Transformation of the Covariant Derivative

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Abstract

The article considers the transformation of the covariant derivative of a rank one contravariant tensor to bring out a conflicting aspect of the theory in that we arrive at an impossible equation.

Introduction

The covariant derivative of a rank one contravariant tensor is a mixed tensor of rank two. Its transformation leads to an impossible equation to bring out a contradiction in the theory.

Calculations

We consider the transformation of the covariant derivative\(^{(\ref{eq:1})}\) of the rank one contravariant tensor[which is a mixed tensor of rank two]

\[
\frac{\partial \tilde{A}^\mu}{\partial \tilde{x}^\rho} + \Gamma^\mu_{\rho\sigma} \tilde{A}^\sigma = \frac{\partial \tilde{x}^\mu}{\partial x^\alpha} \frac{\partial A^\alpha}{\partial x^\beta} \left[ \frac{\partial A^\beta}{\partial x^\gamma} + \Gamma^\gamma_{\beta\gamma} A^\rho \right] \tag{1}
\]

\[
\frac{\partial \tilde{A}^\mu}{\partial \tilde{x}^\rho} \, d\tilde{x}^\rho + \Gamma^\mu_{\rho\sigma} \tilde{A}^\sigma \, d\tilde{x}^\rho = \frac{\partial \tilde{x}^\mu}{\partial x^\alpha} \frac{\partial x^\beta}{\partial x^\gamma} \frac{\partial A^\alpha}{\partial x^\beta} \frac{\partial \tilde{x}^\rho}{\partial x^\gamma} \, dx^k + \Gamma^\mu_{\rho\gamma} A^\gamma \frac{\partial \tilde{x}^\rho}{\partial x^\gamma} \, dx^k
\]

\[
d\tilde{A}^\mu + \Gamma^\mu_{\rho\sigma} \tilde{A}^\sigma \, d\tilde{x}^\rho = \frac{\partial x^\beta}{\partial x^\alpha} \frac{\partial A^\alpha}{\partial x^\beta} \frac{\partial \tilde{x}^\rho}{\partial x^\gamma} \, dx^k + \Gamma^\mu_{\rho\gamma} A^\gamma \frac{\partial \tilde{x}^\rho}{\partial x^\gamma} \, dx^k + \Gamma^\mu_{\rho\gamma} A^\gamma \frac{\partial \tilde{x}^\rho}{\partial x^\gamma} \, dx^k
\]

\[
d\tilde{A}^\mu + \Gamma^\mu_{\rho\sigma} \tilde{A}^\sigma \, d\tilde{x}^\rho = \delta^\mu_k \frac{\partial A^\alpha}{\partial x^\alpha} \frac{\partial \tilde{x}^\rho}{\partial x^\gamma} \, dx^k + \Gamma^\mu_{\rho\gamma} A^\gamma \frac{\partial \tilde{x}^\rho}{\partial x^\gamma} \, dx^k
\]

\[
d\tilde{A}^\mu + \Gamma^\mu_{\rho\sigma} \tilde{A}^\sigma \, d\tilde{x}^\rho = \frac{\partial A^\alpha}{\partial x^\alpha} \frac{\partial \tilde{x}^\rho}{\partial x^\gamma} \, dx^k + \Gamma^\mu_{\rho\gamma} A^\gamma \frac{\partial \tilde{x}^\rho}{\partial x^\gamma} \, dx^k
\]

\[
d\tilde{A}^\mu + \Gamma^\mu_{\rho\sigma} \tilde{A}^\sigma \, d\tilde{x}^\rho = \frac{\partial \tilde{x}^\rho}{\partial x^\gamma} \, dx^k + \Gamma^\mu_{\rho\gamma} A^\gamma \frac{\partial \tilde{x}^\rho}{\partial x^\gamma} \, dx^k
\]

\[
d\tilde{A}^\mu + \Gamma^\mu_{\rho\sigma} \tilde{A}^\sigma \, d\tilde{x}^\rho = \frac{\partial \tilde{x}^\rho}{\partial x^\gamma} \, dx^k + \Gamma^\mu_{\rho\gamma} A^\gamma \frac{\partial \tilde{x}^\rho}{\partial x^\gamma} \, dx^k
\]
\[ d\bar{A}^\mu + \bar{\Gamma}_{\rho\sigma}^\mu \bar{A}^\sigma d\bar{x}^\rho = d\bar{A}^\mu + \Gamma_{\kappa\gamma}^\alpha \partial_{\bar{x}^\alpha} \bar{A}^\mu d\bar{x}^k \]

\[ \Gamma_{\rho\sigma}^\mu \bar{A}^\sigma d\bar{x}^\rho = \Gamma_{\kappa\gamma}^\alpha \partial_{\bar{x}^\alpha} \bar{A}^\mu d\bar{x}^k \quad (2) \]

Equation (2) holds for any manifold. In particular we consider the flat space time manifold.

In the flat space time context the Christoffel symbols [all of them] are zero only in the Cartesian system but non zero [all not zero] in the others. On the right side of equation (2) we consider Cartesian coordinates in flat space time: \( \Gamma_{\kappa\gamma}^\alpha = 0 \). On the left side we consider some other coordinate system manifold being the same that is flat space time.

Therefore from (2) we obtain:

\[ \Gamma_{\rho\sigma}^\mu \bar{A}^\sigma d\bar{x}^\rho = 0 \quad (3) \]

But \( \bar{\Gamma}_{\rho\sigma}^\mu \neq 0 \), and the field \( \bar{A}^\sigma \) is arbitrary! The possibility of equation (3) materializing comes into question.

We may obtain the Christoffel symbols for flat space time in the spherical system by applying \( M = 0 \) to the Schwarzschild Christoffel symbols \([2]\).

We have six non vanishing Christoffel symbols for \( M = 0 \):

\( \Gamma_{r\theta}^r = -r, \Gamma_{r\phi}^r = -r \sin^2 \theta, \Gamma_{\theta\phi}^\phi = \frac{1}{r}, \Gamma_{\phi\phi}^\phi = -\cos \theta \sin \theta, \Gamma_{r\phi}^r = 1/r, \Gamma_{\theta\phi}^\phi = \cot \theta \)

Direct Verification [flat space time, spherical]:

Following the usual technique \([3]\),

\[ \Gamma_{\beta\gamma}^\alpha = \frac{1}{2} g^{\alpha s} \left[ \frac{\partial g_{s\beta}}{\partial x^\gamma} + \frac{\partial g_{s\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^s} \right] \]

\[ g_{ak}\Gamma_{\beta\gamma}^\alpha = \frac{1}{2} g_{ak} g^{\alpha s} \left[ \frac{\partial g_{s\beta}}{\partial x^\gamma} + \frac{\partial g_{s\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^s} \right] \]

\[ g_{ak}\Gamma_{\beta\gamma}^\alpha = \frac{1}{2} \delta^s_k \left[ \frac{\partial g_{s\beta}}{\partial x^\gamma} + \frac{\partial g_{s\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^s} \right] \]

\[ g_{ak}\Gamma_{\beta\gamma}^\alpha = \frac{1}{2} \left[ \frac{\partial g_{s\beta}}{\partial x^\gamma} + \frac{\partial g_{s\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^s} \right] \]

[No summation on \( k \)]

In the orthogonal system the only surviving term on the left side is \( g_{kk}\Gamma_{\beta\gamma}^k \) with no summation on \( k \).

We have,
\[
g_{kk} \Gamma^k_{\beta\gamma} = \frac{1}{2} \left( \frac{\partial g_{k\beta}}{\partial x^\gamma} + \frac{\partial g_{k\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^k} \right)
\]

\[
\Gamma^k_{\beta\gamma} = \frac{1}{2 g_{kk}} \left( \frac{\partial g_{k\beta}}{\partial x^\gamma} + \frac{\partial g_{k\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^k} \right)
\]

As for an example we may have,

\[
\Gamma^\varphi_{r\varphi} = \frac{1}{2 g_{\varphi\varphi}} \left( \frac{\partial g_{\varphi r}}{\partial \varphi} + \frac{\partial g_{\varphi\varphi}}{\partial r} - \frac{\partial g_{r\varphi}}{\partial \varphi} \right)
\]

\[
\Gamma^\varphi_{r\varphi} = \frac{1}{2 g_{\varphi\varphi}} \frac{\partial g_{\varphi\varphi}}{\partial r} = \frac{1}{-2r^2 \sin^2 \theta} \frac{\partial (-r^2 \sin^2 \theta)}{\partial r} = \frac{1}{r}
\]

The other Christoffel symbols may be verified in a similar manner. [Covariant derivative reduces to partial derivative in the fat space time context only in the Cartesian system]

Conclusions

As stated at the outset we have arrived at an impossible equation starting from the transformation of the covariant derivative of a contravariant tensor. This points to difficulties in the basic theory.

References