Abridgment of Cycles in GCS

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Abstract

A shortened English version of new concepts appeared this year in three French papers about cycles in Generalized Collatz Sequences (GCS) and few new developments.

- Reduced and Compact Subseqs
- Shape Vector as Subseq Rank $\rho$
- Monoid of Transition Functions $\omega$ between elements of compact sequences
- Triplet Operator $\langle a, b, c \rangle$ as powerful tool to compose linear functions
- Diophantine Equation $p^m x - r^d y - q = 0$ related to each monoid element
- Shape Class as general solution of the diophantine equation
- Cyclic Solution Requirement
- Universal Rotation Function $R$ on $q$ parameters
- Parallel Rotation Function $V$ on $\rho$ ranks
- Cardinality of Equation Classes
- Cycle Layers
- Layer’s Level
- Cycle Barycenter
- Explicit Role of $\delta$ in Cyclic Process
- Numerical Cycle Occurrence Probability
Sequences and Subseqs

**Definition 1.** Let $F$, $G$ and $H$ be functions. By convention, an $FGH$ chain is assimilated to the compound $H \circ G \circ F$. Thus $FGH(x) = H(G(F(x)))$.

**Definition 2.** Let $p$ and $r$ be relatively prime integers where $p > r > 1$. The reduced form of a Generalized Collatz Sequence is a sequence of integers where the successor to $x$ is

$$D(x) = x/r \quad \text{if } x \mod r = 0$$
$$T(x) = \left\lfloor px/r \right\rfloor \quad \text{otherwise}$$

and the full form is

$$D(x) = x/r \quad \text{if } x \mod r = 0$$
$$C(x) = r \left\lfloor px/r \right\rfloor \quad \text{otherwise}$$

The $C$ and $T$ functions combine $r-1$ simple functions $C_i$ and $T_i$ where $0 < i < r$.

$$C_1(x) = px - 1$$
$$C_2(x) = px - 2$$
$$\vdots$$
$$C_{r-1}(x) = px - r + 1$$

$$T_1(x) = (px - 1)/r$$
$$T_2(x) = (px - 2)/r$$
$$\vdots$$
$$T_{r-1}(x) = (px - r + 1)/r$$

**Definition 3.** A Generalized Collatz subsequence, or “subseq”, is a finite part of a Generalized Collatz Sequence that begins and ends with a non multiple of $r$. It comes in three varieties:

- **full subseq** if extracted from a full sequence ($C \text{ et } D$)
- **reduced subseq** if extracted from a reduced sequence ($T \text{ et } D$)
- **compact subseq** if it does not include multiples of $r$.

**Definition 4.** The shape vector of a subseq is the sequence of numbers that corresponds to the chain of functions between the first and the last term of the equivalent reduced subseq, as per

$$D \rightarrow 0$$
$$T_i \rightarrow i \quad (0 < i < r)$$

**Definition 5.** The shape class of a subseq is the set of all subseqs having the same shape vector.

Functions $T_i$ and $D$ are followed by any function. So any number in a shape vector is allowed. This means that shape classes can be numbered by setting the rank $\rho$ in the $r$ base to the class shape vector.
Definition 6. Let \( \Omega \) be the set of transition functions \( \omega \) between elements of compact subseq. This together with the composition law \( \circ \) of functions and the identity \( I \) define the free monoid \((\Omega, \circ, I)\) whose basis is

\[
\{\beta_{qd}\} = \left\{ T_q D^d \right\} \quad (0 < q < r) \quad (d \geq 0)
\]  

and the set of functions \( \omega \) is

\[
\Omega = \left\{ \prod_{i=1}^{m} \beta_{q_i d_i} \right\} \quad (m > 0)
\]  

The sequences \((q_i)_m\) and \((d_i)_m\) determine a distinct \( \omega \) function:

\[
\omega = \beta_{q_1 d_1} \beta_{q_2 d_2} \cdots \beta_{q_m d_m} = T_{q_1} D_{d_1} T_{q_2} D_{d_2} \cdots T_{q_m} D_{d_m}
\]  

Let \( X = (x_1, x_2, ..., x_{m+1}) \) be a compact subseq. A \( q_i 0^{d_i} \) represents a base element shape vector where an isolated \( q_i \) (not followed by 0) is written as \( q_i 0^0 \).

\[
q_1 0^{d_1} \quad q_2 0^{d_2} \quad q_m 0^{d_m}
\]

\[
x_1 \mapsto x_2 \mapsto x_3 \cdots x_m \mapsto x_{m+1}
\]

\[
x_{m+1} = \omega(x_1)
\]

The rank \( \rho \) in \( r \) base of an \( \omega \) function is the shape vector produced by concatenation of successive \( q_i 0^{d_i} \) but where \( 0^0 \) is not written.

\[
\rho = q_1 0^{d_1} q_2 0^{d_2} \cdots q_m 0^{d_m}
\]

Composition of Functions

Triplet Operator

To any function of the form \( f(x) = (ax + c)/b \) we can associate the triplet \( \langle a, b, c \rangle \) as an operator \( \langle f \rangle \) equivalent to \( f \):

\[
\langle f \rangle (x) = \langle a, b, c \rangle (x) = f(x)
\]

This implies the following composition law

\[
\langle a_1, b_1, c_1 \rangle \langle a_2, b_2, c_2 \rangle = \langle a_1 a_2, b_1 b_2, b_1 c_2 + c_1 a_2 \rangle
\]  

the neutral element \( \langle 1, 1, 0 \rangle \), the inverse \( \langle a, b, c \rangle^{-1} = \langle b, a, -c \rangle \) and \( \langle ka, kb, kc \rangle = \langle a, b, c \rangle \).
In the monoid there will be no inverse. Only descending functions are composed but all their inverse form the monoid of the rising functions. The triplets of the elementary functions are

\[ \langle D \rangle = \langle 1, r, 0 \rangle \quad \langle C_i \rangle = \langle p, 1, -i \rangle \quad \langle T_i \rangle = \langle p, r, -i \rangle \]

and those of the base elements are

\[ \langle \beta_{q_i d_i} \rangle = \langle p, r_{d_i+1}, -q_i \rangle \]

To improve the notations in arrays let put

\[ \langle \beta_{q_i d_i} \rangle = \langle p, r_i, -q_i \rangle \quad \text{where} \quad r_i = r_{d_i+1} \]

The only formula needed to compose the functions of a chain as a block is

\[
\prod_{i=1}^{n} \langle a_i, b_i, c_i \rangle = \langle a, b, c \rangle \\
a = a_1 a_2 a_3 a_4 \cdots a_n \\
b = b_1 b_2 b_3 b_4 \cdots b_n \\
c_1 a_2 a_3 a_4 \cdots a_n \\
+ b_1 c_2 a_3 a_4 \cdots a_n \\
+ b_1 b_2 c_3 a_4 \cdots a_n \\
+ b_1 b_2 b_3 c_4 \cdots a_n \\
\cdots \cdots \cdots \\
+ b_1 b_2 b_3 b_4 \cdots c_n
\]

(10)

**Diophantine Equation**

With \( \langle p, r_i, -q_i \rangle \) the formula gives us \( \langle \omega \rangle \) where the number of divisions \( d = m + \sum d_i \)

\[
\langle \omega \rangle = \prod_{i=1}^{m} \langle p, r_i, -q_i \rangle = \langle p^m, r^d, -q \rangle \\
\quad + q_1 p p p \cdots p \\
\quad + r_1 q_2 p p \cdots p \\
\quad + r_1 r_2 q_3 p \cdots p \\
q = + r_1 r_2 r_3 q_4 \cdots p \\
\quad \cdots \cdots \cdots \\
\quad + r_1 r_2 r_3 r_4 \cdots q_m
\]

(11)
and the corresponding diophantine equation

\[ p^m x - r^d y - q = 0 \]  \hspace{1cm} (12)

whose \( q \) parameter can be written in a concise way

\[ q = \sum_{i=1}^{m} q_i p^{m-i} r^{\sigma_i} \]  \hspace{1cm} (13)

by posing \( \sigma_i = \sum_{k=1}^{i} (1 + d_{k-1}) \) where \( d_0 = -1 \) \hspace{1cm} (14)

The \((d_i)_m\) sequence is thus replaced by the \((\sigma_i)_m\) sequence. For example:

\[(d_i)_5 = (2 \ 3 \ 0 \ 2 \ 1) \rightarrow (d_i + 1)_5 = (3 \ 4 \ 1 \ 3 \ 2) \rightarrow (\sigma_i)_5 = (0 \ 3 \ 7 \ 8 \ 11) \]

\[ q = q_1 p^4 + q_2 p^3 r^3 + q_3 p^2 r^7 + q_4 p r^8 + q_5 r^{11} \]

**General Solution of the equation**

There’s necessarily a solution as \( p \) and \( r \) are relatively prime. In the general solution \( (r^d k + x_0, \ p^m k + y_0) \) where \( k \) is a relative integer we obtain \( (p^m k + y_0) \mod r = 0 \) at the frequency of once on \( r \). These \( k \) values will not be used because \( y \) cannot be multiple of \( r \).

By solving Bezout’s equation

\[ p^m u + r^d v = 1 \]

we have a minimal solution \( y_0 = -q v \mod p^m \) which can be a multiple of \( r \). In this case, we should take the minimum value \( y_0 \) of opposite sign by decreasing or adding \( p^m \). We then derive \( x_0 \) from the equation and calculate the \( k_0 \) value of the test \( k \mod r = k_0 \) to eliminate the non-permitted \( k \).

<table>
<thead>
<tr>
<th>[ n ]</th>
<th>[ \rho ]</th>
<th>[ \langle \omega \rangle ]</th>
<th>[ bk + x ]</th>
<th>[ ak + y ]</th>
<th>[ full ]</th>
<th>[ compact ]</th>
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<td>54</td>
<td>2000</td>
<td>( \langle 4, 81, -2 \rangle )</td>
<td>81k – 40,</td>
<td>4k – 2</td>
<td>(-40 -162 -54 -18 -6 -2)</td>
<td>(-40 -2)</td>
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<td>55</td>
<td>2001</td>
<td>( \langle 16, 81, -35 \rangle )</td>
<td>81k + 68</td>
<td>16k + 13</td>
<td>(68 270 90 30 10 39 13)</td>
<td>(68 10 13)</td>
</tr>
<tr>
<td>56</td>
<td>2002</td>
<td>( \langle 16, 81, -62 \rangle )</td>
<td>81k – 67</td>
<td>16k – 14</td>
<td>(-67 -270 -90 -30 -10 -42 -14)</td>
<td>(-67 -10 -14)</td>
</tr>
<tr>
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<td>2010</td>
<td>( \langle 16, 81, -17 \rangle )</td>
<td>81k – 4</td>
<td>16k – 1</td>
<td>(-4 -18 -6 -2 -9 -3 -1)</td>
<td>(-4 -2 -1)</td>
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<tr>
<td>58</td>
<td>2011</td>
<td>( \langle 64, 81, -95 \rangle )</td>
<td>81k + 23</td>
<td>64k + 17</td>
<td>(23 90 30 10 39 13 51 17)</td>
<td>(23 10 13 17)</td>
</tr>
<tr>
<td>59</td>
<td>2012</td>
<td>( \langle 64, 81, -122 \rangle )</td>
<td>81k + 50</td>
<td>64k + 38</td>
<td>(50 198 66 22 87 29 114 38)</td>
<td>(50 22 29 38)</td>
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<tr>
<td>60</td>
<td>2020</td>
<td>( \langle 16, 81, -26 \rangle )</td>
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<td>16k – 10</td>
<td>(-49 -198 -66 -22 -90 -30 -10)</td>
<td>(-49 -22 -10)</td>
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Cyclic Solution

The cyclic solution \( x = y \) is however quite simple:

\[
x = q / (p^m - r^d)
\]

and the requirement for a numerical cycle as well:

\[
q \mod \delta = 0 \quad \text{where} \quad \delta = p^m - r^d
\]

In a numerical cycle the numbers rotate and so do the functions. The question is, can there be a function rotation cycle when the divisibility requirement by \( \delta \) does not allow a numerical cycle? The answer is yes and the implications are substantial.

Universal Cycles and Derived Cycles

We need to reverse the idea that functional cyclicity derives from numerical cyclicity. In the algebraic elucidation all is ordered and numbered. All is regulated except the \( \delta | q \) possibility. All \( q \) belong to a cycle \( (q_i)_m \) resulting from rotation of \( i \) in the \( \langle p, r_i, -q_i \rangle \) components. It also corresponds to rotation of \( i \) in the \( T_iD^i \) of shape vectors and thus implies \( (\rho_i)_m \) rank cycles.

Rotational cyclicity is a universal algebraic property. It is occasionally reflected in a derived numerical cycle when \( q \mod \delta \) is an integer.

The mod* notation

It’s a kind of modulo in multiplicative context. As well as \( x = rk + q \) where \( q < r \) implies \( q = x \mod r \) as the rest of the division, similarly \( x = r^kq \) where \( q \) is not multiple of \( r \) implies \( q = x \mod* r \) as a non-multiple residue of \( r \). Used to stylishly define functions like cyclical \( R \), where \( \mod* r \) replace an inappropriate \( r^{-\delta v(pq - (pq \mod r)\delta)} \) expression.

The \( N \) function to rotate \( (x_i)_m \)

The usual \( N \) function can be defined as \( N(x) = T(x) \mod* r \). The function \( N \) is not cyclic in itself, but may have a cyclic behavior \( N^m(x) = x \) when \( x \) is an integer \( q / \delta \). Then the function \( N \) rotates the \( x_i \) in the numerical cycle \( (x_i)_m \).

The \( R \) function to rotate \( (q_i)_m \)

The general rotation function to rotate \( q_i \) values contains the parameter \( \delta \) which changes with count \( m \) of multiplications and count \( d \) of divisions. It’s explicitly stated

\[
R(q) = (pq - (pq \mod r)\delta) \mod* r
\]
This can result in the following lines of Pascal code

\[
q := (p * q - ((p * q) \mod r) * \text{delta}) \div r;
\]

\[
\textbf{while} (q \mod r) = 0 \textbf{ do } q := q \div r;
\]

being used with the example \(64x - 81y - q = 0\) where exist eight cycles \(R^3(q) = q\).

\[
\begin{align*}
(37 & \ 55 & \ 79) \ (46 & \ 67 & \ 95) \ (58 & \ 83 & \ 122) \ (74 & \ 110 & \ 158) \\
(49 & \ 71 & \ 106) \ (53 & \ 82 & \ 115) \ (62 & \ 94 & \ 131) \ (65 & \ 98 & \ 142)
\end{align*}
\]

The \textit{V function to rotate} \((\rho_i)_m\)

The shape vectors related to the \(q\)-values in the previous eight cycles are as follows:

\[
\begin{align*}
(1110 & \ 1101 & \ 1011) \ (1120 & \ 1201 & \ 2011) \ (1220 & \ 2201 & \ 2012) \ (2220 & \ 2202 & \ 2022) \\
(1210 & \ 2101 & \ 1012) \ (2110 & \ 1102 & \ 1021) \ (2120 & \ 1202 & \ 2021) \ (2210 & \ 2102 & \ 1022)
\end{align*}
\]

The cyclic function \(V\) outlined by \(V^m(\rho) = \rho\) is easily expressed in Pascal code by

\[
\text{repeat} \quad v := \text{Concat}(v, v[1]); \\
\quad \text{Delete}(v, 1, 1) \\
\text{until } v[1] <> '0';
\]

\textbf{Equation Classes Cardinality}

There is no limit to the amount of equations having the same \(m\) number of multiplications because there is no upper limit to the number of divisions.

\textbf{Definition 7.} Let’s write \(K_d[n]\) the class of equations having the same number \(n\) of divisions.

\textbf{Definition 8.} Let’s write \(K_{dm}[d, m]\) the subclasses of \(K_d[d]\) having \(m\) multiplications.

The geometrical representation of \(K_{dm}[d, m]\) classes as a network enables cardinality calculation easy:

\[
\text{card } K_d[d] = (r - 1) r^d \quad \text{(18)}
\]

\[
\text{card } K_{dm}[d, m] = \left(\frac{d - 1}{m - 1}\right) (r - 1)^m \quad \text{(19)}
\]

In the following figure each circle represents a \(K_{dm}[d, m]\) class at abscissa \(d\) and ordinate \(m\). Each circle contains \(\text{card } K_{dm}[d, m]\) with a number \(n = r - 1\). Diagonal lines represent \(T_i\) and horizontal lines represent \(D\). How to get these amounts? The problem is as simple as how many possible numbers in base \(r\) with \(m\) non-zero digits and \(d - m\) zeros?
Figure 1: Amount of equations in $K_{dm}$ classes with $n = r - 1$.

Subseqs of equal length

How many full subseqs of length $L$ having distinct shape vector? Simply add the amounts in the diagonals where $d + m = L - 1$.

$$n, n, n(n + 1), n(2n + 1), n(n^2 + 3n + 1)\ldots$$
$$r - 1, r - 1, r(r - 1), (2r - 1)(r - 1), (r^2 + r - 1)(r - 1)\ldots$$

It seems complex but dividing by $n = r - 1$ we get the following $(u_k)$ sequences

$$u_0 = 0, \quad u_1 = 1$$
$$u_k = u_{k-1} + nu_{k-2} \quad (20)$$

These are Lucas sequences $U_k(P, Q)$ where $P = 1$ and $Q = 1 - r$ including sequences of Fibonacci for $r = 2$ and Jacobsthal for $r = 3$.

Linking of Cycles

Let $q = Q(\rho)$ be the $q$ parameter in the equation of rank $\rho$. Adding $D$ at the end of a chain or '0' at the end of a shape vector $\rho$ does not change $q$.

$$Q(r^k \times \rho) = Q(\rho) \quad (21)$$
Thus all $q$-values in a $K_{dm}$ class are found in those with same $m$ but higher $d$. In addition by applying the $R$ rotation function with new $\delta$ to all $q$-values of the previous class we get all values of the new class. The monoid $\Omega$ is structured in layers starting from an initial $q$-value in $K_{dm}[m, m]$. Some of the following topics have not yet been addressed in French.

Cycle Layers

**Definition 9.** A cycle layer is the set of all algebraic cycles reachable using only rotation functions starting from a $q$-value in $K_{dm}[m, m]$.

The amount of cycles in a layer has no limit.

**Definition 10.** The seed of a cycle layer is the smallest $q$-value of the layer.

For example, the cycle $(94\ 185\ 122)$ is obtained by $R_\delta[-179]$, the cycle $(94\ 131\ 62)$ by $R_\delta[-17]$, finally $(62\ 58\ 65)$ by $R_\delta[37]$. The seed is $q_0 = 58$. The same, more simply, in terms of shape vectors $\rho$ obtained by $V$ is

$$(12020\ 20201\ 20120) \rightarrow (1202\ 2021\ 2120) \rightarrow (212\ 122\ 221)$$

The seed is $\rho_0 = 122$. With $r = 3$ and $m = 3$ there are four layers, the $\rho$-seeds being

$$\{111, 112, 122, 222\}$$

**Definition 11.** The layer's level $\lambda$ is the mean value of $q_i$ in the shape vector $\rho$

$$\lambda = m^{-1} \sum_{i=1}^{m} (q_i) \quad (22)$$

With $r > 5$ and $m = 4$ these five $\rho$-seeds have the same level $\lambda = 2$

$$\{1115, 1124, 1133, 1223, 2222\}$$

Cycle Barycenter

The $\mu$ function and the balance center $\mu_0$ of a cycle were proposed in the first paper. The definitions are slightly modified to suited the new context of Generalized Collatz Sequences.

**Definition 12.** The $\mu$ function and the balance center $\mu_0$ are defined by

$$\mu(x) = (p - r^x)^{-1} \quad (23)$$

$$\mu_0 = \mu(d/m) \quad (24)$$
Here is the source of these definitions showed first in a restricted context. In some cases the \( q_i \) values of the shape vector \( \rho = (q_i0^d)_m \) are a constant value \( c \). Further on we will introduce the mean value \( \bar{q} \) as a constant \( \lambda \) level. But for now we just suppose that the parameter \( r \) is shifted to \( x \) such that \( d = m \). Then

\[
\langle \omega \rangle = \left(p^m, r^d, -q\right) = \left(p, x, -c\right)^m
\]

(25)

Here is a known example to grasp the meaning of \( \mu_0 \). Let \((p, r)\) be \((3, 2)\). These are the negative Collatz-Kakutani sequences with \( \delta = 139 \) and \( c = 1 \). The \( K_{dm}[11, 7] \) contains the well known cycle \((x_k)_7 = (q_k/139)_7 = (17\ 25\ 37\ 55\ 41\ 61\ 91)\). Now suppose that \( d \) is changed to \( d_k \), or \( m \) is changed to \( m_k \), so that

\[
x_k = \mu(d_k/7) = \mu(11/m_k)
\]

Consistent with \( \mu_0 = \mu(11/7) \approx 35.6997 \) indicated by • we have

\[
(d_k)_7 \approx (10.8948\ 10.9590\ •\ 11.0033\ 11.0335\ 11.0123\ 11.0394\ 11.0577)
\]

\[
(m_k)_7 \approx (7.0676\ 7.0261\ •\ 6.9979\ 6.9789\ 6.9922\ 6.9750\ 6.9635)
\]

The following equations being verified this shows that \( \mu_0 \) is the equilibrium point.

\[
\sum_{k=1}^{m} 11 - d_k = \sum_{k=1}^{m} 7 - m_k = 0
\]

The \( \mu_0 \) value is the cycle barycenter of the minimum layer where \( c = 1 \). For a layer with \( c = 2 \) the barycenter is \( 2\mu_0 \) as this example illustrates.

\[
V^{(k)}(\rho) = (1110\ 1101\ 1011) \rightarrow R^{(k)}(q) = (37\ 55\ 79)
\]

\[
V^{(k)}(\rho) = (2220\ 2202\ 2022) \rightarrow R^{(k)}(q) = (74\ 110\ 158)
\]

Definition 13. The barycenter \( q_0 \) of an algebraic cycle in any layer is the minimum barycenter multiply by the layer’s level. Ditto for the related \( x_0 \) which is the barycenter of \((x_i)_m\) when the cycle exists.

\[
q_0 = \lambda \delta \mu_0 \\
x_0 = q_0 / \delta = \lambda \mu_0
\]
Now we need to show how is derived this general definition for non-constant $q_i$. We have

\begin{align*}
q & = \sum_{i=1}^{m} q_i p^{m-i} \sigma_i \\
& = \sum_{i=1}^{m} q_i \hat{q}_i \quad \text{where } \hat{q}_i = p^{m-i} \sigma_i \\
& = \begin{bmatrix} q_1 & q_2 & \ldots & q_m \end{bmatrix} \begin{bmatrix} \hat{q}_1 & \hat{q}_2 & \ldots & \hat{q}_m \end{bmatrix}^T
\end{align*}

(27)

(28)

(29)

The second matrix changes with division amount $d$ while the first matrix is constant throughout the layer. If we add all $q$-values in a cycle the rotation of $i$ generates a matrix with constant values.

\begin{align*}
\sum_{i=1}^{m} R^i(q) & = \begin{bmatrix} m\lambda & m\lambda & \ldots & m\lambda \end{bmatrix} \begin{bmatrix} \hat{q}_1 & \hat{q}_2 & \ldots & \hat{q}_m \end{bmatrix}^T \\
q_0 & = m^{-1} \sum_{i=1}^{m} R^i(q) = \lambda \begin{bmatrix} 1 & 1 & \ldots & 1 \end{bmatrix} \begin{bmatrix} \hat{q}_1 & \hat{q}_2 & \ldots & \hat{q}_m \end{bmatrix}^T \\
q_0 & = \lambda \delta \mu_0 \quad \text{always} \quad \text{(31)}
\end{align*}

(30)

(31)

(32)

(33)

How can the barycenter $\lambda \mu_0$ be useful? In the context $(p, r) = (3, 2)$ for example, we can say that no cycle $(x_i)_m$ can exist if $d > 2m$, since $|\lambda \mu_0| < 1$.

**Explicit role of $\delta$ in cyclic process**

There is no inner connection between $\delta$ and $q$ but, with the rotation of $q$ in cycles, $\delta$ plays an external role plain to see with the following matrices shifts yet providing a more accurate grasp of the $R$ function implying $q_1$.

\begin{align*}
q & = \begin{bmatrix} p^{m-1} & p^{m-2} & \ldots & p & 1 \end{bmatrix} \begin{bmatrix} q_1 & q_2 \sigma_2 & \ldots & q_m \sigma_m \end{bmatrix}^T \\
pq & = \begin{bmatrix} p^m & p^{m-1} & \ldots & p^2 & p \end{bmatrix} \begin{bmatrix} q_1 & q_2 \sigma_2 & \ldots & q_m \sigma_m \end{bmatrix}^T \\
pq + q_1 r^d & = \begin{bmatrix} p^m & p^{m-1} & \ldots & p^2 & p \end{bmatrix} \begin{bmatrix} q_1 & q_2 \sigma_2 & \ldots & q_m \sigma_m & q_1 r^d \end{bmatrix}^T \\
pq + q_1 r^d - q_1 p^m & = \begin{bmatrix} p^m & p^{m-1} & \ldots & p^2 & p \end{bmatrix} \begin{bmatrix} q_2 \sigma_2 & \ldots & q_m \sigma_m & q_1 r^d \end{bmatrix}^T \\
(pq - q_1 \delta) \mod^* r & = \begin{bmatrix} p^{m-1} & p^{m-2} & \ldots & p & 1 \end{bmatrix} \begin{bmatrix} q_2 & q_3 \sigma_3 - \sigma_2 & \ldots & q_1 r^d - \sigma_2 \end{bmatrix}^T \\
R(q) & = \begin{bmatrix} p^{m-1} & p^{m-2} & \ldots & p & 1 \end{bmatrix} \begin{bmatrix} q_2 & q_3 \sigma_3 & \ldots & q_1 \sigma_m \end{bmatrix}^T
\end{align*}

(34)

(35)

(36)

(37)

(38)

(39)
The rotation of \((d_i)_m\) is less evident than that of \((q_i)_m\) but not less true. I leave it to check. Instead, here is the inverse Collatz-Kakutani example in \((p,r) = (3,2)\) about the cycle \((x_i)_7 = (17 25 37 55 41 61 91)\) with \((1 + d_i)_7 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 2 \end{pmatrix}\) thus \((\sigma_i)_7 = (0 1 2 3 5 6 7)\).

\[
q = \begin{bmatrix} 3^6 & 3^5 & 3^4 & 3^3 & 3^2 & 3^1 & 3^0 \end{bmatrix} \begin{bmatrix} 2^0 & 2^1 & 2^2 & 2^3 & 2^5 & 2^6 & 2^7 \end{bmatrix}^T = 2363
\]

\[
3q = \begin{bmatrix} 3^7 & 3^6 & 3^5 & 3^4 & 3^3 & 3^2 & 3^1 \end{bmatrix} \begin{bmatrix} 2^0 & 2^1 & 2^2 & 2^3 & 2^5 & 2^6 & 2^7 \end{bmatrix}^T = 7089
\]

\[
3q + 2^{11} = \begin{bmatrix} 3^7 & 3^6 & 3^5 & 3^4 & 3^3 & 3^2 & 3^1 \end{bmatrix} \begin{bmatrix} 2^0 & 2^1 & 2^2 & 2^3 & 2^5 & 2^6 & 2^7 & 2^{11} \end{bmatrix}^T = 9137
\]

\[
3q + 2^{11} - 3^7 = \begin{bmatrix} 3^6 & 3^5 & 3^4 & 3^3 & 3^2 & 3^1 & 3^0 \end{bmatrix} \begin{bmatrix} 2^1 & 2^2 & 2^3 & 2^5 & 2^6 & 2^7 & 2^{11} \end{bmatrix}^T = 6950
\]

\[
R(q) = (3q - \delta) \mod 2 = \begin{bmatrix} 3^6 & 3^5 & 3^4 & 3^3 & 3^2 & 3^1 & 3^0 \end{bmatrix} \begin{bmatrix} 2^0 & 2^1 & 2^2 & 2^4 & 2^5 & 2^6 & 2^{10} \end{bmatrix}^T = 3475
\]

The new \((\sigma_i)_7 = (0 1 2 4 5 6 10)\) matches \((1 + d_i)_7 = \begin{pmatrix} 1 & 1 & 1 & 1 & 4 & 1 \end{pmatrix}\) hence the rotated previous \((1 + d_i)_7\).

**Incidental divisibility**

Beside the constant external role of \(\delta\) in \(q\)-parameter cycles there is an independant link between \(\delta\) and \(q\) in the previous example: \(\delta = 139\) divide 2363 giving 17, fulfilling so the requirement to get a numerical cycle, either \(q \mod \delta = 0\). Then, forcibly, \(25 \times \delta = 3475\).

Such numerical cycle occurs very rarely, randomly and presumably never with large integers. Often, in addition, there is no way to predict that a potential \(q \mod \delta\) value will be more frequent than another. Is it therefore justified to assume the equiprobability of the potential values, especially since this would allow us to account what is observed: rarely, randomly and never seen with large numbers? The issue requires investigation to go beyond the conditional calculus of probabilities in the first French paper.

**Equiprobability violation**

With a given \(m\) there exist a class \(K_{dm}\) for which

\[
|\delta| = \left| p^m - r^d \right| \quad \text{and} \quad |\mu_0|^{-1} = \left| p - r^{d/m} \right|
\]

are minima. For \(K_{dm}\) classes with greater \(d\) the \(\mu_0\) values are decreasing and eventually \(|\lambda\mu_0| < 1\). This implies that there is no possibility of a numerical cycle and therefore 0 can’t be a potential value of \(q \mod \delta\). There are a few other indirect ways of knowing that 0 is excluded or required as a potential value, but nothing can be said about \(q \mod \delta\) values between 0 and \(\delta\). On the other hand, if \(\delta\) divide a \(q_i\) in a cycle \((q_i)_m\) it can be shown that \(\delta\) divide all the others \(q_i\) of that cycle.
Numerical Cycle Occurrence Probability

If all possible values of $q \mod \delta$ are allowed in a class $K_{dm}$ and there is no way to predict the relative frequency of the values, we can assume that they are equal and calculate, under this condition, what is the occurrence probability of a cycle in the class.

Let $P_{\text{one}} = |\delta|^{-1}$ be the probability that a $(q_i)_m$ cycle results in a numerical $(x_i)_m$ cycle and $e$ be the number of cycles of length $m$ (which excludes spirals when $d$ and $m$ are not relatively prime). The probability $P_{\text{no}}$ that there is no numerical cycle in this class $K_{dm}$ is

$$P_{\text{no}} = (1 - P_{\text{one}})^e$$ (40)

The number of cycles when $d$ an $m$ are relatively prime is

$$e = m^{-1} \text{ card } K_{dm}$$ (41)

If there are spirals we should subtract from card $K_{dm}$ the elements in the spirals but this is only a minor correction that can be neglected. So we have

$$P_{\text{no}} = \left(1 - \frac{1}{|p^m - rd|}ight)^{\frac{1}{d(m)}(r - 1)^m}$$ (42)

Except when $d$ and $m$ are small we can use the following approximation where $1/N$ means a chance on $N$ of at least one cycle.

$$P_{\text{no}} = \left(\frac{|\delta| - 1}{|\delta|}\right)^e$$

$$= 1 - \left(\frac{e}{1}\right)|\delta|^{-1} + \left(\frac{e}{2}\right)|\delta|^{-2} - \ldots$$

$$1/N = P_{\text{yes}} \simeq e |\delta|^{-1}$$

$$N \simeq \frac{|p^m - rd|d}{\binom{d}{m}(r - 1)^m}$$ (43)

References

