p_{m+1}$, that $P_4(\{n \in \Omega_4 : F' \in \{odd\}\}) =$

$P_4(\{n \in \Omega_4 : F' \in \{even\}\}) = \frac{1}{2}$ and given that by definition $\bigcup_{k_{m+1}\in\mathbb{N}_1\cup k_{m+1}\in\mathbb{N}_2^*}(\bigcup_{k_m\in\mathbb{N}_1\cup k_m\in\mathbb{N}_2^*}(\cdots(\bigcup_{k_i\in\mathbb{N}_1\cup k_i\in\mathbb{N}_2^*}(\cdots(\bigcup_{k_2\in\mathbb{N}_1\cup k_2\in\mathbb{N}_2^*}\{p_1^{k_1} \times p_2^{k_2} \times \ldots \times p_i^{k_i} \cdots \times p_m^{k_m} \times p_{m+1}^{k_{m+1}} : k_1 \in \mathbb{N}^*\})\ldots))\ldots)) = \bigcup_{f_{m+1}\in\mathbb{N}_1}(\bigcup_{k_m\in\mathbb{N}_1\cup k_m\in\mathbb{N}_2^*}(\cdots(\bigcup_{k_i\in\mathbb{N}_1\cup k_i\in\mathbb{N}_2^*}(\cdots(\bigcup_{k_2\in\mathbb{N}_1\cup k_2\in\mathbb{N}_2^*}\{p_1^{k_1} \times p_2^{k_2} \times \ldots \times p_i^{k_i} \cdots \times p_m^{k_m} \times p_{m+1}^{f_{m+1}} : k_1 \in \mathbb{N}^*\})\ldots))\ldots)) \cup \bigcup_{f_{m+1}\in\mathbb{N}_2^*}(\bigcup_{k_m\in\mathbb{N}_1\cup k_m\in\mathbb{N}_2^*}(\cdots(\bigcup_{k_i\in\mathbb{N}_1\cup k_i\in\mathbb{N}_2^*}(\cdots(\bigcup_{k_2\in\mathbb{N}_1\cup k_2\in\mathbb{N}_2^*}\{p_1^{k_1} \times p_2^{k_2} \times \ldots \times p_i^{k_i} \cdots \times p_m^{k_m} \times p_{m+1}^{f_{m+1}} : k_1 \in \mathbb{N}^*\})\ldots))\ldots))$ we can immediately deduce that : $P_{L_6}(\{n \in \Omega_{L_6} : F' \in \{odd\}\}) = P_4(\{n \in \Omega_4 : F' \in \{odd\}\}) = \frac{1}{2}$, $P_{L_6}(\{n \in \Omega_{L_6} : F' \in \{even\}\}) = P_4(\{n \in \Omega_4 : F' \in \{even\}\}) = \frac{1}{2}$ and $P_{L_6}(\Omega_{L_6}) = P_{L_6}(\{n \in \Omega_{L_6} : F' \in \{odd\}\}) + P_{L_6}(\{n \in \Omega_{L_6} : F' \in \{even\}\}) = \frac{1}{2} + \frac{1}{2} = 1$; which means that **Lemma 6** is true for $m+1$.

* * *

Having shown that **Lemma 6** is true for $m+1$, we have completed the proof by mathematical induction and we can therefore deduce that **Lemma 6** is true $\forall m \in \mathbb{N}^*$, $m \geq 2$; which establishes **Lemma 6**.

THEOREM 4. *$\forall m \in \mathbb{N}^*$, $m \geq 2$, to draw randomly any integer n from $\mathbb{B}_m$ then the probability that F being odd is equal to the probability that F being even which is $\frac{1}{2}$. That is, considering the probability space $\langle \Omega_{B_m} = \{n \in \mathbb{B}_m : F' \in \{odd\}\} \cup \{n \in \mathbb{B}_m : F' \in \{even\}\}, \mathcal{F}_{B_m} = \{\emptyset, \{n \in \mathbb{B}_m : F' \in \{odd\}\}, \{n \in \mathbb{B}_m : F' \in \{even\}\}, \Omega_{B_m}\}, P_{B_m} : \mathcal{F}_{B_m} \to [0,1]\rangle$ then :*

$$P_{B_m}(\{n \in \mathbb{B}_m : F' \in \{odd\}\}) = P_{B_m}(\{n \in \mathbb{B}_m : F' \in \{even\}\}) = \tfrac{1}{2}$$

Proof

Let $m \in \mathbb{N}^*$, $m \geq 2$ be any given integer in $\mathbb{N}^*$.

Let us consider the probability space :

$\langle \Omega_{B_m} = \{n \in \mathbb{B}_m : F' \in \{odd\}\} \cup \{n \in \mathbb{B}_m : F' \in \{even\}\}, \mathcal{F}_{B_m} = \{\emptyset, \{n \in \mathbb{B}_m : F' \in \{odd\}\}, \{n \in \mathbb{B}_m : F' \in \{even\}\}, \Omega_{B_m}\}, P_{B_m} : \mathcal{F}_{B_m} \to [0,1]\rangle$.

Since $\mathbb{B}_m = \bigcup_{p_i\in\mathbb{P}, p_1<p_2<\ldots<p_i<\ldots<p_m}(\bigcup_{k_m\in\mathbb{N}_1\cup k_m\in\mathbb{N}_2^*}(\cdots(\bigcup_{k_i\in\mathbb{N}_1\cup k_i\in\mathbb{N}_2^*}(\cdots(\bigcup_{k_2\in\mathbb{N}_1\cup k_2\in\mathbb{N}_2^*}\{(p_1^{k_1} \times p_2^{k_2} \times \ldots \times p_i^{k_i} \times \ldots \times p_m^{k_m}) : k_1 \in \mathbb{N}^*\})\ldots))\ldots))$ and $\forall p_1, p_2, \ldots, p_i, \ldots, p_m \in \mathbb{P}, p_1 < p_2 < \ldots < p_i < \ldots < p_m, \forall p'_1, p'_2, \ldots, p'_i, \ldots, p'_m \in \mathbb{P}, p'_1 < p'_2 < \ldots < p'_i < \ldots < p'_m$, $p_1 \times p_2 \times \ldots \times p_i \times \ldots \times p_m \neq p'_1 \times p'_2 \times \ldots \times p'_i \times \ldots \times p'_m$, $\bigcup_{k_m\in\mathbb{N}_1\cup k_m\in\mathbb{N}_2^*}(\cdots(\bigcup_{k_i\in\mathbb{N}_1\cup k_i\in\mathbb{N}_2^*}(\cdots(\bigcup_{k_2\in\mathbb{N}_1\cup k_2\in\mathbb{N}_2^*}\{(p_1^{k_1} \times p_2^{k_2} \times \ldots \times p_i^{k_i} \times \ldots \times p_m^{k_m}) : k_1 \in \mathbb{N}^*\})\ldots))\ldots) \cap \bigcup_{k_m\in\mathbb{N}_1\cup k_m\in\mathbb{N}_2^*}(\cdots(\bigcup_{k_i\in\mathbb{N}_1\cup k_i\in\mathbb{N}_2^*}(\cdots(\bigcup_{k_2\in\mathbb{N}_1\cup k_2\in\mathbb{N}_2^*}\{(p_1'^{k_1} \times p_2'^{k_2} \times \ldots \times p_i'^{k_i} \times \ldots \times p_m'^{k_m}) : k_1 \in \mathbb{N}^*\})\ldots))\ldots) = \emptyset$ and given **Lemma 6**, $\forall p_1, p_2, \ldots, p_i, \ldots, p_m \in \mathbb{P}, p_1 < p_2 < \ldots < p_i < \ldots < p_m$, $P_{L_6}(\{n \in \Omega_{L_6} : F' \in \{odd\}\}) = \frac{1}{2}$, by applying **Lemma 1** we can deduce that $P_{B_m}(\{n \in \mathbb{B}_m : F' \in \{odd\}\}) = P_{L_6}(\{n \in$

$\Omega_{L_6} : F' \in \{odd\}\}) = \frac{1}{2}$, $P_{B_m}(\{n \in \mathbb{B}_m : F' \in \{even\}\}) = P_{L_6}(\{n \in \Omega_{L_6} : F' \in \{even\}\}) = \frac{1}{2}$ and $P_{B_m}(\Omega_{B_m}) = P_{B_m}(\{n \in \mathbb{B}_m : F' \in \{odd\}\}) + P_{B_m}(\{n \in \mathbb{B}_m : F' \in \{even\}\}) = \frac{1}{2} + \frac{1}{2} = 1$. Having shown for any $m \in \mathbb{N}^*$, $m \geq 2$, **Theorem 4** immediately follows.

5/ n belongs to $\mathbb{N}^*$

THEOREM 5. *To draw randomly any integer n from $\mathbb{N}^*$ then the probability that F being odd is equal to the probability that F being even which is $\frac{1}{2}$. That is, considering the probability space $\langle \Omega_\Omega = \mathbb{N}^*, \mathcal{F}_\Omega = \{\emptyset, \{n \in \mathbb{N}^* : F' \in \{odd\}\}, \{n \in \mathbb{N}^* : F' \in \{even\}\}, \Omega_\Omega\}, P_\Omega : \mathcal{F}_\Omega \to [0,1]\rangle$ then :*

$$P_\Omega(\{n \in \mathbb{N}^* : F' \in \{odd\}\}) = P_\Omega(\{n \in \mathbb{N}^* : F' \in \{even\}\}) = \frac{1}{2}$$

Proof

Since $\mathbb{N}^* = \mathbb{A}_1 \cup \mathbb{A}_2 \cup \bigcup_{m\in\mathbb{N}^*, m\geq 2} \mathbb{B}_m$ and $\mathbb{A}_1 \cap \mathbb{A}_2 = \emptyset$, $\forall m \in \mathbb{N}^*, m \geq 2, \mathbb{A}_1 \cap \mathbb{B}_m = \emptyset$, $\forall m \in \mathbb{N}^*, m \geq 2, \mathbb{A}_2 \cap \mathbb{B}_m = \emptyset$, $\forall m, m' \in \mathbb{N}^*, m, m' \geq 2, m \neq m', \mathbb{B}_m \cap \mathbb{B}_{m'} = \emptyset$ and given **Theorem 1**,**Theorem 2** and **Theorem 4**, by applying **Lemma 1**, we can deduce that $P_\Omega(\{n \in \mathbb{N}^* : F' \in \{odd\}\}) = P_\Omega(\{n \in \mathbb{N}^* : F' \in \{even\}\}) = \frac{1}{2}$ and $P_\Omega(\Omega) = P_\Omega(\{n \in \mathbb{N}^* : F' \in \{odd\}\}) + P_\Omega(\{n \in \mathbb{N}^* : F' \in \{even\}\}) = \frac{1}{2} + \frac{1}{2} = 1$; which establishes **Theorem 5**.

6/ The limit of the summatory Liouville function divided by $\sqrt{\mathbf{N}}$

THEOREM 6. *Let $L(N) = \lambda(1) + \lambda(2) + \ldots + \lambda(i) + \ldots + \lambda(N)$, $\lambda(i)$ being the Liouville function applied to $i \in \mathbb{N}^*$, be the summatory Livouille function up to $N \in \mathbb{N}^*$ then :*

$$\lim_{N\to+\infty} \frac{L(N)}{\sqrt{N}} = 0$$

Proof

Let us consider $(X_1, X_2, \ldots, X_i, \ldots, X_N), N \in \mathbb{N}^*$ a sequence of N independent and identically distributed random variables X_i, where X_i is defined as the random variable consisting in drawing randomly any integer $n \in \mathbb{N}^*$ in order to note as the outcome the value given by the Liouville function such that $X_i(n) = \lambda(n)$. That is to say that $\forall i \in \mathbb{N}^*$ by considering the probability space $\langle \Omega = \mathbb{N}^*, \mathcal{F} = \{\emptyset, \{n \in \mathbb{N}^* : F' \in \{odd\}\}, \{n \in \mathbb{N}^* : F' \in \{even\}\}, \Omega\}, P(\{n \in \mathbb{N}^* : F' \in \{odd\}\}) = \frac{1}{2}\rangle$ – given **Theorem 5** – and the measurable space $\langle E = \{-1, 1\}\ , \mathcal{E} = 2^E\rangle$, X_i is defined as the random variable $X_i : \Omega \to E$ such that $\forall \omega \in \{n \in \mathbb{N}^* : F' \in \{odd\}\}$, $X_i(\omega) = \lambda(\omega) = -1$ and $\forall \omega \in \{n \in \mathbb{N}^* : F' \in \{even\}\}$, $X_i(\omega) = \lambda(\omega) = 1$ with $P(X_i = -1) = P(\{\omega \in \Omega \mid X_i(\omega) = -1\}) = \frac{1}{2}$ and $P(X_i = 1) = P(\{\omega \in \Omega \mid X_i(\omega) = 1\}) = \frac{1}{2}$.

By noting $\mathrm{E}[X_i]$ the mean of X_i and $\mathrm{Var}[X_i]$ the variance of X_i, we then have $\mathrm{E}[X_i] = 0$ and $Var[X_i] = 1$.

Let $\sqrt{N}.S_N$ be the random variable such that $\sqrt{N}.S_N = \frac{X_1+X_2+...+X_i+...+X_N}{\sqrt{N}}$. By applying the classical central limit theorem (Lindeberg-Lévy central limit theorem), it comes that as N tends to infinity, the series of random variables $\sqrt{1}.S_1, \sqrt{2}.S_2, \ldots, \sqrt{N}.S_N, \ldots$, converges in distribution to the random variable X with $X \sim \mathcal{N}(0,1)$. That is to say that :

$$\sqrt{N}.S_N \xrightarrow[N\to+\infty]{d} X \sim \mathcal{N}(0,1)$$

where the random variable X follows the standard normal distribution $\mathcal{N}(0,1)$.

$L(N)$ being the summatory Liouville function up to N, $\frac{L(N)}{\sqrt{N}}$ is a specific value that the random variable $\sqrt{N}.S_N$ takes for the specific outcome sequence where $\forall i \in [1,N], X_i(i) = \lambda(i)$ and $\lim\limits_{N\to+\infty} \dfrac{L(N)}{\sqrt{N}}$ is a specific value that the random variable X takes for the specific outcome sequence where $\forall i \in \mathbb{N}^*, X_i(i) = \lambda(i)$.

By definition the equiprobability over $\mathbb{N}^*$ given by **Theorem 5** means that the probability to draw randomly either $\{-1\}$ or $\{1\}$ from the outcome sequence $(\lambda(1), \lambda(2), \ldots, \lambda(i), \ldots, \lambda(N))$ as N tends to infinity is equiprobable. That is to say that by definition : if one considers the random experiment consisting in drawing randomly any element from the sequence $(\lambda(1), \lambda(2), \ldots, \lambda(i), \ldots, \lambda(N))$ as N tends to infinity then the probability space associated to the latter random experiment is $\langle \Omega_S = \bigcup_{i\in\mathbb{N}^*}\{\lambda(i)\}, \mathcal{F}_S = \{\emptyset, \{\lambda(i) : \lambda(i) = -1, i \in \mathbb{N}^*\}, \{\lambda(i) : \lambda(i) = 1, i \in \mathbb{N}^*\}, \Omega_S\}, P_S : \mathcal{F}_S \to [0,1]\rangle$ with $\{n \in \mathbb{N}^* : F' \in \{odd\}\} \in \mathcal{F}_\Omega \Leftrightarrow \{n \in \mathbb{N}^* : F' \in \{odd\}\} \in \mathcal{F} \Leftrightarrow \{\lambda(i) : \lambda(i) = -1, i \in \mathbb{N}^*\} \in \mathcal{F}_S$ and $P_\Omega(\{n \in \mathbb{N}^* : F' \in \{odd\}\}) = P(X_i = -1) = P_S(\{\lambda(i) : \lambda(i) = -1, i \in \mathbb{N}^*\}) = \frac{1}{2}$.

Thus, by noting $l = \lim\limits_{N\to+\infty} \dfrac{L(N)}{\sqrt{N}}$, l corresponds to the mode of $X \sim \mathcal{N}(0,1)$ by definition. Indeed the standard normal distribution of $X \sim \mathcal{N}(0,1)$ gives us :

$$l > 0 \Leftrightarrow \mathbb{P}(X \leq l) > \tfrac{1}{2} \Rightarrow P(X_i = -1) > P_S(\{\lambda(i) : \lambda(i) = -1, i \in \mathbb{N}^*\}) \ \textbf{(a)}$$

That is : $l > 0$ is equivalent to $\mathbb{P}(X \leq l) > \frac{1}{2}$ which means that it is strictly more probable to have outcomes of X that are lesser or equal to l than to have outcomes of X that are strictly greater than l. Thereafter, given that it is strictly more probable to have outcome sequences of X in which the events $\{X_i = -1\}$ have occurred more than the events $\{\lambda(i) : \lambda(i) = -1, i \in \mathbb{N}^*\}$ have occurred in the outcome sequence of l, than to have outcome sequences of X in which the events $\{X_i = -1\}$ have occurred less than the events $\{\lambda(i) : \lambda(i) = -1, i \in \mathbb{N}^*\}$

have occurred in the outcome sequence of l, therefore $P(X_i = -1) > P_S(\{\lambda(i) : \lambda(i) = -1, i \in \mathbb{N}^*\})$ which establishes **(a)**.

$l < 0 \Leftrightarrow \mathbb{P}(X \leq l) < \frac{1}{2} \Rightarrow P(X_i = -1) < P_S(\{\lambda(i) : \lambda(i) = -1, i \in \mathbb{N}^*\})$ **(b)**

That is : $l < 0$ is equivalent to $\mathbb{P}(X \leq l) < \frac{1}{2}$ which means that it is strictly less probable to have outcomes of X that are lesser or equal to l than to have outcomes of X that are strictly greater than l. Thereafter, given that it is strictly less probable to have outcome sequences of X in which the events $\{X_i = -1\}$ have occurred more than the events $\{\lambda(i) : \lambda(i) = -1, i \in \mathbb{N}^*\}$ have occurred in the outcome sequence of l, than to have outcome sequences of X in which the events $\{X_i = -1\}$ have occurred less than the events $\{\lambda(i) : \lambda(i) = -1, i \in \mathbb{N}^*\}$ have occurred in the outcome sequence of l, therefore $P(X_i = -1) < P_S(\{\lambda(i) : \lambda(i) = -1, i \in \mathbb{N}^*\})$ which establishes **(b)**.

Additionally given **(a)** and **(b)** :

(l is not well-defined) $\Rightarrow l \neq 0 \Leftrightarrow (l > 0) \cup (l < 0) \Rightarrow (P(X_i = -1) > P_S(\{\lambda(i) : \lambda(i) = -1, i \in \mathbb{N}^*\})) \cup (P(X_i = -1) < P_S(\{\lambda(i) : \lambda(i) = -1, i \in \mathbb{N}^*\}))$ **(c)**

That is : if l is not well-defined then by definition $l \neq 0$ which is equivalent to $(l > 0) \cup (l < 0)$ which given **(a)** and **(b)** implies that $(P(X_i = -1) > P_S(\{\lambda(i) : \lambda(i) = -1, i \in \mathbb{N}^*\})) \cup (P(X_i = -1) < P_S(\{\lambda(i) : \lambda(i) = -1, i \in \mathbb{N}^*\}))$ which establishes **(c)**.

It follows that by taking the contraposition of **(c)** :

$\neg((P(X_i = -1) > P_S(\{\lambda(i) : \lambda(i) = -1, i \in \mathbb{N}^*\})) \cup (P(X_i = -1) < P_S(\{\lambda(i) : \lambda(i) = -1, i \in \mathbb{N}^*\}))) \Leftrightarrow ((P(X_i = -1) \leq P_S(\{\lambda(i) : \lambda(i) = -1, i \in \mathbb{N}^*\})) \cap (P(X_i = -1) \geq P_S(\{\lambda(i) : \lambda(i) = -1, i \in \mathbb{N}^*\}))) \Leftrightarrow P(X_i = -1) = P_S(\{\lambda(i) : \lambda(i) = -1, i \in \mathbb{N}^*\}) \Rightarrow \neg((l > 0) \cup (l < 0)) \Leftrightarrow (l = 0) \Rightarrow \neg$($l$ is not well-defined) $\Leftrightarrow$ (l is well-defined)

Said otherwise : $P(X_i = -1) = P_S(\{\lambda(i) : \lambda(i) = -1, i \in \mathbb{N}^*\}) \Rightarrow l = 0$

Given that by definition : $P(X_i = -1) = P_S(\{\lambda(i) : \lambda(i) = -1, i \in \mathbb{N}^*\}) = \frac{1}{2}$, therefore $l = 0$. Thus **Theorem 6** is established.

7/ The Riemann hypothesis

THEOREM 7. *By the Riemann hypothesis it is understood the hypothesis according to which all the complex numbers $s \in \mathbb{C}$, $0 \leq \Re(s) \leq 1$, such that $\zeta(s) = 0$, ζ being the Riemann zeta function (B. Riemann, 1859, in [1]), are located on the abscissa $\frac{1}{2}$ of the complex plane then :*

***Theorem 6** implies the veracity of the Riemann hypothesis.*

Proof

The Dirichlet series for the Liouville function for $s = \sigma + it$, where $\Re(s) = \sigma, \sigma \in \mathbb{R}^{+*}, \Im(s) = it, t \in \mathbb{R}$, gives us the following relation (E. Landau, 1909, in [2]) for $\sigma > 1$:

$$\frac{\zeta(2s)}{\zeta(s)} = \sum_{n=1}^{+\infty} \frac{\lambda(n)}{n^s}$$

furthermore given that $\lambda(0) = L(0) = 0$ as $\lambda(1) = L(1) = 1$ – similarly to E. Landau, 1909, in [3] – :

$$\begin{aligned}\sum_{n=1}^{+\infty} \frac{\lambda(n)}{n^s} &= \sum_{n=1}^{+\infty} \frac{L(n) - L(n-1)}{n^s} \\ &= \sum_{n=1}^{+\infty} \frac{L(n)}{n^s} - \sum_{n=1}^{+\infty} \frac{L(n)}{(n+1)^s} \\ &= \sum_{n=1}^{+\infty} L(n)(\frac{1}{n^s} - \frac{1}{(n+1)^s}) \\ &= \sum_{n=1}^{+\infty} L(n) \int_n^{n+1} \frac{s}{x^{s+1}}\, dx\end{aligned}$$

then – similarly to P. Turán, 1948, in [4] – as $L(x)$ remaining constant for $x \in [n, n+1[$:

$$\sum_{n=1}^{+\infty} \frac{\lambda(n)}{n^s} = s. \int_1^{+\infty} \frac{L(x)}{x^{s+1}}\, dx$$

(Let us remark that P. Turán, 1948, in [4] has exposed a very similar equation where the summatory Liouville function is being substituted by a different step function that remains constant on $[m, m+1[, m \in \mathbb{N}$; namely the function "$L(n) \equiv \sum_{v \le n} \frac{\lambda(v)}{v}$").

* * *

Theorem 6 implies that for $\varepsilon \in \mathbb{R}^{+*}$, ε arbitrarily small, $\exists M \in \mathbb{R}^{+*}, M > 1$, such that $\forall x \in \mathbb{R}^{+*}, x \ge M$ we have :

$$0 \le \left| \frac{L(x)}{x^{\frac{1}{2}}} \right| < \varepsilon$$

then for $\delta \in \mathbb{R}^{+*}$:

$$0 \le \left| \frac{L(x)}{x^{\frac{1}{2}+1+\delta}} \right| < \frac{\varepsilon}{x^{1+\delta}}$$

which implies that – as $L(x)$ being a step-function with changes only at each strictly positive integer and $\frac{1}{x^{\frac{1}{2}+1+\delta}}$ being a function that is differentiable on $[1, +\infty[$ – :

$$0 \leq \int\limits_{M}^{+\infty} \left| \frac{L(x)}{x^{\frac{1}{2}+1+\delta}} \right| dx < \varepsilon. \int\limits_{M}^{+\infty} \frac{1}{x^{1+\delta}} dx$$

so that clearly – as the integral on the right side exists, is convergent and non null – $\exists C \in \mathbb{R}^{+*}$, $C = \varepsilon. \int\limits_{M}^{+\infty} \frac{1}{x^{1+\delta}} dx = \varepsilon \frac{M^{-\delta}}{\delta}$, a constant such that :

$$0 \leq \int\limits_{M}^{+\infty} \left| \frac{L(x)}{x^{\frac{1}{2}+1+\delta}} \right| dx < C$$

Now considering for $\sigma \geq 0$ that :

$$\int\limits_{1}^{+\infty} \left| \frac{L(x)}{x^{s+1}} \right| dx = \int\limits_{1}^{+\infty} \left| \frac{L(x)}{x^{\sigma+1}} \right| dx = \int\limits_{1}^{M} \left| \frac{L(x)}{x^{\sigma+1}} \right| dx + \int\limits_{M}^{+\infty} \left| \frac{L(x)}{x^{\sigma+1}} \right| dx$$

Then for $\sigma \geq 0$, given that $\forall x \in [1, M], 0 \leq |L(x)| \leq M$ by definition and $\forall x \in [1, M], x \neq 0$, it comes that $\exists C' \in \mathbb{R}^{+*}$, C' a constant such that :

$$0 \leq \int\limits_{1}^{M} \left| \frac{L(x)}{x^{\sigma+1}} \right| dx < C'$$

Thus for $\sigma > \frac{1}{2}, \sigma = \frac{1}{2} + \delta, \delta \in \mathbb{R}^{+*}$:

$$0 \leq \int\limits_{1}^{+\infty} \left| \frac{L(x)}{x^{\sigma+1}} \right| dx < C + C'$$

Consequently $s. \int\limits_{1}^{+\infty} \frac{L(x)}{x^{s+1}} dx$ is absolutely convergent for $\sigma > \frac{1}{2}$.

* * *

Let us note by f the complex function of the complex number s, from the half-plane $\Re(s) > \frac{1}{2}$ to $\mathbb{C}$ such that $f(s) = s. \int\limits_{1}^{+\infty} \frac{L(x)}{x^{s+1}} dx$.

Let s_0 be any complex number in the half-plane $\Re(s) > \frac{1}{2}$ and s_1 be a complex number in the half-plane $\Re(s) > \frac{1}{2}$ and in the neighborhood of s_0. Given that $s. \int\limits_{1}^{+\infty} \frac{L(x)}{x^{s+1}} dx$ is obviously non-constant and analytic for $\sigma > 1$, it comes that for the complex number $(s_0 + \frac{1}{2})$ in the half-plane $\Re(s + \frac{1}{2}) > 1$ and the complex number $(s_1 + \frac{1}{2})$ in the half-plane $\Re(s + \frac{1}{2}) > 1$ and in the neighborhood of $(s_0 + \frac{1}{2})$ there exists a series (a_k) of complex coefficients such that :

$$f(s_1 + \tfrac{1}{2}) = \sum_{k=0}^{+\infty} a_k.[(s_1 + \tfrac{1}{2}) - (s_0 + \tfrac{1}{2})]^k$$

$s. \int\limits_{1}^{+\infty} \frac{L(x)}{x^{s+1}} dx$ being absolutely convergent for $\sigma > \frac{1}{2}$, $\exists z_0 \in \mathbb{C}$ such that :

$$f(s_1) = f(s_1 + \tfrac{1}{2}) + z_0 = \sum_{k=0}^{+\infty} a_k.[(s_1 + \tfrac{1}{2}) - (s_0 + \tfrac{1}{2})]^k + z_0$$

By posing $A_0 = a_0 + z_0$ and $\forall k \in \mathbb{N}^*, A_k = a_k$ we then have :

$$f(s_1) = \sum_{k=0}^{+\infty} A_k.(s_1 - s_0)^k$$

This demonstrates that for any complex number s_0 in the half-plane $\Re(s) > \frac{1}{2}$ if one is to consider a complex number s_1 in the half-plane $\Re(s) > \frac{1}{2}$ and in the neighborhood of s_0 then one can write $f(s_1)$ as a convergent power series; which means that $s.\int_1^{+\infty} \frac{L(x)}{x^{s+1}}\,dx$ is analytic for $\sigma > \frac{1}{2}$.

It follows that by analytic continuation to the half-plane $\Re(s) > \frac{1}{2}$, $\frac{\zeta(2s)}{\zeta(s)}$ is analytic for $\sigma > \frac{1}{2}$ (and so is $\frac{1}{\zeta(s)} = \frac{s.\int_1^{+\infty} \frac{L(x)}{x^{s+1}}\,dx}{\zeta(2s)}$ for $\sigma > \frac{1}{2}$ given that $\zeta(2s)$ is by definition analytic and never null for $\sigma > \frac{1}{2}$). Said otherwise $\frac{\zeta(2s)}{\zeta(s)}$ is holomorphic on the half-plane $\Re(s) > \frac{1}{2}$ and has a single zero for $s = 1$ which corresponds to the simple pole of $\zeta(s)$ in the the half-plane $\Re(s) > \frac{1}{2}$.

The fact that $\frac{\zeta(2s)}{\zeta(s)}$ is holomorphic for $\sigma > \frac{1}{2}$ implies that $\zeta(s)$ can never be null for $\sigma > \frac{1}{2}$: indeed if $\exists z \in \mathbb{C}$, $\Re(z) > \frac{1}{2}$, $\zeta(z) = 0$ then necessarily $\zeta(2z) = 0$, $\Re(2z) > 1$, – in that if $\frac{\zeta(2z)}{\zeta(z)}$ is not a pole of $\frac{\zeta(2s)}{\zeta(s)}$ in the half-plane $\Re(s) > \frac{1}{2}$ then necessarily $\zeta(2z) = 0$, $\Re(2z) > 1$ – which is a contradiction given that by definition $\zeta(2s)$ is absolutely convergent and never null for $\sigma > \frac{1}{2}$. That is to say that $\zeta(s)$ has no non-trivial zeros – i.e. zeros in the strip $[0, 1]$ of the complex plane – whenever $\sigma > \frac{1}{2}$.

(Let us remark that – as stated above – the holomorphism of $\frac{\zeta(2s)}{\zeta(s)}$ for $\sigma > \frac{1}{2}$ can constitute an alternate proof of the results of J. Hadamard, 1896, in [5] and Ch. J. de la Vallée Poussin, 1896, in [6] according to which there cannot be zeros of $\zeta(s)$ on the abscissa $\Re(s) = 1$).

It follows that by the symmetry of the non-trivial zeros of $\zeta(s)$ with regard to the abscissa $\frac{1}{2}$ in the complex plane (E. Landau, 1909, in [7]), there are no non-trivial zeros of $\zeta(s)$ whenever $\sigma \neq \frac{1}{2}$. Consequently **Theorem 7** is established.

* * *

Additionally one can note that :

THEOREM 8. *Let $L(N) = \lambda(1) + \lambda(2) + \ldots + \lambda(i) + \ldots + \lambda(N)$, $\lambda(i)$ being the Liouville function applied to $i \in \mathbb{N}^*$, be the summatory Livouille function up to $N \in \mathbb{N}^*$ then $\forall \rho \in \left]0, \frac{1}{2}\right]$:*

$$\lim_{N \to +\infty} \left| \frac{L(N)}{N^{\frac{1}{2}-\rho}} \right| = +\infty$$

Proof

Following the exact same steps of the reasoning previously exposed in this section we have :

$$
\begin{aligned}
&\exists A \in \mathbb{R}^{+*}, \lim_{N\to+\infty} \left|\frac{L(N)}{N^{\frac{1}{2}-\rho}}\right| < A,\ \rho \in \left]0, \frac{1}{2}\right] \\
&\Rightarrow \exists B \in \mathbb{R}^{+*}, \exists T \in \mathbb{R}^{+*}, T > 1, \forall x \in \mathbb{R}^{+*}, x \geq T : 0 \leq \left|\frac{L(x)}{x^{\frac{1}{2}-\rho}}\right| < B \\
&\Rightarrow s. \int_1^{+\infty} \frac{L(x)}{x^{s+1}} dx \text{ is absolutely convergent for } \sigma > \frac{1}{2} - \rho \\
&\Rightarrow s. \int_1^{+\infty} \frac{L(x)}{x^{s+1}} dx \text{ is analytic for } \sigma > \frac{1}{2} - \rho \\
&\Rightarrow \frac{\zeta(2s)}{\zeta(s)} \text{ is holomorphic on the half-plane } \Re(s) > \frac{1}{2} - \rho \\
&\Rightarrow \zeta(2z) = 0, \Re(2z) = 1 \text{ for } z \in \mathbb{C}, \Re(z) = \frac{1}{2}, \zeta(z) = 0 \\
&\Rightarrow \zeta(4z) = 0, \Re(4z) = 2 \text{ for } 2z \in \mathbb{C}, \Re(2z) = 1, \zeta(2z) = 0, \\
&\quad \text{with } \zeta(4z) = 0, \Re(4z) = 2, \text{ being a contradiction.}
\end{aligned}
$$

That is to say that : if $\exists A \in \mathbb{R}^{+*}$, $\lim_{N\to+\infty} \left|\frac{L(N)}{N^{\frac{1}{2}-\rho}}\right| < A$, $\rho \in \left]0, \frac{1}{2}\right]$, then $\exists B \in \mathbb{R}^{+*}$, $\exists T \in \mathbb{R}^{+*}, T > 1$ such that $\forall x \in \mathbb{R}^{+*}, x \geq T$, $0 \leq \left|\frac{L(x)}{x^{\frac{1}{2}-\rho}}\right| < B$, then $s. \int_1^{+\infty} \frac{L(x)}{x^{s+1}} dx$ is absolutely convergent for $\sigma > \frac{1}{2} - \rho$, then $s. \int_1^{+\infty} \frac{L(x)}{x^{s+1}} dx$ is analytic for $\sigma > \frac{1}{2} - \rho$, then by analytic continuation $\frac{\zeta(2s)}{\zeta(s)}$ is analytic or equivalently holomorphic in the half-plane $\Re(s) > \frac{1}{2} - \rho$ as $s. \int_1^{+\infty} \frac{L(x)}{x^{s+1}} dx$ is obviously non-constant and analytic in the the half-plane $\Re(s + \frac{1}{2} + \rho) > 1$, then $\zeta(2z) = 0$, $\Re(2z) = 1$, for $z \in \mathbb{C}$, $\Re(z) = \frac{1}{2}$, $\zeta(z) = 0$, the existence of which can be admitted given the result of J. P. Gram, 1903, in [8], – in that if $\frac{\zeta(2z)}{\zeta(z)}$ is not a pole of $\frac{\zeta(2s)}{\zeta(s)}$ in the half-plane $\Re(s) > \frac{1}{2} - \rho$ then necessarily $\zeta(2z) = 0$, $\Re(2z) = 1$ – , then $\zeta(4z) = 0$, $\Re(4z) = 2$, for $2z \in \mathbb{C}$, $\Re(2z) = 1$, $\zeta(2z) = 0$, – in that if $\frac{\zeta(4z)}{\zeta(2z)}$ is not a pole of $\frac{\zeta(2s)}{\zeta(s)}$ in the half-plane $\Re(s) > \frac{1}{2} - \rho$ then necessarily $\zeta(4z) = 0$, $\Re(4z) = 2$, – which is a contradiction given that by definition $\zeta(4z)$ is absolutely convergent and never null for $\Re(4z) > 1$; which establishes that $\lim_{N\to+\infty} \left|\frac{L(N)}{N^{\frac{1}{2}-\rho}}\right| = +\infty$, $\rho \in \left]0, \frac{1}{2}\right]$ and therefore **Theorem 8**.

Using big $\mathcal{O}$ Landau notation : **Theorem 6** means that $L(N) \in o(\sqrt{N})$ while **Theorem 8** means that $L(N) \notin \mathcal{O}(N^{\frac{1}{2}-\rho}), \rho \in \left]0, \frac{1}{2}\right]$.

* * * * *

Besides, let us remark that L. Menici, 2012, in [9] has briefly mentioned as a theorem (1.4.2) the equivalence between the Riemann hypothesis and the limit $\lim_{N\to+\infty} \frac{L(N)}{N^{\frac{1}{2}+\varepsilon}} = 0, \varepsilon > 0$, and further commented that "RH is equivalent to the statement that a natural integer n has equal probability of having an odd or even number of distinct prime factors (counted with multiplicity)."

However L. Menici has not provided any specific reference nor further information on how such results can be obtained. It was therefore unknown how the said equiprobability could have been possible. Furthermore, given **Theorem 6** one can note that the comment by L. Menici in [9] is in fact not totally correct in that the equiprobability that any non null natural integer has an odd or even number of primes factor(s) counted with multiplicity actually implies that $\lim_{N\to+\infty} \frac{L(N)}{N^{\frac{1}{2}}} = 0$ precisely and not $\lim_{N\to+\infty} \frac{L(N)}{N^{\frac{1}{2}+\varepsilon}} = 0, \varepsilon > 0$. Nonetheless, it was in L. Menici's writing that the present author has encountered for the first time the mentioning of a relation between probability and the Riemann hypothesis.

Last but not least, let us remark that the proofs of **Theorem 5** and **Theorem 6** are self-standing and elementary in that the proofs do only rely on : enumerations performed on probability spaces defined in accordance with the axioms of probability, the **Lemma 1**, the **Lemma 2**, the **Lemma 3**, the **Lemma 4**, the classical central limit theorem and the standard normal distribution. In this regard, it is very much remarkable that **Theorem 5** and **Theorem 6** being self-standing and elementary, do actually imply the veracity of the Riemann hypothesis.

APPENDIX

A posteriori, let us remark that the expression "outcome sequences of X in which the events $\{X_i = -1\}$ have occurred more than the events $\{\lambda(i) : \lambda(i) = -1, i \in \mathbb{N}^*\}$ have occurred in the outcome sequence of l" on the page 16 of the present article, can be reformulated more formally.

Let $O \in \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$ be a given outcome of the random variable X. Let us note by $t_O(N)$ the function that counts the number of occurrences of $\{-1\}$ in the outcome sequence of O up to N, including N, $N \in \mathbb{N}^*$, and by $t_l(N)$ the function that counts the number of occurrences of $\{-1\}$ in the outcome sequence of l up to N, including N, $N \in \mathbb{N}^*$.

For $O \leq l$ we then have :

$$
\begin{aligned}
& O \leq l, \text{ by definition,} \\
\Leftrightarrow & \lim_{N\to+\infty} \frac{[N - t_O(N)] - t_O(N)}{\sqrt{N}} \leq \lim_{N\to+\infty} \frac{[N - t_l(N)] - t_l(N)}{\sqrt{N}} \\
\Leftrightarrow & \lim_{N\to+\infty} \frac{[[N - t_O(N)] - t_O(N)] - [[N - t_l(N)] - t_l(N)]}{\sqrt{N}} \leq 0 \\
\Leftrightarrow & \lim_{N\to+\infty} \frac{-2.t_O(N) + 2.t_l(N)}{\sqrt{N}} \leq 0, \text{ as } \lim_{N\to+\infty} \frac{1}{\sqrt{N}} = 0^+, \\
\Leftrightarrow & \lim_{N\to+\infty} [-2.t_O(N) + 2.t_l(N)] \leq 0 \\
\Leftrightarrow & \lim_{N\to+\infty} t_O(N) \geq \lim_{N\to+\infty} t_l(N)
\end{aligned}
$$

Thus more formally : a given outcome O of X "in which the events $\{X_i = -1\}$ have occurred more than the events $\{\lambda(i) : \lambda(i) = -1, i \in \mathbb{N}^*\}$ have occurred in the outcome sequence of l" is an outcome such that $\lim_{N\to+\infty} t_O(N) \geq \lim_{N\to+\infty} t_l(N)$.

Similarly, a given outcome O of X "in which the events $\{X_i = -1\}$ have occurred less than the events $\{\lambda(i) : \lambda(i) = -1, i \in \mathbb{N}^*\}$ have occurred in the outcome sequence of l" is an outcome such that $\lim_{N\to+\infty} t_O(N) \leq \lim_{N\to+\infty} t_l(N)$.

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