On the Physics of Interval and Phase Invariance of Lorentz Transform

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Abstract

The early pioneers who worked or derived Lorentz transform, like Voigt, Lorentz, Poincare, and Einstein used an additional common scaling factor to start with. We review the arguments of these pioneers and analyze them to deduce which aspects by dropping this scaling factor are affected. The Interval and phase invariance of the Lorentz transform contradicts its own clock time relationship, but the same is shown to disappear if we translate them to the real domain. Moreover, the invariance does not imply the values of interval and phase are preserved as Lorentz transforms do scale them across frames. Behind the success of the four-vector-based covariant formulation of Lorentz transform is the fact that they operate in the Minkowski domain that maps the positions and computes the times accordingly, but the same makes them difficult to interpret. The real domain view makes their interpretation easy where time dilation persists but the synchronization term disappears.

1. A brief review

Voigt [1] was the first who demanded covariance of the homogeneous wave equation in inertial reference frames, assumed the invariance of lightspeed in these frames, and obtained Voigt transforms (VT),

$$X' = x-vt,\ T' = t-vx/c^2\ ,\ Y' = y/\gamma\ ,\ Z' = z/\gamma, \qquad (1)$$

where $\gamma = 1/\sqrt{1-v^2/c^2}$, v is the relative velocity between primed and unprimed frames. Lorentz [2] used a similar set of transforms that leave the Maxwell formulations and Lorentz force covariant,

$$X' = l\gamma x,\ T' = l(t/\gamma-\gamma vx/c^2),\ Y' = ly\ ,\ Z' = lz, \qquad (2)$$

where l is assumed to be a function of v. Poincare [3] corrected it to,

$$X' = l\gamma(x-vt),\ T' = l\gamma(t-vx/c^2),\ \ Y' = ly,\ \ Z' = lz \qquad (3)$$

Thus, Lorentz in his original form (2) did not even use today's much celebrated *x,t* symmetry like *x-vt* and *ct-vx/c*, but he instead tweaked the *t* and *x* parts of the temporal transforms. Einstein [4] by using the two famous postulates of relativity arrived at a similar form as (3) except for use of variable $\varphi(v)$ instead of *l(v)*. Lorentz in [2] explicitly assumed l to differ from 1 in second-order only like γ, Above all equations are claimed to preserve the lightspeed and leave the formulation of electrodynamics covariant i.e. they retain the shape and form of waves and laws, and so are conformal transforms. Also, it is obvious if we use $l=1/\gamma$ in (3) it reduces to Voigt's transform (1) and if we use l=1 in (3) we arrive at the modern-day Lorentz transforms (LT) of (4), named so by Poincare in [3].

$$X' = \gamma(x-vt),\ T' = \gamma(t-vx/c2),\ \ Y' = y,\ \ Z' = z \qquad (4)$$

Voigt was the earliest user of the above classes of covariant or conformal transforms, but the other three pioneers in their respective papers provided their own different arguments to fix l=1 to get (4). Two of them Lorentz and Einstein were physicists and Poincare was a mathematician, who very early discovered the mathematical elegance of (4) as they formed a group. We are interested to know the exact physical reasons of these pioneers for arriving at l=1 apart from mathematical necessity. Later in 1906 while proposing a Lorentz covariant gravitational theory, Poincare briefly introduced the idea of four-dimensional spacetime, which is elaborated by another mathematician Minkowski in 1908 [5], incited by symmetrical coupling of x and t, a feature of all the above transforms. However LT of (4) also offers the invariance of spacetime interval and phase for l=1,

$$S'^2 - c^2T'^2 = l^2 (s^2 - c^2t^2), \ \phi' = l^2 \phi \quad (5)$$

where S', s are spatial vectors and ϕ', ϕ are phases in the two frames. Thus conformal equations (1) to (3) do not preserve the interval and phase as they homogeneously and isotropically scale them by a common factor l. However, this non-invariance least deterred Minkowski to say in 1908 that 'the transformations which play the main role in the principle of relativity were first examined by Voigt in 1887' because the key to formulate the Minkowski space in these equations is the symmetrical coupling of x and t that helps to establish the space-time equivalence, not the invariance. Also, non-invariance of interval does not mean the lightspeed is not preserved for (3), nor it means the violation of the first postulate if l like lorentzian factor γ is a function of second orders in v/c as is the case for VT. Moreover, l=1 also does not mean LT preserves the values of interval and phases across the frames as LT still scales them as is shown later. It seems after the popularity of LT when physicists came to know about similar transforms by Voigt, it was found while VT succeeds to produce many convincing results such as it conserves the energy and mass, reproduces the realtivistc velocity-addition, the oblate shape of sphere, aberration angle, but misses on certain aspects such as time dilation, Doppler by a factor $l=1/\gamma$, and thus on Doppler frequencies, mass, etc by a factor γ [6]. This success of VT was attributed to their resemblance with LT, and its failures to the non-invariance of the interval, but wrongly so because the exact cause of the failures of VT was very different as we shall see. In this connection, it is important to carefully review and analyze the arguments of the three pioneers in favor of l=1, from the physical point of view.

2. The earliest arguments for *l* to be one

Lorentz was the first to deduce l=1 based on his calculation for longitudinal mass of electrons moving in the Ether that is an obsolete concept. He first derives the expressions for axial and transverse electromagnetic masses for electrons in the ether as a function of v and other variables, and then on the basis of his expression for the axial mass, which shows a γ^3 dependence, he concludes dl/dv=0, and hence the constancy of l [2]. Finally, he uses his awareness that l can only have a second or even order dependence, and l should reduce to one for v=0, so l=1. Recall, Voigt also used $l=1/\gamma$, showing a second-order dependence in v/c. Lorentz's decision for l=1 seems to have least concern for invaraince and more related to extracting the correct dependence of electron's mass on v/c.

Poincare was introduced to the above group of transforms through Lorentz's paper. The mathematician not only advocated the case for l=1 more forcefully than anyone else but also slightly corrected the original transform, and named them after Lorentz. However, his reasons for l to be 1, as expected, were purely mathematical [3], as he writes, *"The sum of all these transformations, together with the set of all rotations of space, must form a group; but for this to occur, we need l=1; so one is forced to suppose l=1 and this is a consequence that Lorentz has obtained by another way"*, in his June abstract to the proceedings. A month later in section 4 of his detailed submission to the same proceedings, by carrying different operations on the frames, he produces two constraints on l for (3) to be a group, $l(v)=l(-v)$ and $l(v).l(-v)$=1, yielding l=1. However Poincare is not the only one to reach these two constraints on l, Einstein later obtains them in reverse order using two operations of reflection and back transform [4]. However, on careful analysis it can be seen that Einstein has either ignored the inverted transform (IT) altogether or has treated IT identical to backward transform (BT). To see this, let us obtain three different transforms from the forward transform (FT) of (3), and name them as reflected transform (RT), BT, and IT to understand Einstein's logic. Set v=-v in (3) to get RT,

$$x' = l(-v)\gamma(x+vt),\ t' = l(-v)\gamma(t+vx/c^2),$$
$$y' = l(-v)y,\ z' = l(-v)z, \qquad (6)$$

By considering a rod along y' in the moving frame, Einstein argues that its length can not depend on the frame's direction of motion to rightly deduce,

$$l(v)=l(-v) \qquad (7)$$

Next is BT that can be derived in a similar fashion as FT of (3) are derived,

$$x = l\gamma(x'+vt'),\ t' = l\gamma(t'+vx'/c^2),\ y= ly',\ z =lz', \qquad (8)$$

where we have used (7). For the IT, invert (3),

$$x = \gamma(x'+vt')/l,\ t' = \gamma(t'+vx'/c^2)/l,\ y=y'/l,\ z=z'/l \qquad (9)$$

Einstein uses BT in (8) on the transformed coordinates of (3) and argues to restore original coordinates *(x,y,z,t)* in the rest frame, the following relation must hold true.

$$l = 1/l\ , \qquad (10)$$

or l=1. However, had Einstein used IT of eq (9) to transform back to the rest frame, they would have restored the original coordinates without the constraint of (10). Thus, (10) also implies that roles of BT and IT are interchanged forcing them to be identical, but there is no reason to do so especially when they are obtained very differently: BT is obtained by following a similar derivation for the other frame as was done for FT, and IT is just the inversion of (3). They both apply to different physical scenarios. Did they realize there are two ways to map back, one is using BT and the other using IT, and so a physical justification is required to decide on which one to apply in which scenario? Perhaps they were aware of the availability of two transforms to map back but saw it as an ambiguity to be resolved by setting $l=1$, without realizing the BT and IT addressed two different physical scenarios. This merging of BT and IT simplifies the usage of LT greatly as RT being a special case of (3) is not separate from (3), and by reducing BT to IT it effectively leaves eq (3) alone to cater to all the physical scenarios, bringing great ease for the user. LT as a group is one of the most elegant tools in the hands of physicists. It is so simple to use that sometimes the user bewilders at its simplicity fearing if he is doing any mistake but LT works. Besides mathematical elegance, l=1 also set the right factor for time dilation which VT missed, and that is behind the successful physics of LT.

However, this remarkable mathematical elegance of LT comes at a little cost of its physical elegance because $l=1$ also leads to the invariance of spacetime interval, causing some physical discrepancy. Voigt however was either consciously or unconsciously aware of this fact as he retained the separate identity of IT and BT. Further, he chose a value of $l=1/\gamma$ that generates the correct interval relationship between the two frames, but the same choice took VT away from the right factor needed for the time dilation.

3. Interval invariance discrepancy

Time dilation is validated experimentally and can also be deduced from LT in (5) using $x=vt$ for a clock placed in the moving frame,

$$t' = t/\gamma = \sqrt{1-v^2/c^2}\,t \qquad (11)$$

Eq (11) relates the time of the clocks of the two frames which were set to $t'=t=0$ when their origins overlapped. Also recall, the demand on (3) to be a group leads to l=1, which in turn leads to the merger of BT and IT, and also from (5) to spacetime interval invariance for LT,

$$s'^2 - c^2t'^2 = s^2 - c^2t^2\ , \qquad (12)$$

Next, consider a light ray that originated at the origin of the moving frame at $t=t'=0$ and detected at time t in the rest frame at $s = ct$. When the clock of the rest frame shows t, the corresponding time in the moving frame clock is $t'=t/\gamma$ by (11), so the ray has traversed $s'=ct'/\gamma$. Constructing the spacetime interval relationship based on (11) gives us,

$$s'^2-c^2t'^2=(1/\gamma)^2(s^2-c^2t^2) \qquad (13)$$

But, the interval invariance of LT (12) contradicts the one deduced from its own clock-time of (13). Eq (12) is the result of mathematical demand on (3) to be a group and (13) comes from the experimentally verified physics of time-dilation given by (11). Which one will physicists choose, a mathematical-demand on LT or the demand put by the Physics? However, if the issue of different physical scenarios of IT and BT that we raised in section 2 is solved then for sure it will help to pave a way to resolve this contradiction. Principally, there are three distinct physical scenarios each having two subcategories related by respective IT. Let us list them in the context of the above example of viewing or detecting the emitted photons.

1. The source of photons lies in the moving frame, which are detected in either of rest or moving frame. Then we wish to know the corresponding position and time of the photons in the other frame. Subcase one, suppose photons are detected in the rest frame, so we have s,t, and wish to know s',t' in the moving frame. This one we have discussed above, where FT of (3) is employed. Subcase two is the scenario when they are detected in the moving frame, so we have s',t' and wish to calculate s,t. This is the inverse problem of case one, so the inverse of (3) is employed. In both subcases, the moving frame observer is not viewing the photons coming from the rest frame, which is devoid of the source, so the same clock relationship of (11) applies here. Thus, (3) or the IT of (3) suffice here. Erroneous use of BT in this case led to the constraint l=1.

2. Contrast the above with the cases when rays originate in the rest frame and are detected in either of the frames. This is the case of BT given in eq (8) for transforming from moving frame to the rest and Inverse of BT for transforming back. In this scenario, the clock time relation is given by $t=t'/\gamma$, easily obtainable from backward LT, and the intervals are related by $s'^2-c^2t'^2=\gamma(s^2-c^2t^2)$.

3. Third is the trivial case where an experiment done in one frame is repeated in the other using an identical copy of the setup. The results of the experiment have to be identical for the two frames from the first postulate, and therefore coordinates, interval frequency, phase, etc are mapped by an identity matrix: $s'=s$, $t'=t$, $s'^2-c^2t'^2 = s^2-c^2t^2$, $\nu'=\nu$, and $\phi'=\phi$, etc. where ν is the frequency and ϕ is the phase-acquired. It seems the interval and phase invariance which is only valid for the third scenario is enforced by LT over the first two scenarios as well, though this is a small price to pay for the large benefit of the four-vector-based covariant formulation it offers by operating in the Minkowski spacetime domain. Is it so? Does LT preserve the values of interval and phase across frames or just their forms? Let's examine.

4. LT preserves the forms and not the values

Interval swept by the lightray or photon is not suitable to test the preservation of value because for lightlike intervals both (5) and (13) produce zero in both frames. Similarly, the causality is preserved under (13) also, in the sense a lightlike spacelike or timelike interval gets transformed to their types in the other frame. Thus values of interval need not be preserved for preserving causality. Like a lightlike interval, the spatial and temporal part of the accumulated phases while propagating an interval also add to zero. Therefore with photons, we have to resort to a strategy of focussing on spatial and temporal parts of interval separately. Consider a photon originating in the moving frame at the common origin at $t=t'=0$ and found at (x,t) in the rest frame, generating (x,ct) the two components of the spacetime interval. Use LT to transform it to $x'=\gamma(1-v/c)x$, $ct'=\gamma(1-v/c).ct$, confirming LT does scale the value of the interval. Similarly, the values of spatial and temporal parts of phase are scaled by LT by a common factor $(1-v/c)^2$ for reduced interval and frequency. Thus LT does not preserve the values but the forms of interval or phase. However, this does not solve the basic contradiction of invariance of LT not agreeing with its clock relation (11). The real cause of this discrepancy is that the LT does not map the clock times in the two frames, what they map are the positions, and this technique of working in the split time domain or Minkowski domain enables LT for covariant formulation in spacetime but makes it difficult to interpret them. Also, as shown below this contradiction can not be fixed in the Minkowski domain i.e. while retaining the symmetric coupling in *x,t*. However, both these problems of agreement with clock time and of interpretation are resolved by translating the results from Minkowski to clock domain.

Efforts to fix the problem in Minkowski domain

From eq (13), it may sound easy to fix this invariance contradiction by taking $l=1/\gamma$ in (3), but that gives us the VT of (1). Voigt arrived at a transform that preserved the lightspeed and also rightly mapped the spacetime intervals without falling for invariance that too so early in the pre-relativity era. By taking $x=vt$, it can be seen they fail on the correct time dilation factor, yielding $t'=t/\gamma^2$. That's why VT, despite succeeding on many fronts including conservation laws, yields an extra factor wherever temporal part is of concern such as doppler frequency. The cause of the failure of VT is not the non-invariance of intervals, but the fact that fixing the latter resulted in disturbing the time dilation. We need to fix both, however, while retaining the basic structure of symmetric coupling between *x* and *t*, we can never achieve all the three conditions of preserving the light speed, correct time-dilation, and interval-invariant relation. Thus, joining the pioneers, we provide one more reason why $l=1$ is the best option for (3).

5. Real domain solution and reinterpretation

The clock and inariance discrepancy arises from ignoring two facts about LT. First, LT does not map the clock times, it maps the positions and recalculates the time associated with that position in the other frame. Let us call his recalculated time as split time, and so obtained *(X',T')* to lie in the Minkowski domain, opposed to the real domain which is obtained by mapping the clock times and recalculating the positions. Thus, when we calculated intervals based on LT's own clock time relationship of (11), it contradicted its own interval-invariance. Second, at any instant, the particles exist at different positions in different frames (DPDF) in the real domain, not at overlapped positions in different frames (OPDF). Thus, when LT maps or overlaps the position in one frame to the other and calculates the time for occupying that position, due to DPDF it results in mixing of the past and future and also the invarince of interval and phase in the Minkowski domain, but this does not imply these two effects also happen in the real domain. To understand, let us transform the LT in (4) to the real domain from the Minkowski one and see if the discrepancy still remains. The real domain transform (RDT) that maps the clock times and calculates positions accordingly are derived in [7],

$$x' = em(x - vt)\ ,\ y' = e^2 my\ ,\ z' = e^2 mz\ , \tag{14}$$

$$t' = e\,t\ , \tag{15}$$

where, $e=1/\gamma$, $m = 1/\left[1 - vx/(c^2 t)\right]$, *(x',t')* are real domain parameters related to *(X',T') of (4)* in the Minkowski domain. Voila, various apparent discrepancies, synchronization term, and the relativity of simultaneity (RoS) which were visible in the Minkowski domain disappear in the real domain. (14-15) provides the FT for the first scene together with its IT. Similarly, BT related to backward LT, for the other scenarios mentioned in section 3, is listed in [7].

To see how interval invariance discrepancy appears in the Minkowski domain and disappears when translated to the real domain and also to understand the relation between LT and its counterpart RDT, consider a photon that originated at the common origin in the moving frame at $t'=t=0$ and detected at *(x,t)* in the rest frame. Use $x=ct$ and RDT gives $x'=ex$ and $t'=et$ in the moving frames. It means RDT mapped t with the correct clock time in the moving frame $t'=et$ and calculated the position of the photon at t' in the moving frame which gives $x'=et'$. Now use LT in (3) to get $X'=x(c-v)/ec$, $T'=t(c-v)/ec$ in the split domain. Use these X', T' to compute the position of the photon in the moving frame at $t'=et$ to get real-time position $x'=ex$ in the moving frame, confirming from LT the results of RDT, eq (7) and the DPDF i.e. the photon is not at the overlapped position X' when it is detected in the rest frame but it is at x'. LT however in its scheme of split domain discards the journey of the photon after (X',T') till (x',t') to retain its group-status and the interval and phase invariance, twhich does not meanthe invariance of values but the forms. This mathematical trick gives LT extraordinary mathematical capabilities to realize four-vector-based covariant formulation by mixing spacetime in the Minkowski domain, but also makes it difficult to interpret them. So caution is advised while interpreting them literally.

The RDT is a useful tool to provide the correct interpretation of LT as it translates back the results of LT from split to the real domain. They are shown in [8] to preserve the lightspeed, shape of the lightsphere, and rightly predict time-dilation, length-contraction, aberration angle, the Doppler shift for the correctly mapped phases $\phi'=e^2m\phi$, and they are free from the interval-invariance discrepancy as they correctly generate (12). However, they lack mathematical elegance for the covariant formulation, but they readily reduce to LT, using $x'=e^2mX'$ and $t'=e^2mT'$, which suits the best for the purposen. One of the greatest impacts of RDT is on the current interpretation of LT based on the relativity of simultaneity and synchronization [8,9]. RDT offers the relativity of spatial concurrence (RSC) as the explanation. It also predicts many stunning phenomena like relativistic non localization of the particle [8-11].

5. Conclusion

LTs that form a group offer a huge mathematical advantage in terms of four vector-based covariant formulation. This capability of LT is achieved by operating in Minkowski or split domain that is by not mapping the clock-times, but mapping the positions instead and calculating the times accordingly. However, this mathematical elegance of LT comes at the cost of physical elegance as it becomes difficult to interpret them because of the mixing of spacetime in the split domain. The solution is to translate the outcomes back to the real domain for correct interpretation using real domain transforms that map the clock times and recalculate the positions, which are easy to interpret. The interval and phase discrepancy and the effects like synchronization term and relativity of simultaneity that appear in the split domain, disappear in the real domain. Besides, RDT reveals many physical phenomena that remained hidden behind the mathematical elegance of LT and thus not explored so far [7-11]. However, the latter is not suitable for four-vector formulation, but they can readily reduce to LT that offers that advantage. Physicists are appealed to reconsider the current interpretation of LT based on the relativity of simultaneity. Lastly, LTs do not preserve the values of intervals and phases but their form.

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