The $\infty$-manifolds
The $\infty$-bundles

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Abstract
We introduce the $\infty$-manifolds and the $\infty$-bundles which are spaces of
dimension the cardinality of the continuum.

1 The classical tensor calculus
For a differential manifold $M$ [B][K], it is possible to make a tensor calculus
[A][BC][S] with tensor products of the tangent and cotangent spaces. We
tensorize the spaces and introduce local coordinates $(x_i)$. A tensor is then
an expression like:

\[ R_{ijkl} \]

It is possible to transform the tensor under coordinates changes $\tilde{x}_j$ by the
matrix:

\[ \frac{\partial \tilde{x}_i}{\partial x_j} \]

We obtain new expressions, for example:

\[ \tilde{A}^i = \sum_j A^j \frac{\partial \tilde{x}_i}{\partial x_j} \]

2 The $\infty$-manifolds
It is possible to make a tensor calculus when the index of the tensor is
continuous instead of being discreet. For example, is $\tilde{x}^t$ are the local
coordinates: the tensor $\tilde{A}^i$ transforms under the change of coordinates
$\tilde{x}'^t$, according to:

\[ \tilde{A}^i = \int_{-\infty}^{+\infty} A^t (\frac{\partial \tilde{x}^t}{\partial x'^t}) dt' \]

We have the coherence rule for the change of coordinates:

\[ \int_{-\infty}^{+\infty} (\frac{\partial x'^t}{\partial x''^t})(\frac{\partial \tilde{x}'^t}{\partial x''^t}) dt' = \delta(t - t'') \]
With $\delta$, the Dirac function. If $\tilde{x}' = x'$, we obtain the equation:

$$\int_{-\infty}^{+\infty} \delta(t - t')\delta(t' - t'') dt' = \delta(t - t'')$$

The basic space is the Fréchet space of Schwartz functions [M] (smooth real functions with polynomial decreasing at infinity of the functions and all their derivatives). So that we have:

$$x'(f) = f(t) = \delta(t)(f)$$

The functions over this space are functionals over the smooth Schwartz functions. A functional $F$ is derivable if the following limit exists:

$$\lim_{\epsilon \to 0} \frac{F(g + \epsilon h) - F(g)}{\epsilon} = dF_g(h)$$

and if the differential is a distribution over the Schwartz space. The functional $F$ is smooth if we can infinitely iterate the differentials. The derivations are identified with the Schwartz functions and we have:

$$X F(g) = dF_g(X)$$

The differential of a functional is:

$$dF = \int_{-\infty}^{+\infty} \frac{\partial F}{\partial x'} dx' dt$$

We have, under a change of coordinates:

$$\frac{\partial F}{\partial x'} = \int_{-\infty}^{+\infty} \left( \frac{\partial F}{\partial x'} \right) \left( \frac{\partial x'}{\partial \tilde{x}} \right) dt'$$

The metric $g$ is a 2-tensor such that:

$$g(X, Y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g_{t't'} X' Y' dt' dt$$

The metric is a riemannian metric [J] if the quadratic form is definite positiv. The inverse of the metric $g_{t't'}$ is $g^{t't'}$ such that:

$$\int_{-\infty}^{+\infty} g_{t't'} g^{t't'} dt' = \delta(t - t'')$$

**Definition:**

The manifolds which are modeled over the Schwartz space are called the $\infty$-manifolds.
3 The $\infty$-bundles

Definition:
The $\infty$-bundles over an $\infty$-manifold $M$ are projectiv modules over the ring of smooth functionals of $M$.

The connections over an $\infty$-bundle are defined by the fact that they are linear and the Leibniz condition:

$$\nabla_X(F.s) = X(F.s) + F.\nabla_X(s)$$

with $F$ a smooth functional over $M$, and $s$ an element of the $\infty$-bundle. The Levi-Civita connection can be defined by the condition of zero torsion and that it conserves the riemannian metric.

References


