Norm of Four Acceleration in Flat Space Time

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Abstract

This article in Section I brings out the fact that the norm-squared of the acceleration four vector in flat space time is not a real number if the acceleration (of a particle)is non zero. This leads to the fiasco that the time derivative of the Lorentz factor is a complex quantity with a non zero imaginary part. For uniform motion the norm projects a consistent picture . Section II establishes the fact that zero proper acceleration ,spatial, is equivalent to zero coordinate acceleration,spatial and vice versa.

Introduction

Considering four acceleration in the flat space time context we show that the norm-squared of the stated four vector cannot be represented by a real number unless the particle moves uniformly[with a constant velocity]. For non zero acceleration this leads to the fact that the time derivative of the Lorentz factor is a complex quantity with a non zero imaginary part. This fiasco has been portrayed in the paper. For uniform motion the norm projects a consistent picture. Section II establishes the fact that zero proper acceleration ,spatial is equivalent to zero coordinate acceleration,spatial.

Section I

Four acceleration[1] in flat space time:

$$\left(c\frac{d^2t}{d\tau^2}, \frac{d^2x}{d\tau^2}, \frac{d^2y}{d\tau^2}, \frac{d^2z}{d\tau^2}\right) \quad (1)$$

$\big[[ct] = [x] = [y] = [z]\big]$

Flat space time metric[2]

$$c^2 d\tau^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 \ (2.1)$$

$$c^2 = c^2\left(\frac{dt}{d\tau}\right)^2 - \left(\frac{dx}{d\tau}\right)^2 - \left(\frac{dy}{d\tau}\right)^2 - \left(\frac{dy}{d\tau}\right)^2 \ (2.2)$$

Differentiating equation (2.1) with respect to proper time we have

$$c^2\frac{dt}{d\tau}\frac{d^2t}{d\tau^2} - \frac{dx}{d\tau}\frac{d^2x}{d\tau^2} - \frac{dy}{d\tau}\frac{d^2y}{d\tau^2} - \frac{dt}{d\tau}\frac{d^2y}{d\tau^2} = 0 \ (3)$$

$$\Rightarrow v.a = 0 \ (4)$$

$$\Longrightarrow \frac{dx}{d\tau}\frac{d^2x}{d\tau^2} + \frac{dy}{d\tau}\frac{d^2y}{d\tau^2} + \frac{dt}{d\tau}\frac{d^2y}{d\tau^2} = c^2\frac{dt}{d\tau}\frac{d^2t}{d\tau^2} \ (5)$$

Norm of acceleration four vector

$$Nc^2 = c^2\left(\frac{d^2t}{d\tau^2}\right)^2 - \left(\frac{d^2x}{d\tau^2}\right)^2 - \left(\frac{d^2y}{d\tau^2}\right)^2 - \left(\frac{d^2z}{d\tau^2}\right)^2 \ (6)$$

We transform to an inertial frame of reference[boosted frame] where the particle is momentarily at rest. That means momentarily $\bar{v}_x = \bar{v}_y = \bar{v}_z = 0$ but $\frac{d\,\bar{v}_x}{d\tau}, \frac{d\,\bar{v}_y}{d\tau}$ and $\frac{d\,\bar{v}_z}{d\tau}$ are not necessarily zero[even momentarily].[The barred quantities are those in the new frame of reference where we have transformed to]This effect of transformation may be achieved at any point of time during the motion of the particle.

In the boosted frame of reference where the particle is momentarily at rest, equation (5) reduces to

$$\frac{d\bar{t}}{d\tau}\frac{d^2\bar{t}}{d\tau^2} = 0 \ (7)$$

$$\frac{d\bar{t}}{d\tau} = \gamma = 1$$

$$\Longrightarrow \frac{d^2\bar{t}}{d\tau^2} = 0 \ (8)$$

In the boosted frame norm does not change

$$Nc^2 = c^2\left(\frac{d^2\bar{t}}{d\tau^2}\right)^2 - \left(\frac{d^2\bar{x}}{d\tau^2}\right)^2 - \left(\frac{d^2\bar{y}}{d\tau^2}\right)^2 - \left(\frac{d^2\bar{z}}{d\tau^2}\right)^2$$

$$Nc^2 = -\left(\frac{d^2\bar{x}}{d\tau^2}\right)^2 - \left(\frac{d^2\bar{y}}{d\tau^2}\right)^2 - \left(\frac{d^2\bar{z}}{d\tau^2}\right)^2$$

$$Nc^2 < 0 \ (9)$$

The inequality (9) holds for (6) also, norm being an invariant.

Equation (9) holds for non zero spatial acceleration in all inertial frames of reference.

$$\Rightarrow c^2\left(\frac{d^2t}{d\tau^2}\right)^2-\left(\frac{d^2x}{d\tau^2}\right)^2-\left(\frac{d^2y}{d\tau^2}\right)^2-\left(\frac{d^2z}{d\tau^2}\right)^2<0$$

$$\Rightarrow -\sqrt{\left(\frac{d^2x}{d\tau^2}\right)^2-\left(\frac{d^2y}{d\tau^2}\right)^2-\left(\frac{d^2z}{d\tau^2}\right)^2}<\frac{d^2t}{d\tau^2}c<\sqrt{\left(\frac{d^2x}{d\tau^2}\right)^2-\left(\frac{d^2y}{d\tau^2}\right)^2-\left(\frac{d^2z}{d\tau^2}\right)^2}$$

$$\Rightarrow -\int_{\tau_1}^{\tau_2}\sqrt{\left(\frac{d^2x}{d\tau^2}\right)^2-\left(\frac{d^2y}{d\tau^2}\right)^2-\left(\frac{d^2z}{d\tau^2}\right)^2}\,d\tau<c\int_{\tau_1}^{\tau_2}\frac{d\gamma}{d\tau}d\tau<\int_{\tau_1}^{\tau_2}\sqrt{\left(\frac{d^2x}{d\tau^2}\right)^2-\left(\frac{d^2y}{d\tau^2}\right)^2-\left(\frac{d^2z}{d\tau^2}\right)^2}\,d\tau$$

Integrating both sides with respect to proper time

$$-\int_{\tau_1}^{\tau_2}\sqrt{\left[\left(\frac{d^2x}{d\tau^2}d\tau\right)^2+\left(\frac{d^2y}{d\tau^2}d\tau\right)^2+\left(\frac{d^2z}{d\tau^2}d\tau\right)^2\right]}<c\int_{\tau_1}^{\tau_2}c\int_{\tau_1}^{\tau_2}\frac{d\gamma}{d\tau}$$
$$<\int_{\tau_1}^{\tau_2}\sqrt{\left[\left(\frac{d^2x}{d\tau^2}d\tau\right)^2+\left(\frac{d^2y}{d\tau^2}d\tau\right)^2+\left(\frac{d^2z}{d\tau^2}d\tau\right)^2\right]}$$

$$\Rightarrow -\int_{\tau_1}^{\tau_2}\sqrt{\left[\left(\frac{d^2x}{d\tau^2}d\tau\right)^2+\left(\frac{d^2y}{d\tau^2}d\tau\right)^2+\left(\frac{d^2z}{d\tau^2}d\tau\right)^2\right]}<c\gamma(v_2)-c\gamma(v_1)$$
$$<\int_{\tau_1}^{\tau_2}\sqrt{\left[\left(\frac{d^2x}{d\tau^2}d\tau\right)^2+\left(\frac{d^2y}{d\tau^2}d\tau\right)^2+\left(\frac{d^2z}{d\tau^2}d\tau\right)^2\right]}$$

$$\Rightarrow -\int_{\tau_1}^{\tau_2}\sqrt{\left[\left(\frac{d}{d\tau}\left(\frac{dx}{d\tau}\right)d\tau\right)^2+\left(\frac{d}{d\tau}\left(\frac{dy}{d\tau}\right)d\tau\right)^2+\left(\frac{d}{d\tau}\left(\frac{dz}{d\tau}\right)d\tau\right)^2\right]}<c\gamma(v_2)-c\gamma(v_1)$$
$$<\int_{\tau_1}^{\tau_2}\sqrt{\left[\left(\frac{d}{d\tau}\left(\frac{dx}{d\tau}\right)d\tau\right)^2+\left(\frac{d}{d\tau}\left(\frac{dy}{d\tau}\right)d\tau\right)^2+\left(\frac{d}{d\tau}\left(\frac{dz}{d\tau}\right)d\tau\right)^2\right]}$$

$$\Rightarrow -\int_{\tau_1}^{\tau_2} \sqrt{\left[\left(d\left(\frac{dx}{d\tau}\right)\right)^2 + \left(d\left(\frac{dy}{d\tau}\right)\right)^2 + \left(d\left(\frac{dz}{d\tau}\right)\right)^2\right]} < c\gamma(v_2) - c\gamma(v_1)$$

$$< \int_{\tau_1}^{\tau_2} \sqrt{\left[\left(d\left(\frac{dx}{d\tau}\right)\right)^2 + \left(d\left(\frac{dy}{d\tau}\right)\right)^2 + \left(d\left(\frac{dz}{d\tau}\right)\right)^2\right]}$$

$$\Rightarrow -\int_{\tau_1}^{\tau_2} \sqrt{\left[(dv_x)^2 + \left(dv_y\right)^2 + (dv_z)^2\right]} < c\gamma(v_2) - c\gamma(v_1) < \int_{\tau_1}^{\tau_2} \sqrt{\left[(dv_x)^2 + \left(dv_y\right)^2 + (dv_z)^2\right]}$$

$$\Rightarrow -\int_{v_1}^{v_2} \sqrt{(dv)^2} < c\gamma(v_2) - c\gamma(v_1) < \int_{v_1}^{v_2} \sqrt{(dv)^2}$$

$$\Rightarrow -\int_{v_1}^{v_2} dv < c\gamma(v_2) - c\gamma(v_1) < \int_{v_1}^{v_2} dv \ (11)$$

We make $v_1 = 0$

Therefore

$$-v_2 < c\gamma(v_2) - c < v_2$$

$$-v_2 < c[\gamma(v_2) - 1] < v_2 \ (12)$$

As $v_2 \longrightarrow c, \gamma(v_2) \longrightarrow \infty$. Consequently $c[\gamma(v_2) - 1] \longrightarrow \infty$

Equation (12) breaks down.

Thus for non zero proper acceleration components, spatial, [at least one component zero] the norm-squared leads to a discrepancy in that it cannot be positive ,negative or zero that it cannot be represented by a real number.

$$Nc^2 = c^2\left(\frac{d^2t}{d\tau^2}\right)^2 - \left(\frac{d^2x}{d\tau^2}\right)^2 - \left(\frac{d^2y}{d\tau^2}\right)^2 - \left(\frac{d^2z}{d\tau^2}\right)^2 \neq Real\ number$$

$$c^2\left(\frac{d^2t}{d\tau^2}\right)^2 - \left(\frac{d^2x}{d\tau^2}\right)^2 - \left(\frac{d^2y}{d\tau^2}\right)^2 - \left(\frac{d^2z}{d\tau^2}\right)^2 = u + iv; v \neq 0 \ (13)$$

$$\left(\frac{d^2x}{d\tau^2}\right)^2 + \left(\frac{d^2y}{d\tau^2}\right)^2 + \left(\frac{d^2z}{d\tau^2}\right)^2 = c^2\left(\frac{d^2t}{d\tau^2}\right)^2 - (u + iv)$$

Now, $\left(\frac{d^2x}{d\tau^2}\right)^2 + \left(\frac{d^2y}{d\tau^2}\right)^2 + \left(\frac{d^2z}{d\tau^2}\right)^2 = Real\ number$

$$\Rightarrow c^2\left(\frac{d^2t}{d\tau^2}\right)^2 = complex\ number[with\ non\ zero\ nimginary\ part](14.1)$$

$$\frac{d^2t}{d\tau^2} = complex\ number[with\ non\ zero\ imginary\ part]\ (14.2)$$

$$\frac{d}{dt}\left(\frac{dx}{dt}\right) = complex\ number[with\ non\ zero\ imginary\ part]\ (15.1)$$

$$\frac{d\gamma}{dt} = complex\ number[with\ non\ zero\ imginary\ part]\ (15.2)$$

Equation (15.2) is an impossible equation. It results from non uniform motion[motion with a non zero acceleration]

Now,

$$\left(\frac{d^2x}{d\tau^2}\right)^2 + \left(\frac{d^2y}{d\tau^2}\right)^2 + \left(\frac{d^2z}{d\tau^2}\right)^2 = 0$$

$$\Rightarrow \frac{d^2x}{d\tau^2} = 0, \frac{d^2y}{d\tau^2} = 0, \frac{d^2z}{d\tau^2} = 0\ (16)$$

We have uniform motion and the concept of norm is now consistent.

Section II

In a general manner we do have,

$$\frac{d^2x}{d\tau^2} = \frac{d}{d\tau}\left(\frac{dx}{d\tau}\right) = \frac{d}{dt}\left(\frac{dx}{dt}\frac{dt}{d\tau}\right)\frac{dt}{d\tau} = \frac{d}{dt}\left(\frac{dx}{dt}\gamma\right)\gamma = \gamma\left[\frac{d^2x}{dt^2}\gamma + \frac{dx}{dt}\frac{d\gamma}{dt}\right] = \gamma\left[\frac{d^2x}{dt^2}\gamma - v_x\gamma^3\left(\frac{-v}{c^2}\right)\frac{dv}{dt}\right]$$

[v_x is the time derivative, [$v_x = \frac{dx}{dt}; v_i = \frac{dx^i}{dt}$]as opposed to what we have in the last section v_x being the time derivative with respect to proper time there]

$$\frac{d^2x}{d\tau^2} = \gamma\left[\frac{d^2x}{dt^2}\gamma + \gamma^3\left(\frac{vv_x}{c^2}\right)\frac{dv}{dt}\right]\ (17.1)$$

Similarly

$$\frac{d^2y}{d\tau^2} = \gamma\left[\frac{d^2y}{dt^2}\gamma + \gamma^3\left(\frac{vv_y}{c^2}\right)\frac{dv}{dt}\right]\ (17.2)$$

$$\frac{d^2z}{d\tau^2} = \gamma\left[\frac{d^2z}{dt^2}\gamma + \gamma^3\left(\frac{vv_z}{c^2}\right)\frac{dv}{dt}\right]\ (17.3)$$

if $\frac{d^2x}{d\tau^2} = \frac{d^2x}{d\tau^2} = \frac{d^2x}{d\tau^2} = 0$ (18) *momentarily*

$$\frac{d^2x}{dt^2} + \gamma^2\left(\frac{vv_x}{c^2}\right)\frac{dv}{dt} = 0 \ (19)$$

$$\frac{dv_x}{dt} + \gamma^2\left(\frac{vv_x}{c^2}\right)\frac{dv}{dt} = 0$$

$$dv_x + \gamma^2\left(\frac{vv_x}{c^2}\right)dv = 0$$

$$\frac{dv_x}{v_x} + \gamma^2\left(\frac{v}{c^2}\right)dv = 0 \ (20.1)$$

Similarly

$$\frac{dv_y}{v_y} + \gamma^2\left(\frac{v}{c^2}\right)dv = 0 \ (20.2)$$

$$\frac{dv_z}{v_z} + \gamma^2\left(\frac{v}{c^2}\right)dv = 0 \ (20.3)$$

$$\frac{dv_x}{v_x} = \frac{dv_y}{v_y} = \frac{dv_z}{v_z} = -\gamma^2\left(\frac{v}{c^2}\right)dv \ (21)$$

$$\frac{dv_x}{v_x} = \frac{dv_y}{v_y} = \frac{dv_z}{v_z} \ (22)$$

$$\frac{v_x dv_x}{{v_x}^2} = \frac{v_y dv_y}{{v_y}^2} = \frac{v_z dv_z}{{v_z}^2}$$

Keeping in mind $|\vec{v}| = v$

$$\frac{1}{2}\frac{d{v_x}^2}{{v_x}^2} = \frac{1}{2}\frac{d{v_y}^2}{{v_y}^2} = \frac{1}{2}\frac{d{v_z}^2}{{v_z}^2} = \frac{1}{2}\frac{d\left[{v_x}^2 + {v_y}^2 + {v_z}^2\right]}{{v_x}^2 + {v_y}^2 + {v_z}^2} = \frac{1}{2}\frac{d|\vec{v}|^2}{v^2} = \frac{1}{2}\frac{dv^2}{v^2} = \frac{vdv}{v^2}$$

$$\frac{dv_x}{v_x} = \frac{dv_y}{v_y} = \frac{dv_z}{v_z} = \frac{dv}{v}$$

Considering (21) with the last equation,

$$\gamma^2\left(\frac{v}{c^2}\right)dv = -\frac{dv}{v}$$

For $v \neq 0$ we have

$$\gamma^2\left(\frac{v^2}{c^2}\right) = -1 \Rightarrow \gamma^2 = -\frac{c^2}{v^2}$$

which is not true for real values

For maintaining consistency we have to consider

$$dv_x = dv_y = dv_z = dv = 0$$

$$\Rightarrow \frac{dv_x}{dt} = \frac{dv_y}{dt} = \frac{dv_z}{dt} = 0$$

Suppose $v_x = 0$with (21)[even momentarily]. Then from (19), $\frac{dv_x}{dt} = 0$.Also,

$$\frac{dv_y}{v_y} = \frac{dv_z}{v_z} = \frac{dv}{v} = -\gamma^2\left(\frac{v^2}{c^2}\right)dv; \frac{dv_y}{dt} = \frac{dv_z}{dt} = 0$$

Finally,as before, we obtain

$$\frac{dv_x}{dt} = \frac{dv_y}{dt} = \frac{dv_z}{dt} = 0 \; (23)$$

$$\frac{d^2x}{dt^2} = \frac{d^2x}{dt^2} = \frac{d^2x}{dt^2} = 0 \;\; (24) momentarily$$

Thus (even for an instant)

$$\frac{d^2x}{d\tau^2} = \frac{d^2y}{d\tau^2} = \frac{d^2z}{d\tau^2} = 0 \Rightarrow \frac{d^2x}{dt^2} = \frac{d^2x}{dt^2} = \frac{d^2x}{dt^2} = 0 \; (25)$$

Zero proper acceleration [spatial] cannot lead to non zero coordinate acceleration]

The other way round if $\frac{d^2x}{dt^2} = \frac{d^2x}{dt^2} = \frac{d^2x}{dt^2} = 0$ then from (17.1),(17.2) and (17.3)

$$\frac{d^2x}{d\tau^2} = \gamma^4\left(\frac{vv_x}{c^2}\right)\frac{dv}{dt} \; (26.1)$$

$$\frac{d^2y}{d\tau^2} = \gamma^4\left(\frac{vv_y}{c^2}\right)\frac{dv}{dt} \; (26.2)$$

$$\frac{d^2z}{d\tau^2} = \gamma^4\left(\frac{vv_z}{c^2}\right)\frac{dv}{dt} \; (26.3)$$

Now

$$\frac{d^2x}{dt^2} = \frac{d^2x}{dt^2} = \frac{d^2x}{dt^2} = 0 \Longrightarrow \frac{dv_x}{dt} = \frac{dv_y}{dt} = \frac{dv_z}{dt} = 0 \Longrightarrow \frac{dv}{dt} = 0[see\ appendix]$$

It follows from (26.1), (26.2)and (26.3)

$$\frac{d^2x}{dt^2} = \frac{d^2y}{dt^2} = \frac{d^2z}{dt^2} = 0 \Longrightarrow \frac{d^2x}{d\tau^2} = \frac{d^2y}{d\tau^2} = \frac{d^2z}{d\tau^2} = 0\ (28)$$

Appendix

$$\vec{v} = v\hat{n}; v = |\vec{v}|$$

$\hat{n}$ is the unit vector in the direction of the velocity vector

$$\frac{d\vec{v}}{dt} = \frac{dv}{dt}\hat{n} + v\frac{d\hat{n}}{dt}$$

W consider $\frac{d\vec{v}}{dt} = 0$[even momentarily]that is $\frac{d^2x}{dt^2} = \frac{d^2y}{dt^2} = \frac{d^2z}{dt^2} = 0$[even for a moment]

$$\frac{d\vec{v}}{dt} = 0 \Longrightarrow \frac{dv}{dt}\hat{n} + v\frac{d\hat{n}}{dt} = 0$$

Since $\frac{d\hat{n}}{dt} \perp \hat{n}$, $\frac{dv}{dt}\hat{n} = -v\frac{d\hat{n}}{dt}$ is an impossible equation unless $\frac{dv}{dt} = 0, \frac{d\hat{n}}{dt} = 0$

Now if $\frac{dv}{dt} = 0$

$$\frac{d\vec{v}}{dt} = \boldsymbol{v}\frac{d\hat{n}}{dt}$$

We cannot claim $\frac{d\vec{v}}{dt} = 0$ from $\frac{dv}{dt} = 0$ unless we have extra equations like (19) that lead to equations like (23) and (24).

Conclusions

As claimed at the outset the concept of norm projects a consistent picture only if the motion is uniform in nature. Else the norm square cannot be represented by a real number. This leads to a fiasco in that the time derivative of the Lorentz factor becomes a complex quantity with a non zero imaginary part as portrayed through equations (15.1) and (15.2). We must always keep in our mind that we cannot return to the classical laws: the laws in reality are of a more complex nature than envisaged in the current state of physics.

References

1. Wikipedia, Four Acceleration, Equation 2a, https://en.wikipedia.org/wiki/Acceleration_(special_relativity)#Four-acceleration, Accessed on 9th Jan,2021.