SMALLEST NUMBERS WHOSE NUMBER OF DIVISORS IS A PERFECT NUMBER

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Abstract. We present a formula for smallest possible numbers whose number of divisors is the n-th perfect number. The formula, that produces an integer sequence $a(n)$, involves the n-th Mersenne prime that appears both in an exponent of a power of 2 and in the product of consecutive odd primes (the odd primorial). While smallest in some sense, these numbers are among largest one can run into through an exercise in elementary number theory.

1. Introduction

The perfect number is a simple idea that uses the concept of divisors. By definition, for a natural number n to be perfect, the sum of its proper divisors must be n, or, if all divisors are included, $2n$, i.e., $sigma(n) = 2n$, where $sigma(n)$ stands for the sum of all divisors of n.

Perfect numbers have been exciting the imagination of professional mathematicians and amateurs alike for centuries now and the interest in them does not seem to be abating. They were studied already in antiquity, particularly by one of the era mathematical greats, Euclid. It was Euclid who realized that perfect numbers are triangular numbers of the form $M_p(M_p + 1)/2$ (1), where M_p is what is now called the Mersenne prime, i.e., a prime of the form $2^p - 1$, where p is also prime. Not all primes p generate Mersenne primes (for instance, this is not so for $p = 11$), but if M_k is to be prime, then it can be shown that k must be prime too.

It was not until the 18-th century that Leonhard Euler proved that (1) generates all even perfect numbers. It is not clear if there are any odd perfect numbers.

The first four Mersenne primes are thus $2^2 - 1 = 3$, $2^3 - 1 = 7$, $2^5 - 1 = 31$, and $2^7 - 1 = 127$. They give rise to the four perfect numbers known already to the ancients, namely 6, 28, 496, and 8128, respectively.

In what follows, we are not interested in perfect numbers but in smallest numbers whose number of divisors, often denoted as τ, is a perfect number. By focusing on smallest such numbers, we are able to put forward a simple formula for them, which turns out to contain expressions using both the Mersenne primes and, which was less expected, the primorial function.

In a way, simplifying a bit, this formula is an analog of Euclid's formula for perfect numbers (1), but for $\tau(n)$ while Euclid's is for n.

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2. The formula

The n-th term of the sequence consisting of smallest numbers whose number of divisors is the n-th perfect number can be cast in the form

$$a(n) = 2^{(M_n-1)} oddprimorial(log_2((M_n + 1)/2)),$$

where M_n is the n-th Mersenne prime and $oddprimorial(k)$ is the product of the first k odd primes.

For future use and as an example, let us list a few first values of $oddprimorial(k)$. For $k = 1$ to $k = 4$ we have: 3, $3*5 = 15$, $3*5*7 = 105$, and $3*5*7*11 = 1155$.

The first two values of $a(n)$ can easily be found by searching for the smallest numbers whose number of divisors is 6 and 28. Using Mathematica [1] (see the next section), we find them to be 12 and 960. Based on these two data points, it is relatively straightforward to predict the general pattern for $a(n)$. Using Mathematica again, we first verify the formula for $n = 1$ and $n = 2$, and then for the smallest numbers with 496 and 8128 divisors (terms 3 and 4 of our sequence). Subsequently, we can do this for even larger Mersenne primes.

Hence, for the first Mersenne prime of 3: $a(1) = 2^2 * oddprimorial(log_2(2)) = 4 * 3 = 12$, and for the second Mersenne prime of 7: $a(2) = 2^6 * oddprimorial(log_2(4)) = 64 * 3 * 5 = 960$.

Next, we get $a(3) = 2^{30} * oddprimorial(log_2(16)) = 2^{30} * 3 * 5 * 7 * 11$ and $a(4) = 2^{126} * oddprimorial(log_2(64)) = 2^{126} * 3 * 5 * 7 * 11 * 13 * 17$.

These numbers are much larger then the first two, containing 13 and 44 digits, respectively. Listed below in their full glory, they are:

1240171806720,

21714693892101036872835921354998998703800320.

Their very form in terms of the powers of 2 and the product of consecutive odd primes already suggests that they have the required number of divisors, but we can also easily check this using Mathematica.

The next term of the sequence studied has 2480 digits and the last one (the 8-th) that we were able to produce without getting overflow problems contains nearly 650 million digits.

3. The Wolfram Mathematica code

The following simple Mathematica code can be used to search for the first number whose number of divisors is 28 (the second perfect number)

```
SelectFirst[Range@10000, DivisorSigma[0,#]==28&]
```

Of course, changing 28 to 6 in this piece of code, let us quickly find the smallest number whose number of divisors is 6 (the first perfect number).

The following code can be used to test if the first four terms of $a(n)$ have the correct number of divisors:

```
a={2^2*3, 2^6*3*5, 2^30*3*5*7*11, 2^126*3*5*7*11*13*17};
DivisorSigma[0,#]&/@a
```

The next piece of code let us produce $a(n)$ for even higher values of n. However, for $n > 8$, Mathematica (version 12) starts having overflow problems.

```
Table[2^(2^MersennePrimeExponent[n]-2)*Product[Prime[i],
{i,2,1+Log [2,(2^MersennePrimeExponent[n])/2]}], {n,5}]
```

The last piece of code is used to find out the number of digits of $a(n)$. As noted earlier, we can do this only up to $n = 8$; the last term in this table is 646457041.

```
Table[IntegerLength[2^(2^MersennePrimeExponent[n]-2)*Product[Prime[i],
{i,2,1+Log [2,(2^MersennePrimeExponent[n])/2]}]], {n,8}]
```

4. Conclusion

What started as an innocent exercise in elementary number theory, quickly escalated into a problem that even Mathematica finds too hard to handle. The numbers this exercise leads to are truly staggering, especially if one realizes that Mersenne primes are among the biggest numbers ever explored by humanity and here we have them employed not only as exponents of powers of 2 but also in the product of consecutive odd primes (the odd primorial), where they determine how many of the odd primes occur in it.

The formula this simple problem leads to is quite rich but still very elementary; certainly elementary enough to be explored even in high school setting. While the appearance of Mersenne primes may not be unexpected, the primorial function that shows up in the formula may be harder to anticipate, yet seems pretty natural when things are thought through. Moreover, it appears in a rather novel context; for another appearance of this function in some other problem of elementary number theory most likely previously not studied see a recent paper by this author [2].

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References

[1] Mathematica, Wolfram Research Inc., http://www.wolfram.com.

[2] Waldemar Puszkarz, *Primorials and a formula for odd abundant numbers*, https://www.researchgate.net/publication/339644520_PRIMORIALS_AND_A_FORMULA_FOR_ODD_ABUNDANT_NUMBERS.

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