

Evaluating the Alignment of Astronomical Linear Polarization Data, introductory software

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Abstract

This article is a Mathematica notebook that is meant to serve as a template. User-supplied astronomical observations of transverse vectors on the sky can be evaluated, their alignment judged by the so-called Hub test. The test can be applied to any set of transverse vectors on a spherical surface, but the language here applies to linear polarization directions of electromagnetic radiation from astronomical sources. This article presents a simulation, analyzing artificial data as an illustration of the process. The analysis produces a numerical value quantifying the alignment and its significance. A visual representation of the alignment is developed, mapping regions of convergence and divergence on the Celestial sphere.

Keywords: Polarization ; Alignment ; Computer Program

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UPDATES:

- (1) The formula for “probMAX0” on page 6 of the ViXra article 2101.0073v1 is fixed in version 3.
- (2) The date and time for the most recent running of this notebook follows.

In[1]:= **Print["The date and time that this statement was evaluated: ", Now]**

The date and time that this statement was evaluated: Tue 23 Feb 2021 13:30:01 GMT-5.

0. Preface

This notebook is intended to be used as a template. In order to use the notebook, it must be recognized and interpreted by Wolfram Mathematica’s proprietary software. Since this file is published as a pdf, it must be somehow translated into the Mathematica computer language. You can simply copy the text here keystroke-by-keystroke into an active Mathematica notebook. A link⁰ to the Mathematica notebook is provided in the references.

Replace the simulated data in *Sec. 3* and run the notebook. One needs the location of the sources on the sky and a position angle at each source.

Transverse vectors on the sky can be observed for many situations, linear polarization, major/minor axes, jets and others. These observed asymmetries may be analyzed for their mutual alignment.

The work is based on an article¹ “Indirect polarization alignment with points on the sky, the Hub Test”. This notebook was created using Wolfram Mathematica², Version Number: 12.1 which is running on Microsoft Windows(64-bit).

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1. Introduction

Given a collection of astronomical sources with linearly polarized electromagnetic emissions, one may ask whether the polarization directions align.

The Hub test answers the question of alignment indirectly. Instead of attempting to find direct correlations of the polarization directions of a number of sources, an alternative process is applied.

The basic idea is illustrated in the figures below. The Celestial sphere is pictured on the left and on the right is the plane tangent to the sphere at the source S . The linear polarization direction \hat{v}_ψ lies in the tangent plane and determines the purple great circle on the sphere. A point H on the sphere and the location S of the source determine a second great circle, the blue circle drawn on the sphere at the left. Clearly, H and S must be distinct points on the sphere. The angle η , with $0^\circ \leq \eta \leq 90^\circ$, measures the “alignment of the polarization direction with the point H .” Perfect alignment occurs when $\eta = 0^\circ$ and the two great circles form a single circle.

The basic concept includes “avoidance”, as well as alignment. Avoidance is high when the two directions \hat{v}_ψ and \hat{v}_H differ by a large angle, $\eta \rightarrow 90^\circ$. Perpendicular great circles at S , $\eta = 90^\circ$, would indicate the maximum avoidance of the polarization direction and the point on the sphere.

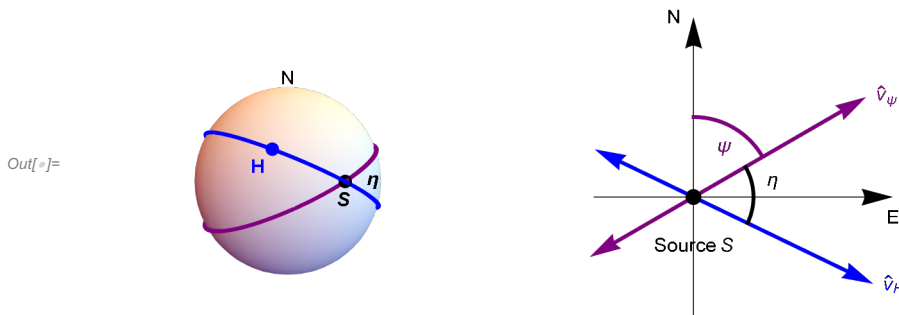


Figure 1. Left: The source S and an arbitrary point H plotted on the Celestial Sphere. The great circle determined by S and H is drawn in Blue. The polarization direction of electromagnetic radiation from S points along a second great circle that is drawn in Purple. The angle η separates the great circles where they meet at S . The angle η is the alignment angle. Right: The 2-dimensional plane tangent to the sphere at S .

With many sources $S_i, i = 1, \dots, N$, there are N alignment angles η_{iH} for the point H . To quantify the alignment of the N sources with the point H , calculate the arithmetic average alignment angle at H ,

$$\bar{\eta}(H) = \frac{1}{N} \sum_{i=1}^N \eta_{iH} . \quad (1)$$

The alignment angle $\bar{\eta}(H)$ is a function of position H on the sphere. The polarization directions are best aligned with the hub point H_{\min} where the alignment angle is a minimum $\bar{\eta}_{\min}$. The polarization directions most avoid the hub point H_{\max} where the function $\bar{\eta}(H)$ takes its maximum value $\bar{\eta}_{\max}$. For a visual aid, see the map generated near the end of the notebook.

The Hub test is based on the idea that the polarization directions are well-aligned with each other when they are well-aligned with some point H_{\min} . The point H_{\max} is also distinguished by the collection of polarization directions; it is the most avoided point.

The hub test calculates $\bar{\eta}_{\min}$ and $\bar{\eta}_{\max}$ for a given collection of polarized sources. The smaller the value of $\bar{\eta}_{\min}$, the better aligned the sources are. The larger the value of $\bar{\eta}_{\max}$, the more significant their avoidance of H_{\max} .

For more on the Hub test, see the article¹.

2. Preliminary

We work on a sphere in 3 dimensional Euclidean space. See the figures in the Introduction. The sphere is called the “Celestial sphere” or simply the “sphere”. The center of the sphere is the origin of a 3D Cartesian coordinate system with coordinates (x, y, z) . The direction of the positive z -axis is associated with “North”. Right ascension, RA or α , and declination, dec or δ , are measured as usual with the direction of the positive x -axis along $(\text{RA}, \text{dec}) = (0^\circ, 0^\circ)$ and $\delta = 90^\circ$ in the direction from the origin $(0,0,0)$ to $(0,0,1)$.

From a point-of-view located outside the sphere, as in the figure in the Introduction, one pictures a source S plotted on the sphere so that local North is upward and local East is to the right. A “position angle” at the point S on the sphere is measured in the 2D plane tangent to the sphere at S . The position angle ψ is measured clockwise from local North with East to the right.

Definitions:

(α, δ)	Right Ascension RA and declination dec of a point on the sphere. Sometimes we use radians, sometimes degrees.
$e_r(\alpha, \delta)$	radial unit vector in a Cartesian coordinate system from the Origin to the point on the sphere with $(\text{RA}, \text{dec}) = (\alpha, \delta)$, with α, δ in radians
$e_N(\alpha, \delta)$	unit vector along local North at the point (α, δ) on the sphere, with α, δ in radians
$e_E(\alpha, \delta)$	unit vector along local East at the point (α, δ) on the sphere, with α, δ in radians
$\alpha_{\text{FROM}}(\hat{r})$	RA for the point on the sphere determined by radial unit vector \hat{r} , result in radians
$\delta_{\text{FROM}}(\hat{r})$	dec for the point on the sphere determined by radial unit vector \hat{r} , result in radians

```
In[2]= er[α_, δ_] := er[α, δ] = {Cos[α] Cos[δ], Sin[α] Cos[δ], Sin[δ]} (* α,δ in radians *)
eN[α_, δ_] := eN[α, δ] = {-Cos[α] Sin[δ], -Sin[α] Sin[δ], Cos[δ]}
eE[α_, δ_] := eE[α, δ] = {-Sin[α], Cos[α], 0}
{"Check er.er = 1, er.eN = 0, er.eE = 0, eN.eN
 = 1, eN.eE = 0, eE.eE = 1, erXeE = eN, eEXeN = er, eNXer = eE: ",
 {0} = Union[Flatten[Simplify[{er[α, δ].er[α, δ] - 1, er[α, δ].eN[α, δ], er[α, δ].eE[α, δ],
 eN[α, δ].eN[α, δ] - 1, eN[α, δ].eE[α, δ], eE[α, δ].eE[α, δ] - 1, Cross[er[α, δ], eE[α, δ]] -
 eN[α, δ], Cross[eE[α, δ], eN[α, δ]] - er[α, δ], Cross[eN[α, δ], er[α, δ]] - eE[α, δ]}]]]}
```

```
Out[5]= {Check er.er = 1, er.eN = 0, er.eE = 0, eN.eN
 = 1, eN.eE = 0, eE.eE = 1, erXeE = eN, eEXeN = er, eNXer = eE: , True}
```

Get (α, δ) in radians from radial vector r , with $-\pi < \alpha < +\pi$ and $-\frac{\pi}{2} < \delta < \frac{\pi}{2}$

```
In[6]= αFROMr[r_] := N[ArcTan[Abs[r[[2]]/r[[1]]]] /; (r[[2]] ≥ 0 && r[[1]] > 0)
αFROMr[r_] := N[π - ArcTan[Abs[r[[2]]/r[[1]]]] /; (r[[2]] ≥ 0 && r[[1]] < 0)
αFROMr[r_] := N[-π + ArcTan[Abs[r[[2]]/r[[1]]]] /; (r[[2]] < 0 && r[[1]] < 0)
αFROMr[r_] := N[-ArcTan[Abs[r[[2]]/r[[1]]]] /; (r[[2]] < 0 && r[[1]] > 0)
αFROMr[r_] := π/2. /; (r[[2]] ≥ 0 && r[[1]] == 0)
αFROMr[r_] := -π/2. /; (r[[2]] < 0 && r[[1]] == 0)
```

```
In[12]= δFROMr[r_] := N[ArcTan[r[[3]]/(sqrt[r[[1]]^2 + r[[2]]^2)]] /; (sqrt[r[[1]]^2 + r[[2]]^2 > 0)
δFROMr[r_] := Sign[r[[3]]] π/2. /; (sqrt[r[[1]]^2 + r[[2]]^2 == 0)
```

3. Input and Settings

This section is where you would enter your data for analysis. You can input source locations in various ways using the functions in Section 2 above.

Be careful of units. The angles α , δ , ψ are all expected to be in radians.

Definitions:

`gridSpacing` separation in degrees between grid points on a constant latitude circle and separation of constant latitude circles. There is no bunching at the poles.

`ρRegion` estimated radius of the region containing the sources, choose from $\rhoRegion = \{90^\circ$ (whole sphere), 48° , 24° , 12° , 5° , 0° (point-like)}.

`nSrc` number of sources in the region

`αSrc` Right Ascension (RA) at the sources, in radians

δ Src declinations (dec) at the sources, in radians
 rSrc radial unit vectors in Cartesian coordinates from origin to sources S_i
 ψ_n the polarization position angles for the EM radiation from the sources, in radians
 $d\eta$ ContourPlot separation of successive contour lines on the map in Sec. 7, in degrees
 mapDirectory folder on the computer where the map is to be saved

Settings

```
In[14]:= gridSpacing = 2. (*, in degrees. This is a setting.*);
Print["The grid points are separated by ",
      gridSpacing, "° arcs along latitude and longitude."]
```

The grid points are separated by 2.° arcs along latitude and longitude.

```
In[16]:= regionRadiusChoices = {90, 48, 24, 12, 5, 0}; (*Do not change this statement*)
regionChoice = 3; (*This is a setting. The choice 24° is 3rd in the list. *)
rgnRadius = regionRadiusChoices[[regionChoice]];
Print["The region radius controls the constants  $c_i$  and  $a_i$  for statistics in Sec. 4."]
Print["The region radius  $\rho$  is set at ", rgnRadius, "°."]
```

The region radius controls the constants c_i and a_i for statistics in Sec. 4.

The region radius ρ is set at 24°.

```
In[21]:= nSrc = 16; (*The number of sources. This is a setting.*)
```

```
In[22]:= dηContourPlot = 4; (*, in degrees. This is a setting.*)
```

```
In[23]:= mapDirectory =
  "C:\\Users\\shurt\\Dropbox\\HOME_DESKTOP-0MRE50J\\SendXXX_CJP_CEPetc\\SendViXra\\
  20200715AlignmentMethod\\20200715AlignmentMMAnotebooks\\StarterKit";
(*This is a setting.*)
```

Inputs

```
In[24]:= (*The locations of the sources  $S_i$ . Here (RA,dec) are the inputs and Cartesian
coordinates are calculated. Alternatively, you can input rSrc and calculate  $\alpha$ Src,
 $\delta$ Src with the functions  $\alpha$ FROMr and  $\delta$ FROMr in Sec. 1.*)
```

```
 $\alpha$ Src = {1.0245, 0.2994, 0.8584, 0.4293, 0.7828, 0.7407, 1.1216, 0.5534,
           0.7863, 1.0897, 0.9064, 0.7216, 0.3302, 0.3788, 1.1390, 0.5709}; (*Input*)
```

```
 $\delta$ Src = {0.8400, 0.6266, 0.2472, 0.2780, 0.3821, 0.3826, 0.5953, 0.9090, 0.6663,
           0.6634, 0.4188, 0.6961, 0.5614, 0.7652, 0.8050, 0.2800}; (*Input*)
```

```
rSrc = Table[er[  $\alpha$ Src[[i]],  $\delta$ Src[[i]] ], {i, nSrc}]; (*calculated from Input.*)
```

```
In[27]:= (*The polarization position angles for the
EM radiation from the sources. This is an input.*)
```

```
 $\psi_n$  = {2.2816, 1.3406, 2.6725, 1.9480, 1.7352, 2.2421, 0.1986, 2.1445,
         2.3088, 2.0109, 1.6127, 0.3118, 1.6390, 2.3304, 2.4428, 1.8222};
```

4. Significance

When 5% or fewer results with random data are better than a result with observed data, the observed result is called “significant” by definition or by convention.

When 1% or fewer random results are better, then a result is called “very significant” by definition or by convention.

To determine the probability distributions and related formulas, we made many runs with random data and fit the results. There were 2000 runs for each combination of N sources in regions of radii ρ , with $N = \{8, 16, 32, 64, 128, 181, 256, 512\}$ and with radii $\rho = \{0^\circ, 5^\circ, 12^\circ, 24^\circ, 48^\circ, 90^\circ\}$. That makes $(2000)(8)(6) = 96000$ runs. For more details see the article¹.

Definitions:

probMIN0, probMAX0	probability distributions for alignment (MIN) and avoidance (MAX), functions of η, η_0, σ
probMIN, probMAX	same as above except these are functions of η and N , using $\eta_0(N, c1, a1)$ and $\sigma(N, c2, a2)$ to get η_0 and σ
signiMIN0, signiMAX0	significance as a function of (η, η_0, σ)
signiMIN, signiMAX	significance as a function of (η, N) using $\eta_0(N, c1, a1)$ and $\sigma(N, c2, a2)$ to get η_0 and σ
norm	a constant used to normalize the distribution (the integral of probability must be 1)
η	alignment angle
η_0	“mean”, a parameter with a value near the peak of the probability distribution
σ	“half-width”, a parameter with a value near the distribution’s half-width
c1MIN, a1MIN, ...	parameters needed to find η_0 and σ from the number of sources N .
c1MINplusMinus, ...	standard error (plus/minus) in parameters found in fitting random data
η_0 MIN, η_0 MAX	functions for finding the mean η_0
σ MIN, σ MAX	functions for half-width σ

```
In[28]:= (* y = ( (eta - eta0) / sigma ) *)
(* dy = ( d eta / sigma ) *)
(* The normalization factor "norm" is needed for the probability density *)
norm = ( 1 / ( (2 pi)^(1/2) NIntegrate[ (1 + e^(4 (y-1)))^-1 e^(-y^2/2), {y, -infinity, infinity}] ) )^-1;
norm (*Constant needed for Eq. (10) and (11) in the article1.*)
```

Out[29]= 1.22029

```
In[30]:= probMIN0[eta_, eta0_, sigma_] := ( norm / ( sigma (2 pi)^(1/2) (1 + e^(4 (eta-eta0)/sigma))^-1 ) )
e^(-1/2 ( (eta-eta0)/sigma )^2) (*A Gaussian modified by an S-function (1 + e^(4 (eta-eta0)/sigma))^-1 .*)
```

```
In[31]:= signiMIN0[eta_, eta0_, sigma_] := NIntegrate[probMIN0[eta1, eta0, sigma], {eta1, -infinity, eta}]
```

Next, check that the normalization constant does not change from the alignment (MIN) case to the avoidance (MAX) case:

```
In[129]:= normMAX =  $\left( \frac{1}{(2\pi)^{1/2}} \text{NIntegrate} \left[ \left( 1 + e^{-4(y+1)} \right)^{-1} e^{-\frac{y^2}{2}}, \{y, -\infty, \infty\} \right] \right)^{-1}$  ;
```

```
Print["The normalization constant for probMIN and probMAX are equal: ",
norm, " ?=? ", normMAX, " ",  $\theta = \text{normMAX} - \text{norm}$ ]
```

The normalization constant for probMIN and probMAX are equal: 1.22029 ?=? 1.22029 True

```
In[34]:= probMAX0[ $\eta_-$ ,  $\eta\theta_-$ ,  $\sigma_-$ ] :=  $\frac{\text{norm}}{\sigma (2\pi)^{1/2}} \left( 1 + e^{-4 \frac{(\eta-\eta\theta+\sigma)}{\sigma}} \right)^{-1} e^{-\frac{1}{2} \left( \frac{\eta-\eta\theta}{\sigma} \right)^2}$ 
```

```
In[35]:= signiMAX0[ $\eta_-$ ,  $\eta\theta_-$ ,  $\sigma_-$ ] := NIntegrate[probMAX0[ $\eta1$ ,  $\eta\theta$ ,  $\sigma$ ], { $\eta1$ ,  $\eta$ ,  $\infty$ }]
```

The significance signiMIN0[η , $\eta\theta$, σ] is the integral of probMIN0, i.e. $\text{signiMIN0} = \int_{-\infty}^{\eta} P_{\text{MIN}}(\eta) d\eta$.

The significance signiMAX0[η , $\eta\theta$, σ] is the integral of probMAX0, i.e. $\text{signiMAX0} = \int_{\eta}^{\infty} P_{\text{MAX}}(\eta) d\eta$.

The formulas for mean $\eta_0 = \frac{\pi}{4} \pm \frac{c1}{N^{a2}}$ and half-width $\sigma = \frac{c2}{4 N^{a2}}$ estimate η_0 and σ by functions of the number N of sources.

These formulas depend on the size of the region (radius ρ) by the choice of parameters c_i and a_i , $i = 1, 2$. The following values for the parameters c_i and a_i are based on random runs. For each combination of $N = \{8, 16, 32, 64, 128, 181, 256, 512\}$ and $\rho = \{0^\circ, 5^\circ, 12^\circ, 24^\circ, 48^\circ, 90^\circ\}$, there were 2000 random runs completed.

A notation conflict between this notebook and the article¹ should be noted. We doubled the exponent “a” so $N^{a/2}$ appears in the article, whereas in the random runs and here we see N^a . Thus $a \approx 1/2$ here and in the random run fits, but the paper has $a_{\text{Article}} \approx 1$. That explains the “/2” in the following arrays.

```
In[36]:=  $\rho$ ciaiMIN =
```

" ρ "	"c1"	"a1"	"c2"	"a2"
90	0.9423	1.0046 / 2	1.061	0.954 / 2
48	0.9505	1.0156 / 2	1.166	0.9956 / 2
24	0.9235	1.0069 / 2	1.127	0.964 / 2 ;
12	0.8912	1.0054 / 2	1.238	1.021 / 2
5	0.8363	1.0088 / 2	1.076	0.940 / 2
0	0.5031	1.0153 / 2	1.522	1.053 / 2

```
In[37]:=  $\rho$ ciaiMAX =
```

" ρ "	"c1"	"a1"	"c2"	"a2"
90	0.9441	1.0055 / 2	1.000	0.931 / 2
48	0.9572	1.0165 / 2	1.090	0.958 / 2
24	0.927	1.0068 / 2	1.101	0.964 / 2 ;
12	0.9049	1.0090 / 2	1.228	1.018 / 2
5	0.8424	1.0062 / 2	1.168	0.992 / 2
0	0.4982	1.0093 / 2	1.543	1.060 / 2

```

      "ρ"   "c1"   "a1"   "c2"   "a2"
      90  0.0050 0.0036/2 0.026 0.016/2
      48  0.0079 0.0057/2 0.016 0.0095/2
In[38]:= ρΔciaiMIN = 24  0.0024 0.0018/2 0.022 0.013/2 ;
      12  0.0034 0.0026/2 0.039 0.021/2
      5   0.0035 0.0028/2 0.030 0.019/2
      0   0.0059 0.0080/2 0.052 0.024/2

      "ρ"   "c1"   "a1"   "c2"   "a2"
      90  0.0061 0.0044/2 0.038 0.025/2
      48  0.0063 0.0045/2 0.026 0.016/2
In[39]:= ρΔciaiMAX = 24  0.011 0.0079/2 0.019 0.011/2 ;
      12  0.0069 0.0052/2 0.039 0.022/2
      5   0.0038 0.0031/2 0.022 0.013/2
      0   0.0058 0.0080/2 0.057 0.025/2

```

If you have trouble translating the arrays from the pdf version into a viable Mathematica notebook, the following cells are equivalent. To activate a cell, remove the remark brackets (* and *).

```

In[40]:= (*ρciaiMIN={{ "ρ", "c1", "a1", "c2", "a2"},
      {90,0.9423`,0.5023`,1.061`,0.477`}, {48,0.9505`,0.5078`,1.166`,0.4978`},
      {24,0.9235`,0.50345`,1.127`,0.482`}, {12,0.8912`,0.5027`,1.238`,0.5105`},
      {5,0.8363`,0.5044`,1.076`,0.47`}, {0,0.5031`,0.50765`,1.522`,0.5265`}}*)

In[41]:= (*ρciaiMAX={{ "ρ", "c1", "a1", "c2", "a2"},
      {90,0.9441`,0.50275`,1.`,0.4655`}, {48,0.9572`,0.50825`,1.09`,0.479`},
      {24,0.927`,0.5034`,1.101`,0.482`}, {12,0.9049`,0.5045`,1.228`,0.509`},
      {5,0.8424`,0.5031`,1.168`,0.496`}, {0,0.4982`,0.50465`,1.543`,0.53`}}*)

In[42]:= (*ρΔciaiMIN={{ "ρ", "c1", "a1", "c2", "a2"},
      {90,0.005`,0.0018`,0.026`,0.008`}, {48,0.0079`,0.00285`,0.016`,0.00475`},
      {24,0.0024`,0.0009`,0.022`,0.0065`}, {12,0.0034`,0.0013`,0.039`,0.0105`},
      {5,0.0035`,0.0014`,0.03`,0.0095`}, {0,0.0059`,0.004`,0.052`,0.012`}}*)

In[43]:= (*ρΔciaiMAX={{ "ρ", "c1", "a1", "c2", "a2"},
      {90,0.0061`,0.0022`,0.038`,0.0125`}, {48,0.0063`,0.00225`,0.026`,0.008`},
      {24,0.011`,0.00395`,0.019`,0.0055`}, {12,0.0069`,0.0026`,0.039`,0.011`},
      {5,0.0038`,0.00155`,0.022`,0.0065`}, {0,0.0058`,0.004`,0.057`,0.0125`}}*)

In[131]:= (*Change the region radius, if necessary, in Section 3 Inputs and Settings. *)
iρ = regionChoice + 1; (* Parameters ci, ai, i = 1,2. *)
Print["These constants are for sources confined to regions with radii ρ = ",
      ρciaiMIN[[iρ, 1]], "°."]
{c1MIN, a1MIN, c2MIN, a2MIN} = Table[ρciaiMIN[[iρ, j]], {j, 2, 5}];
{c1MAX, a1MAX, c2MAX, a2MAX} = Table[ρciaiMAX[[iρ, j]], {j, 2, 5}];
Clear[iρ]

```

These constants are for sources confined to regions with radii $\rho = 24^\circ$.


```
In[136]:= (*Change the region radius, if necessary, in Section 3 Inputs and Settings. *)
iρ = regionChoice + 1; (* ± uncertainty for the parameters ci and ai, i = 1,2. *)
Print["These uncertainties are for sources confined to regions with radii ρ = ",
  ρciaiMAX[[iρ, 1]], "°."]
{c1MINplusMinus, a1MINplusMinus, c2MINplusMinus, a2MINplusMinus} =
  Table[ρΔciaiMIN[[iρ, j]], {j, 2, 5}];
{c1MAXplusMinus, a1MAXplusMinus, c2MAXplusMinus, a2MAXplusMinus} =
  Table[ρΔciaiMAX[[iρ, j]], {j, 2, 5}];
Clear[
  iρ]
```

These uncertainties are for sources confined to regions with radii $\rho = 24^\circ$.

$$\text{In[54]:= } \eta_{\theta\text{MIN}}[n\text{Src}_-, c1_-, a1_-] := \frac{\pi}{4} - \frac{c1}{n\text{Src}^{a1}}$$

$$\sigma_{\text{MIN}}[n\text{Src}_-, c2_-, a2_-] := \frac{c2}{4 n\text{Src}^{a2}}$$

$$\text{In[56]:= } \eta_{\theta\text{MAX}}[n\text{Src}_-, c1_-, a1_-] := \frac{\pi}{4} + \frac{c1}{n\text{Src}^{a1}}$$

$$\sigma_{\text{MAX}}[n\text{Src}_-, c2_-, a2_-] := \frac{c2}{4 n\text{Src}^{a2}}$$

The following probability distributions and significances make use of the above formulas for mean η_0 and half-width σ . They are functions of the alignment angle η and the number of sources N .

```
In[58]:= probMIN[η_, nSrc_] := probMIN0[η, ηθMIN[nSrc, c1MIN, a1MIN], σMIN[nSrc, c2MIN, a2MIN] ]
```

```
In[59]:= signiMIN[η_, nSrc_] := signiMIN0[η, ηθMIN[nSrc, c1MIN, a1MIN], σMIN[nSrc, c2MIN, a2MIN] ]
```

```
In[60]:= probMAX[η_, nSrc_] := probMAX0[η, ηθMAX[nSrc, c1MAX, a1MAX], σMAX[nSrc, c2MAX, a2MAX] ]
signiMAX[η_, nSrc_] := signiMAX0[η, ηθMAX[nSrc, c1MAX, a1MAX], σMAX[nSrc, c2MAX, a2MAX] ]
```

5. Grid

We avoid bunching at the poles by taking into account the diminishing radii of constant latitude circles as the latitude approaches the poles. Successive grid points along any latitude or along any longitude make an arc that subtends the same central angle $d\theta$.

We grid one hemisphere at a time, then they are combined.

Definitions:

gridSpacing separation in degrees between grid points on a constant latitude circle and separation of constant latitude circles.

Set by the user in Sec. 2.

$d\theta$ grid spacing in radians

$\alpha_{\text{pointH}}, \delta_{\text{pointH}}$ RA and dec of the grid points H_j

grid see listing below for "grid" table entries

nGrid number of grid points $H_j, j = 1, 2, \dots, n\text{Grid}$

rGrid radial unit vectors from origin to grid points, in 3D Cartesian coordinates

α_{Grid} RAs for grid points

δ Grid decs for grid points

Tables:

grid, gridN and grids

1. sequential point # 2. RA index 3. dec index 4. RA (rad) 5. dec (rad) 6. Cartesian coordinates of the grid point

```
In[62]:= (*When gridSpacing = 2°, we get a 2°x2° grid.*)
Print["The grid spacing has been chosen in Sec. 3 to be gridSpacing = ", gridSpacing, "°."]
dθ =  $\frac{2 \cdot \pi}{360}$  gridSpacing; (*Convert gridSpacing to radians*)
```

The grid spacing has been chosen in Sec. 3 to be gridSpacing = 2.°.

```
In[64]:= (*The Northern Grid "gridN". *)
gridN = {}; idN = 1;

For[ $\delta j = 0.$ ,  $\delta j < \frac{\pi}{2 \cdot d\theta}$ ,  $\delta j ++$ ,  $\delta pointH = \delta j d\theta$ ;

  For[ $ai = 0.$ ,  $ai < Ceiling[\frac{2 \cdot \pi}{d\theta} (\cos[\delta pointH] + 0.01)]$ ,  $ai ++$ ,  $\alpha pointH = ai d\theta / (\cos[\delta pointH] + 0.01)$ ;

    AppendTo[gridN, {idN, ai,  $\delta j$ ,  $\alpha pointH$ ,  $\delta pointH$ , er[ $\alpha pointH$ ,  $\delta pointH$ ]}];

    idN = idN + 1

  ]]
```

```
In[66]:= (*The Southern Grid "gridS". *)
gridS = {}; idS = 1;

For[ $\delta j = 1.$ ,  $\delta j < \frac{\pi}{2 \cdot d\theta}$ ,  $\delta j ++$ ,  $\delta pointH = -\delta j d\theta$ ;

  (*Print["{ $\delta j$ ,  $\delta pointH$ } = ", { $\delta j$ ,  $\delta pointH$ }]];*)

  For[ $ai = 0.$ ,  $ai < Ceiling[\frac{2 \cdot \pi}{d\theta} (\cos[\delta pointH] + 0.01)]$ ,  $ai ++$ ,  $\alpha pointH = ai d\theta / (\cos[\delta pointH] + 0.01)$ ;

    (*Print["{ $ai$ ,  $\alpha pointH$ } = ", { $ai$ ,  $\alpha pointH$ }]];*)

    AppendTo[gridS, {idS, ai,  $\delta j$ ,  $\alpha pointH$ ,  $\delta pointH$ , er[ $\alpha pointH$ ,  $\delta pointH$ ]}];

    idS = idS + 1

  ]]
```

```
In[68]:= grid = {}; j = 1;
For[jN = 1, jN ≤ Length[gridN], jN++, AppendTo[grid,
  {j, gridN[[jN, 2]], gridN[[jN, 3]], gridN[[jN, 4]], gridN[[jN, 5]], gridN[[jN, 6]]}];
  j = j + 1]
For[jS = 1, jS ≤ Length[gridS], jS++, AppendTo[grid,
  {j, gridS[[jS, 2]], gridS[[jS, 3]], gridS[[jS, 4]], gridS[[jS, 5]], gridS[[jS, 6]]}];
  j = j + 1]
nGrid = Length[grid];
```

```
In[72]:=  $\alpha$ Grid = Table[ $\alpha$ FROMr[grid[[j, 6]]], {j, Length[grid]};
 $\delta$ Grid = Table[ $\delta$ FROMr[grid[[j, 6]]], {j, Length[grid]};
rGrid = Table[grid[[j, 6]], {j, Length[grid]}];
```

```
In[75]:= Print["There are ", nGrid, " points on the grid. "]
There are 10518 points on the grid.
```

6. Analysis

Definitions:

v ψ Src	unit vectors along the polarization directions in the tangent planes of the sources
j η BarHj	{j, $\bar{\eta}(H)$ }, where j is the index for grid point H_j and $\bar{\eta}(H)$ is the average alignment angle at H_j . See Eq. (1) in the Introduction.
sortj η BarHj	{j, $\bar{\eta}(H)$ }, rearranged by value of $\bar{\eta}(H)$, with smallest angles $\bar{\eta}(H)$ first.
j η BarMin	{j, $\bar{\eta}(H)$ }, the j and $\bar{\eta}$ for the smallest value of $\bar{\eta}(H)$, best alignment
η BarMin	the smallest value of $\bar{\eta}(H)$, measures alignment of the polarization directions
j η BarMax	{j, $\bar{\eta}(H)$ }, the j and $\bar{\eta}$ for the largest value of $\bar{\eta}(H)$, most avoided
η BarMax	the largest value of $\bar{\eta}(H)$, measures avoidance
sig η BarMin	significance of the smallest alignment angle
sigRanger η BarMin	using the plus/minus values on the parameters c_i and a_i , the table collects corresponding values of the significance
sigSmall η BarMin	the smallest of the values in sigRanger η BarMin
sigBig η BarMin	the largest of the values in sigRanger η BarMin
sig η BarMax	significance of the largest alignment angle (i.e. avoidance)
sigRanger η BarMax	using the plus/minus values on the parameters c_i and a_i , the table collects corresponding values of the significance
sigSmall η BarMax	the smallest of the values in sigRanger η BarMax
sigBig η BarMax	the largest of the values in sigRanger η BarMax
α HminDegrees	RA of the point H_{\min} where $\bar{\eta}(H)$ is the smallest
δ HminDegrees	dec of the point H_{\min} where $\bar{\eta}(H)$ is the smallest
α HmaxDegrees	RA of the point H_{\max} where $\bar{\eta}(H)$ is the largest
δ HmaxDegrees	dec of the point H_{\max} where $\bar{\eta}(H)$ is the largest

```
In[76]:= (*Analysis using Eq (5) in the article1 to get  $\eta_{iH}$ ,  $\cos(\eta) = |\hat{v}_H \cdot \hat{v}_\psi|$ ,
then {j,  $\bar{\eta}(H_j)$ }, which are sorted to get the extreme values*)
v $\psi$ Src = Table[Cos[ $\psi$ n[[i]]] eN[  $\alpha$ Src[[i]],  $\delta$ Src[[i]] ] +
Sin[  $\psi$ n[[i]] ] eE[  $\alpha$ Src[[i]],  $\delta$ Src[[i]] ], {i, nSrc};
j $\eta$ BarHj = Table[{j, (1/nSrc) Sum[ArcCos[ Abs[ rGrid[[j]]].v $\psi$ Src[[i]] /
((rGrid[[j]] - (rGrid[[j]].rSrc[[i])) rSrc[[i]]) . (rGrid[[j]] - (rGrid[[j]].
rSrc[[i])) rSrc[[i]])1/2] - 0.000001 ], {i, nSrc}], {j, nGrid};
sortj $\eta$ BarHj = Sort[j $\eta$ BarHj, #1[[2]] < #2[[2]] &]; j $\eta$ BarMin = sortj $\eta$ BarHj[[1]];
 $\eta$ BarMin = j $\eta$ BarMin[[2]];
j $\eta$ BarMax = sortj $\eta$ BarHj[[-1]];
 $\eta$ BarMax = j $\eta$ BarMax[[2]];
```

```
In[81]= (*Alternate analysis using Eq (7) in the article1 to get  $\eta_{iH}$ ,  $\cos(\eta) = |\hat{n}_{Sx\psi} \cdot \hat{n}_{SxH}| \cdot *$ )
(*nSx $\psi$ n = Table[ Sin[ $\psi$ n[[n]]]eN[ $\alpha$ Src[[n]], $\delta$ Src[[n]]]-
Cos[ $\psi$ n[[n]]]eE[ $\alpha$ Src[[n]], $\delta$ Src[[n]]], {n,nSrc}];
nSxHn[j_]:=nSxHn[j]=Table[ Cross[ rSrc[[n]],rGrid[[j]] ]/
( $\sqrt{((Cross[ rSrc[[n]],rGrid[[j]] ]).(Cross[ rSrc[[n]],rGrid[[j]] ]))}$ ), {n,
nSrc}];
 $\eta$ nHj[j_]:= $\eta$ nHj[j]=Table[ ArcCos[ Abs[ nSx $\psi$ n[[n]].nSxHn[j][[n]] ] -
0.000001 ], {n,nSrc}];
 $\eta$ BarHj[j_]:= $\eta$ BarHj[j]=Sum[ $\eta$ nHj[j][[n]],{n,nSrc}]/nSrc
j $\eta$ BarHj=Table[{j, $\eta$ BarHj[j]},{j,Length[grid]}];
sortj $\eta$ BarHj=Sort[j $\eta$ BarHj,#1[[2]]<#2[[2]]&];
j $\eta$ BarMin=sortj $\eta$ BarHj[[1]];
 $\eta$ BarMin=j $\eta$ BarMin[[2]]
j $\eta$ BarMax=sortj $\eta$ BarHj[[-1]];
 $\eta$ BarMax=j $\eta$ BarMax[[2]]*)
```

```
In[82]= (*Significance of the alignment of the polarization directions with hub point  $H_{min}$ .*)
sig $\eta$ BarMin = signiMIN[ $\eta$ BarMin, nSrc];
sigRanger $\eta$ BarMin = Sort[Partition[Flatten[Table[
{signiMIN0[ $\eta$ BarMin,  $\eta$ 0MIN[nSrc, c1MIN +  $\gamma$ 1 c1MINplusMinus, a1MIN +  $\alpha$ 1 a1MINplusMinus],
 $\sigma$ MIN[nSrc, c2MIN +  $\gamma$ 2 c2MINplusMinus, a2MIN +  $\alpha$ 2 a2MINplusMinus]],  $\gamma$ 1,  $\alpha$ 1,  $\gamma$ 2,  $\alpha$ 2},
{ $\gamma$ 1, -1, 1}, { $\alpha$ 1, -1, 1}, { $\gamma$ 2, -1, 1}, { $\alpha$ 2, -1, 1} ]], 5] ];
{sigRanger $\eta$ BarMin[[1]], sigRanger $\eta$ BarMin[[-1]]};
sigSmall $\eta$ BarMin = sigRanger $\eta$ BarMin[[1, 1]];
sigBig $\eta$ BarMin = sigRanger $\eta$ BarMin[[-1, 1]];
Print["The best value for the significance of alignment is sig. = ", sig $\eta$ BarMin,
". Using the uncertainties +/- of the  $c_i, a_i$ , the lowest and highest values are ",
sigSmall $\eta$ BarMin, " and ", sigBig $\eta$ BarMin, " giving the range from sig. = ",
sigSmall $\eta$ BarMin, " to ", sigBig $\eta$ BarMin, " . "]

The best value for the significance of alignment is sig. = 0.0111662
. Using the uncertainties +/- of the  $c_i, a_i$ , the lowest and highest values are
0.00832443 and 0.0146188 giving the range from sig. = 0.00832443 to 0.0146188 .
```

```
In[88]= (*Significance of the polarization directions' avoidance of the hub point  $H_{max}$ .*)
sig $\eta$ BarMax = signiMAX[ $\eta$ BarMax, nSrc];
sigRanger $\eta$ BarMax = Sort[Partition[Flatten[Table[
{signiMAX0[ $\eta$ BarMax,  $\eta$ 0MAX[nSrc, c1MAX +  $\gamma$ 1 c1MAXplusMinus, a1MAX +  $\alpha$ 1 a1MAXplusMinus],
 $\sigma$ MAX[nSrc, c2MAX +  $\gamma$ 2 c2MAXplusMinus, a2MAX +  $\alpha$ 2 a2MAXplusMinus]],  $\gamma$ 1,  $\alpha$ 1,  $\gamma$ 2,  $\alpha$ 2},
{ $\gamma$ 1, -1, 1}, { $\alpha$ 1, -1, 1}, { $\gamma$ 2, -1, 1}, { $\alpha$ 2, -1, 1} ]], 5] ];
{sigRanger $\eta$ BarMax[[1]], sigRanger $\eta$ BarMax[[-1]]};
sigSmall $\eta$ BarMax = sigRanger $\eta$ BarMax[[1, 1]];
sigBig $\eta$ BarMax = sigRanger $\eta$ BarMax[[-1, 1]];
Print["The best value for the significance of avoidance is sig. = ", sig $\eta$ BarMax,
". Using the uncertainties +/- of the  $c_i, a_i$ , the lowest and highest values are ",
sigSmall $\eta$ BarMax, " and ", sigBig $\eta$ BarMax, " giving the range from sig. = ",
sigSmall $\eta$ BarMax, " to ", sigBig $\eta$ BarMax, " . "]

The best value for the significance of avoidance is sig. = 0.00636211
. Using the uncertainties +/- of the  $c_i, a_i$ , the lowest and highest values are
0.00397639 and 0.00975809 giving the range from sig. = 0.00397639 to 0.00975809 .
```

```

In[94]:= {jηBarMin, jηBarMax}; (* {1. grid#, 2. alignment angle η} at Min and Max η .*)
αHminDegrees0 = grid[[ jηBarMin[[1]] ]][[4]] (360 / (2 π));
δHminDegrees0 = grid[[ jηBarMin[[1]] ]][[5]] (360 / (2 π));
If[ (180 < αHminDegrees0 < 361), αHminDegrees = αHminDegrees0 - 180;
  δHminDegrees = -δHminDegrees0, αHminDegrees = αHminDegrees0;
  δHminDegrees = δHminDegrees0];
αHmaxDegrees0 = grid[[ jηBarMax[[1]] ]][[4]] (360 / (2 π));
δHmaxDegrees0 = grid[[ jηBarMax[[1]] ]][[5]] (360 / (2 π));
If[ (180 < αHmaxDegrees0 < 361), αHmaxDegrees = αHmaxDegrees0 - 180;
  δHmaxDegrees = -δHmaxDegrees0, αHmaxDegrees = αHmaxDegrees0;
  δHmaxDegrees = δHmaxDegrees0];
Print["The alignment hub Hmin is located at (RA,dec) = ", {αHminDegrees, δHminDegrees },
  " and at ", {αHminDegrees - 180, -δHminDegrees }, " , in degrees"]
Print["The avoidance hub Hmax is located at (RA,dec) = ", {αHmaxDegrees, δHmaxDegrees },
  " and at ", {αHmaxDegrees - 180, -δHmaxDegrees }, " , in degrees"]

The alignment hub Hmin is located at (RA,dec) =
{106.408, -20.} and at {-73.5915, 20.} , in degrees
The avoidance hub Hmax is located at (RA,dec) =
{9.93072, -22.} and at {-170.069, 22.} , in degrees

```

7. Plot of the alignment function $\bar{\eta}(H)$

Definitions

$\alpha_j \delta_j \eta_{\text{BarHj}} \text{Table}$ $\{RA_j, \text{dec}_j, \bar{\eta}(H)\}$ at each grid point $H = H_j$, in degrees
 $\eta_{\text{BarHj}} \text{Smooth}$ interpolation of $\alpha_j \delta_j \eta_{\text{BarHj}} \text{Table}$ yields $\bar{\eta}(H)$ as a smooth function of the (RA,dec) of H
 $xy \eta_{\text{BarAitoff}} \text{Table}$ $\{x, y, \bar{\eta}(x,y)\}$, where x,y are Aitoff coordinates and $\bar{\eta}(x,y)$ is the alignment angle
 $d\eta \text{ContourPlot}$ separation of successive contour lines, in degrees
 listCP list contour plot of $\bar{\eta}(H)$, from $xy \eta_{\text{BarAitoff}} \text{Table}$
 $xy \text{AitoffSources}$ $\{x,y\}$ Aitoff coordinates for the sources' locations on the sphere
 $\text{mapOf} \eta_{\text{Bar}}$ contour plot listCP of the alignment angle $\bar{\eta}(H)$, with source locations and labels

$\alpha H(\alpha, \delta)$, $xH(\alpha, \delta)$, $yH(\alpha, \delta)$ are functions needed when making a 2-D map of the Celestial sphere. The origin xH, yH is centered on $\alpha = \delta = 0$. Notice the naming conflict: $\alpha H(\alpha, \delta)$ is an Aitoff parameter which, in general, differs from the Right Ascension α .

```
In[103]= (*The following table  $\alpha_j \delta_j \eta$ BarHjTable is interpolated below
to yield a smooth function of the alignment angle over the sphere.*)
(* Table Entries: 1. RA at jth grid point (degrees) 2. dec at jth grid
point (degrees) 3. alignment angle  $\eta$ BarRgnkj at jth grid point (degrees)*)
 $\alpha_j \delta_j \eta$ BarHjTable = ( $\alpha_j \delta_j \eta$ BarHjTable0 = {});
For[j = 1, j ≤ Length[j $\eta$ BarHj], j++,
AppendTo[ $\alpha_j \delta_j \eta$ BarHjTable0, {grid[[j, 4]] * (360. / (2.  $\pi$ )), grid[[j, 5]] * (360. / (2.  $\pi$ )),
j $\eta$ BarHj[[j, 2]] * (360. / (2.  $\pi$ ))}]; If[360 ≥ grid[[j, 4]] * (360. / (2.  $\pi$ )) > 354.,
AppendTo[ $\alpha_j \delta_j \eta$ BarHjTable0, {grid[[j, 4]] * (360. / (2.  $\pi$ )) - 360.,
grid[[j, 5]] * (360. / (2.  $\pi$ )), j $\eta$ BarHj[[j, 2]] * (360. / (2.  $\pi$ ))}];
If[6. > grid[[j, 4]] * (360. / (2.  $\pi$ )) ≥ 0., AppendTo[ $\alpha_j \delta_j \eta$ BarHjTable0,
{grid[[j, 4]] * (360. / (2.  $\pi$ )) + 360, grid[[j, 5]] * (360. / (2.  $\pi$ )),
j $\eta$ BarHj[[j, 2]] * (360. / (2.  $\pi$ ))}];
 $\alpha_j \delta_j \eta$ BarHjTable0];
```

```
In[104]=  $\eta$ BarHjSmooth = Interpolation[ $\alpha_j \delta_j \eta$ BarHjTable]
(*The smooth alignment angle function for the region.*)
```

... **Interpolation:** Interpolation on unstructured grids is currently only supported for InterpolationOrder→1 or InterpolationOrder→All. Order will be reduced to 1.

```
Out[104]= InterpolatingFunction[ Domain: {{-5.92, 366.}, {-88., 88.}}
Output: scalar]
```

The following Aitoff Plot formulas³ were be found in, for example, Wikipedia contributors. "Aitoff projection." Wikipedia, The Free Encyclopedia. Wikipedia, The Free Encyclopedia, 25 May. 2017. Web. 3 Jan. 2018.

```
In[105]=  $\alpha H[\alpha_, \delta_] := \alpha H[\alpha, \delta] = \text{ArcCos}[\text{Cos}[(2. \pi) / 360.] \delta] \text{Cos}[(2. \pi) / 360.] \alpha / 2.]$ 
(*angles  $\alpha$  and  $\delta$  are in degrees*)
 $xH[\alpha_, \delta_] := xH[\alpha, \delta] = (2. \text{Cos}[(2. \pi) / 360.] \delta) \text{Sin}[(2. \pi) / 360.] \alpha / 2.] / \text{Sinc}[\alpha H[\alpha, \delta]]$ 
 $yH[\alpha_, \delta_] := yH[\alpha, \delta] = \text{Sin}[(2. \pi) / 360.] \delta / \text{Sinc}[\alpha H[\alpha, \delta]]$ 
```

```
In[108]=  $xy\eta$ BarAitoffTable = Partition[Flatten[Table[{ $xH[\alpha - 180, -\delta]$ ,  $yH[\alpha - 180, -\delta]$ ,  $\eta$ BarHjSmooth[ $\alpha, \delta]$ },
{ $\alpha, 0, 360., 2.$ }, { $\delta, -88., 88., 2.$ }], 3];
(* The smooth alignment angle function  $\eta$ BarHjSmooth mapped onto a 2D
Aitoff projection of the sphere. *)
```

```
 $xy$ AitoffSources = Table[{ $xH[\alpha \text{Src}[[n]] (360 / (2 \pi))$ ,  $\delta \text{Src}[[n]] (360 / (2 \pi))$ },
 $yH[\alpha \text{Src}[[n]] (360 / (2 \pi))$ ,  $\delta \text{Src}[[n]] (360 / (2 \pi))$  ]}, {n, nSrc}];
(*The Aitoff coordinates for the sources' locations.*)
```

```
 $xy$ AitoffOppositeSources =
Table[{ $xH[\text{If}[\theta < \alpha \text{Src}[[n]] (360 / (2 \pi)) < +180, \alpha \text{Src}[[n]] (360 / (2 \pi)) - 180,$ 
 $\text{If}[\theta > \alpha \text{Src}[[n]] (360 / (2 \pi)) > -180, \alpha \text{Src}[[n]] (360 / (2 \pi)) + 180]$ ],  $-\delta \text{Src}[[n]] (360 / (2 \pi))$  ],
 $yH[\text{If}[\theta < \alpha \text{Src}[[n]] (360 / (2 \pi)) < +180, \alpha \text{Src}[[n]] (360 / (2 \pi)) - 180,$ 
 $\text{If}[\theta > \alpha \text{Src}[[n]] (360 / (2 \pi)) > -180, \alpha \text{Src}[[n]] (360 / (2 \pi)) + 180]$ ],
 $-\delta \text{Src}[[n]] (360 / (2 \pi))$  ]}, {n, nSrc}];
```

```

In[111]:= (* Contour plot of the alignment function  $\eta$ BarHjSmooth. *)
listCP = ListContourPlot[Union[xy $\eta$ BarAitoffTable(*,
  {{xH[ $\alpha$ HminDegrees, $\delta$ HminDegrees],yH[ $\alpha$ HminDegrees, $\delta$ HminDegrees], $\eta$ BarMin*(360./(2. $\pi$ ))-1.0}},
  {{xH[ $\alpha$ HmaxDegrees, $\delta$ HmaxDegrees],yH[ $\alpha$ HmaxDegrees, $\delta$ HmaxDegrees],
 $\eta$ BarMax*(360./(2. $\pi$ ))+1.0}}*], AspectRatio  $\rightarrow$  1/2,
Contours  $\rightarrow$  Table[ $\eta$ , { $\eta$ , Floor[j $\eta$ BarMin[[2]]*(360./(2. $\pi$ ))] + 1,
  Ceiling[j $\eta$ BarMax[[2]]*(360./(2. $\pi$ ))] - 1, d $\eta$ ContourPlot}],
ColorFunction  $\rightarrow$  "TemperatureMap", PlotRange  $\rightarrow$  {{-7, 7}, {-3, 3}}, Axes  $\rightarrow$  False, Frame  $\rightarrow$  False];

In[112]:= (*Construct the map of  $\bar{\eta}(H)$ .*)
mapOf $\eta$ Bar =
Show[{listCP, Table[ParametricPlot[{xH[ $\alpha$ ,  $\delta$ ], yH[ $\alpha$ ,  $\delta$ ]},
  { $\delta$ , -90, 90}, PlotStyle  $\rightarrow$  {Black, Thickness[0.002]}, (*Mesh $\rightarrow$ {11,5,0}
  (*{23,11,0}*) , MeshStyle  $\rightarrow$  Thick, *) PlotPoints  $\rightarrow$  60], { $\alpha$ , -180, 180, 30}], Table[
ParametricPlot[{xH[ $\alpha$ ,  $\delta$ ], yH[ $\alpha$ ,  $\delta$ ]}, { $\alpha$ , -180, 180}, PlotStyle  $\rightarrow$  {Black, Thickness[0.002]},
  (*Mesh $\rightarrow$ {11,5,0} (*{23,11,0}*) , MeshStyle  $\rightarrow$  Thick, *) PlotPoints  $\rightarrow$  60], { $\delta$ , -60, 60, 30}],
Graphics[{PointSize[0.007], Text[StyleForm["N", FontSize  $\rightarrow$  10, FontWeight  $\rightarrow$  "Plain"],
  {0, 1.85}], (*Sources S:*)Purple, Point[xyAitoffSources ],
  (*Opposite from sources, -S:*)Magenta, Point[xyAitoffOppositeSources],
  Black, Text[StyleForm["Max", FontSize  $\rightarrow$  8, FontWeight  $\rightarrow$  "Bold"],
  {xH[-180, 0], yH[0, -60]}], {Arrow[BezierCurve[{{xH[-180, 0], yH[0, -70]}, {-2.3, -2.0},
  {xH[ $\alpha$ HmaxDegrees - 180, - $\delta$ HmaxDegrees], yH[ $\alpha$ HmaxDegrees - 180, - $\delta$ HmaxDegrees]}]}]],
Text[StyleForm["Min", FontSize  $\rightarrow$  8, FontWeight  $\rightarrow$  "Bold"], {xH[180, 0], yH[0, -60]}],
{Arrow[BezierCurve[{{xH[180, 0], yH[0, -70]}, {2.3, -2.0},
  {xH[ $\alpha$ HminDegrees,  $\delta$ HminDegrees], yH[ $\alpha$ HminDegrees,  $\delta$ HminDegrees]}]}]],
Text[StyleForm["Min", FontSize  $\rightarrow$  8, FontWeight  $\rightarrow$  "Bold"], {xH[-180, 0], yH[0, 60]}],
{Arrow[BezierCurve[{{xH[-180, 0], yH[0, 70]}, {-2.3, 2.0},
  {xH[ $\alpha$ HminDegrees - 180, - $\delta$ HminDegrees], yH[ $\alpha$ HminDegrees - 180, - $\delta$ HminDegrees]}]}]],
Text[StyleForm["Max", FontSize  $\rightarrow$  8, FontWeight  $\rightarrow$  "Bold"], {xH[180, 0], yH[0, 60]}],
{Arrow[BezierCurve[{{xH[180, 0], yH[0, 70]}, {2.3, 2.0}, {xH[ $\alpha$ HmaxDegrees,  $\delta$ HmaxDegrees],
  yH[ $\alpha$ HmaxDegrees,  $\delta$ HmaxDegrees]}]}]]}], ImageSize  $\rightarrow$  432];

```

```

In[113]:= mapOfηBar
Print["Figure 2: The alignment angle function  $\bar{\eta}(H)$ , Eq. (1).
      The map is centered on (RA,dec) = ( $0^\circ, 0^\circ$ ). Source dots are Purple ",
      Purple, ", and the dots opposite the sources are Magenta ", Magenta, " ."]
Print["Notes: The map is symmetric across diameters, i.e.
      diametrically opposite points  $-H$  and  $H$  have the same alignment angle."]
Print["The contour lines are separated by ", dηContourPlot,
      "°. This choice can be reset in Sec. 3."]
Print["The best alignment angle (min) is  $\bar{\eta}_{\min} =$ ", jηBarMin[[2]] (360./ (2. π)), "°."]
Print["The best avoidance angle (max) is  $\bar{\eta}_{\max} =$ ", jηBarMax[[2]] (360./ (2. π)), "°."]
Print["The alignment hubs  $H_{\min}$  and  $-H_{\min}$  are located at (RA,dec) = ",
      {αHminDegrees, δHminDegrees}, " and at ",
      {αHminDegrees - 180, -δHminDegrees}, " , in degrees."]
Print["The avoidance hubs  $H_{\max}$  and  $-H_{\max}$  are located at (RA,dec) = ",
      {αHmaxDegrees, δHmaxDegrees}, " and at ",
      {αHmaxDegrees - 180, -δHmaxDegrees}, " , in degrees."]

```

Out[113]=

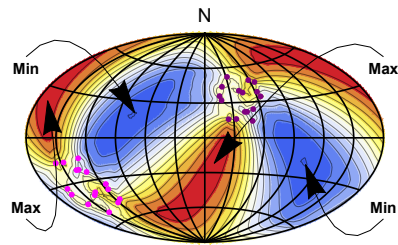


Figure 2: The alignment angle function $\bar{\eta}(H)$, Eq. (1).

The map is centered on (RA,dec) = ($0^\circ, 0^\circ$). Source dots are Purple ■, and the dots opposite the sources are Magenta ■.

Notes: The map is symmetric across diameters, i.e.

diametrically opposite points $-H$ and H have the same alignment angle.

The contour lines are separated by 4° . This choice can be reset in Sec. 3.

The best alignment angle (min) is $\bar{\eta}_{\min} = 21.8882^\circ$.

The best avoidance angle (max) is $\bar{\eta}_{\max} = 68.769^\circ$.

The alignment hubs H_{\min} and $-H_{\min}$ are located at (RA,dec) = $\{106.408, -20.\}$ and at $\{-73.5915, 20.\}$, in degrees.

The avoidance hubs H_{\max} and $-H_{\max}$ are located at (RA,dec) = $\{9.93072, -22.\}$ and at $\{-170.069, 22.\}$, in degrees.


```
In[121]:= Print["The number of sources: N = ", nSrc]
Print["The min alignment angle is  $\eta_{\min} =$ ", j $\eta$ BarMin[[2]] * (360. / (2.  $\pi$ )),
      "° , which has a significance of sig. = ", sig $\eta$ BarMin, ", plus/minus = + ",
      sigBig $\eta$ BarMin - sig $\eta$ BarMin, " and - ", sig $\eta$ BarMin - sigSmall $\eta$ BarMin,
      " , giving a range from sig. = ", sigSmall $\eta$ BarMin, " to ", sigBig $\eta$ BarMin, " ."]
Print["The max avoidance angle is  $\eta_{\max} =$ ", j $\eta$ BarMax[[2]] * (360. / (2.  $\pi$ )),
      "° , which has a significance of sig. = ", sig $\eta$ BarMax, ", plus/minus = + ",
      sigBig $\eta$ BarMax - sig $\eta$ BarMax, " and - ", sig $\eta$ BarMax - sigSmall $\eta$ BarMax,
      " , giving a range from sig. = ", sigSmall $\eta$ BarMax, " to ", sigBig $\eta$ BarMax, " ."]
```

The number of sources: N = 16

The min alignment angle is $\eta_{\min} = 21.8882$

° , which has a significance of sig. = 0.0111662, plus/minus = + 0.0034526
and - 0.00284176 , giving a range from sig. = 0.00832443 to 0.0146188 .

The max avoidance angle is $\eta_{\max} = 68.769$

° , which has a significance of sig. = 0.00636211, plus/minus = + 0.00339597
and - 0.00238572 , giving a range from sig. = 0.00397639 to 0.00975809 .

```
In[124]:= (*Export the map "mapOf $\eta$ Bar" as a pdf. The export location can be reset in Sec. 3.*)
(*To activate, remove the remark brackets "(*" and "*)". *)
(*SetDirectory[mapDirectory];
Export["mapForStarterKit.pdf", Show[mapOf $\eta$ Bar, ImageSize $\rightarrow$ 432], "PDF", ImageSize $\rightarrow$ {480, Automatic} ] *)
```

References

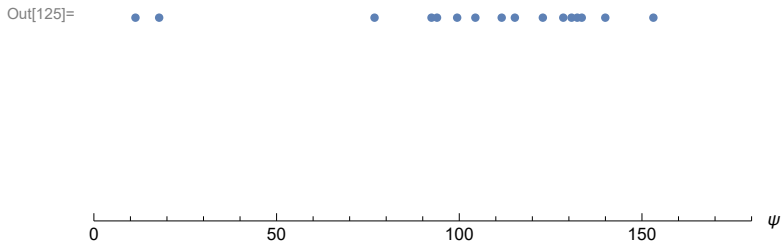
0. R. Shurtleff, the viable Mathematica notebook is available, for a limited time, at the following URL:
<https://www.dropbox.com/s/31f8jak2e73re3c/20201211StarterKitForHubTest4.nb?dl=0> (2021)
1. R. Shurtleff, "Indirect polarization alignment with points on the sky, the Hub Test", <https://vixra.org/abs/2011.0026> (2020).
2. Wolfram Research, Inc., Mathematica, Version 12.1, Champaign, IL (2020).
3. Wikipedia contributors. "Aitoff projection." Wikipedia, The Free Encyclopedia. Wikipedia, The Free Encyclopedia, 25 May. 2017. Web. 3 Jan. 2018.

Appendix

The map of the alignment angle $\bar{\eta}(H)$ above sections the sphere into four regions, two of alignment (blue) and two of avoidance (red). The hubs H_{\min} and H_{\max} are far from the sources. It follows that the polarization directions should align with the direction toward or away from H_{\min} and there should be few sources with polarization directions toward or away from H_{\max} .

The following plot confirms those expectations. The plot shows the occupancy of polarization directions ψ . A dot is placed at $\# = 1$ where the source has a PPA ψ . Clearly there is a bunching of sources near $\psi = 130^\circ$, which is the angle from local North, i.e. South and East. A glance at the $\bar{\eta}(H)$ map shows that H_{\min} is southeast of the sources. Likewise the gap between $\psi = 20^\circ$ to 75° corresponds to an avoidance of the northeast, i.e. the direction of H_{\max} from the sources.

```
In[125]= ListPlot[Table[{ψn[[i]] (360./2.π), 1}, {i, nSrc}],
  PlotLabel → "A gap from 20° to 75° and a bunching at 130°",
  AxesLabel → {"ψ", None}, Axes → {True, False}, PlotRange → {{0, 180}, {0, 2}}]
Print["Caption: The gap and bunching of the polarization directions corresponds
  to the directions of divergence and convergence, respectively, of the
  great circles determined by the polarization directions of the sources."]
Print["In Fig. 2, divergence regions (red) appear Northeast (PA = 45°) and
  Southwest from the source region; the sources are purple dots in Fig. 2."]
Print["Convergence regions (blue) appear Southeast (PA =
  135°) and Northwest from the source region."]
  A gap from 20° to 75° and a bunching at 130°
```



Caption: The gap and bunching of the polarization directions corresponds to the directions of divergence and convergence, respectively, of the great circles determined by the polarization directions of the sources.

In Fig. 2, divergence regions (red) appear Northeast (PA = 45°) and Southwest from the source region; the sources are purple dots in Fig. 2.

Convergence regions (blue) appear Southeast (PA = 135°) and Northwest from the source region.