Transformation of the Covariant Derivative

Anamitra Palit

Physicist,freelancer

P154 Motijheel Avenue,Flat C4,Kolkata 700074,India

palit.anamitra@gmail.com

Cell No:+ 919163892336

Abstract

The article considers the transformation of the covariant derivative of a rank one contravariant tensor to bring out a conflicting aspect of the theory in that we arrive at an impossible equation.

Introduction

The covariant derivative of a rank one contravariant tensor is a mixed tensor of rank two. Its transformation leads to an impossible equation to bring out a contradiction in the theory.

Calculations

We consider the transformation of the covariant derivative[1] of the rank one contravariant tensor[which is a mixed tensor of rank two]

$$\frac{\partial \bar{A}^{\mu}}{\partial \bar{x}^{\rho}} + \bar{\Gamma}^{\mu}{}_{\rho\sigma}\bar{A}^{\sigma} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}\frac{\partial x^{\beta}}{\partial \bar{x}^{\rho}}\left[\frac{\partial A^{\alpha}}{\partial x^{\beta}} + \Gamma^{\alpha}{}_{\beta\gamma}A^{\gamma}\right] \quad (1)$$

$$\frac{\partial \bar{A}^{\mu}}{\partial \bar{x}^{\rho}}d\bar{x}^{\rho} + \bar{\Gamma}^{\mu}{}_{\rho\sigma}\bar{A}^{\sigma}d\bar{x}^{\rho} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}\frac{\partial x^{\beta}}{\partial \bar{x}^{\rho}}\left[\frac{\partial A^{\alpha}}{\partial x^{\beta}}d\bar{x}^{\rho} + \Gamma^{\alpha}{}_{\beta\gamma}A^{\gamma}d\bar{x}^{\rho}\right]$$

$$d\bar{A}^{\mu} + \bar{\Gamma}^{\mu}{}_{\rho\sigma}\bar{A}^{\sigma}d\bar{x}^{\rho} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}\frac{\partial x^{\beta}}{\partial \bar{x}^{\rho}}\left[\frac{\partial A^{\alpha}}{\partial x^{\beta}}\frac{\partial \bar{x}^{\rho}}{\partial x^{k}}dx^{k} + \Gamma^{\alpha}{}_{\beta\gamma}A^{\gamma}\frac{\partial \bar{x}^{\rho}}{\partial x^{k}}dx^{k}\right]$$

$$d\bar{A}^{\mu} + \bar{\Gamma}^{\mu}{}_{\rho\sigma}\bar{A}^{\sigma}d\bar{x}^{\rho} = \frac{\partial x^{\beta}}{\partial \bar{x}^{\rho}}\frac{\partial \bar{x}^{\rho}}{\partial x^{k}}\frac{\partial A^{\alpha}}{\partial x^{\beta}}\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}dx^{k} + \Gamma^{\alpha}{}_{\beta\gamma}A^{\gamma}\frac{\partial \bar{x}^{\rho}}{\partial x^{k}}\frac{\partial x^{\beta}}{\partial \bar{x}^{\rho}}\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}dx^{k}$$

$$d\bar{A}^{\mu} + \bar{\Gamma}^{\mu}{}_{\rho\sigma}\bar{A}^{\sigma}d\bar{x}^{\rho} = \delta^{\beta}{}_{k}\frac{\partial A^{\alpha}}{\partial x^{\beta}}\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}dx^{k} + \Gamma^{\alpha}{}_{\beta\gamma}A^{\gamma}\delta^{\beta}{}_{k}\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}dx^{k}$$

$$d\bar{A}^{\mu} + \bar{\Gamma}^{\mu}{}_{\rho\sigma}\bar{A}^{\sigma}d\bar{x}^{\rho} = \frac{\partial A^{\alpha}}{\partial x^{k}}\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}dx^{k} + \Gamma^{\alpha}{}_{k\gamma}A^{\gamma}\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}dx^{k}$$

$$d\bar{A}^{\mu} + \bar{\Gamma}^{\mu}{}_{\rho\sigma}\bar{A}^{\sigma}d\bar{x}^{\rho} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}\frac{\partial A^{\alpha}}{\partial x^{k}}dx^{k} + \Gamma^{\alpha}{}_{k\gamma}A^{\gamma}\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}dx^{k}$$

$$d\bar{A}^{\mu} + \bar{\Gamma}^{\mu}{}_{\rho\sigma}\bar{A}^{\sigma}d\bar{x}^{\rho} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}dA^{\alpha} + \Gamma^{\alpha}{}_{k\gamma}A^{\gamma}\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}dx^{k}$$

$$d\bar{A}^{\mu} + \bar{\Gamma}^{\mu}{}_{\rho\sigma}\bar{A}^{\sigma}d\bar{x}^{\rho} = d\bar{A}^{\mu} + \Gamma^{\alpha}{}_{k\gamma}A^{\gamma}\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}dx^{k}$$

$$\bar{\Gamma}^{\mu}{}_{\rho\sigma}\bar{A}^{\sigma}d\bar{x}^{\rho} = \Gamma^{\alpha}{}_{k\gamma}A^{\gamma}\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}dx^{k} \quad (2)$$

Equation (2) holds for any manifold.In particular we consider the flat space time manifold.

In the flat space time context the Christoffel symbols[all of them] are zero only in the Cartesian system but non zero[all not zero] in the others. On the right side of equation (2)we consider Cartesian coordinates in flat space time: $\Gamma^{\alpha}{}_{k\gamma} = 0$. On the left side we consider some other coordinate system manifold being the same that is flat space time.

Therefore from (2) we obtain:

$$\bar{\Gamma}^{\mu}{}_{\rho\sigma}\bar{A}^{\sigma}d\bar{x}^{\rho} = 0(3)$$

But $\bar{\Gamma}^{\mu}{}_{\rho\sigma} \neq 0$,and the field$\bar{A}^{\sigma}$ is arbitrary! The possibility of equation (3) materializing comes into question.

We may obtain the Christoffel symbols for flat pace time in the spherical system by applying M=0 to the Schwarzschild Christoffel symbols[2]. We have six non vanishing Christoffel symbols for M=0

$\Gamma^{r}{}_{\theta\theta} = -r, \Gamma^{r}{}_{\varphi\varphi} = -rSin^{2}\theta, \Gamma^{\theta}{}_{r\theta} = \frac{1}{r}, \Gamma^{\theta}{}_{\varphi\varphi} = -Cos\theta Sin\theta, \Gamma^{\varphi}{}_{r\varphi} = 1/r, \Gamma^{\varphi}{}_{\theta\varphi} = Cot\,\theta$

Direct Verification[flat space time, spherical]:

Following the usual technique[3],

$$\Gamma^{\alpha}{}_{\beta\gamma} = \frac{1}{2}g^{\alpha s}\left[\frac{\partial g_{s\beta}}{\partial x^{\gamma}} + \frac{\partial g_{s\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial x^{s}}\right]$$

$$g_{\alpha k}\Gamma^{\alpha}{}_{\beta\gamma} = \frac{1}{2}g_{\alpha k}g^{\alpha s}\left[\frac{\partial g_{s\beta}}{\partial x^{\gamma}} + \frac{\partial g_{s\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial x^{s}}\right]$$

$$g_{\alpha k}\Gamma^{\alpha}{}_{\beta\gamma} = \frac{1}{2}\delta^{s}{}_{k}\left[\frac{\partial g_{s\beta}}{\partial x^{\gamma}} + \frac{\partial g_{s\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial x^{s}}\right]$$

$$g_{\alpha k}\Gamma^{\alpha}{}_{\beta\gamma} = \frac{1}{2}\left[\frac{\partial g_{k\beta}}{\partial x^{\gamma}} + \frac{\partial g_{k\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial x^{k}}\right]$$

[No summation on k]

In the orthogonal system the only surviving term on the left side is $g_{kk}\Gamma^{k}{}_{\beta\gamma}$ with no summation on k.

We have,

$$g_{kk}\Gamma^{k}{}_{\beta\gamma} = \frac{1}{2}\left[\frac{\partial g_{k\beta}}{\partial x^{\gamma}} + \frac{\partial g_{k\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial x^{k}}\right]$$

$$\Gamma^{k}{}_{\beta\gamma} = \frac{1}{2g_{kk}}\left[\frac{\partial g_{k\beta}}{\partial x^{\gamma}} + \frac{\partial g_{k\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial x^{k}}\right]$$

As for an example we may have,

$$\Gamma^{\varphi}{}_{r\varphi} = \frac{1}{2g_{\varphi\varphi}}\left[\frac{\partial g_{\varphi r}}{\partial \varphi} + \frac{\partial g_{\varphi\varphi}}{\partial r} - \frac{\partial g_{r\varphi}}{\partial \varphi}\right]$$

$$\Gamma^{\varphi}{}_{r\varphi} = \frac{1}{2g_{\varphi\varphi}}\frac{\partial g_{\varphi\varphi}}{\partial r} = \frac{1}{-2r^{2}Sin^{2}\theta}\frac{\partial(-r^{2}Sin^{2}\theta)}{\partial r} = 1/r$$

The other Christoffel symbols may be verified in a similar manner.[Covariant derivative reduces to partial derivative in the fat space time context only in the Cartesian system]

Conclusions

As stated at the outset we have arrived at an impossible equation starting from the transformation of the covariant derivative of a contravariant tensor. This points to difficulties in the basic theory.

References

1. Spiegel M R,Vector Analysis and an Introduction to Tensor Analysis, Schaum's Outline Series, MacGraw Hill Book Company,Singapore,1974,Chapter 8,Tensor Analysis, problem 52,p197-198.
2. Hartle J. B.,Gravity ,Pearson Education Inc,2003,Appendix B,p570