IS THE abc CONJECTURE TRUE?

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ABSTRACT. In this paper, we consider the abc conjecture. In the first part, we give the proof of the conjecture $c < \text{rad}^{1.63}(abc)$ that constitutes the key to resolve the abc conjecture. The proof of the abc conjecture is given in the second part of the paper, supposing that the abc conjecture is false, we arrive in a contradiction.

1. INTRODUCTION AND NOTATIONS

Let $a$ be a positive integer, $a = \prod a_i^{\alpha_i}$, $a_i$ prime integers and $\alpha_i \geq 1$ positive integers. We call radical of $a$ the integer $\prod a_i$ noted by $\text{rad}(a)$. Then $a$ is written as:

$$a = \prod a_i^{\alpha_i} = \text{rad}(a). \prod a_i^{\alpha_i-1}$$

We denote:

$$\mu_a = \prod a_i^{\alpha_i-1} \implies a = \mu_a. \text{rad}(a)$$

The abc conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Oesterlé of Pierre et Marie Curie University (Paris 6) [8]. It describes the distribution of the prime factors of two integers with those of their sum. The definition of the abc conjecture is given below:

**Conjecture 1.1. (abc Conjecture):** For each $\varepsilon > 0$, there exists $K(\varepsilon)$ such that if $a, b, c$ positive integers relatively prime with $c = a + b$, then:

$$c < K(\varepsilon).\text{rad}^{1+\varepsilon}(abc)$$

where $K$ is a constant depending only of $\varepsilon$.

We know that numerically, $\frac{\log c}{\log(\text{rad}(abc))} \leq 1.629912 [5]$. It concerned the best example given by E. Reyssat [5]:

$$2 + 3^{10}.109 = 23^5 \implies c < \text{rad}^{1.629912}(abc)$$

A conjecture was proposed that $c < \text{rad}^2(abc)$ [3]. In 2012, A. Nitaj [4] proposed the following conjecture:

**Conjecture 1.2.** Let $a, b, c$ be positive integers relatively prime with $c = a + b$, then:

$$c < \text{rad}^{1.63}(abc)$$

$$abc < \text{rad}^{4.42}(abc)$$

Firstly, we will give the proof of the conjecture given by (5) that constitutes the key to obtain the proof of the abc conjecture. Secondly, we present in section three of the paper the proof that the abc conjecture is true.

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2. A Proof of the Conjecture (1.2) Case $c = a + b$

Let $a, b, c$ be positive integers, relatively prime, with $c = a + b$, $b < a$ and $R = \rad(abc)$, $c = \prod_{j \in R} \beta_j^j, \beta_j \geq 1$.

In a previous paper [1], we have given, for the case $c = a + 1$, the proof that $c < \rad^{1.63}(ac)$. In the following, we will give the proof for the case $c = a + b$.

**Proof.** If $c < \rad(abc)$, then we obtain:

$$c < \rad(abc) < \rad^{1.63}(abc) \implies c < R^{1.63}$$

and the condition (5) is satisfied.

If $c = \rad(abc)$, then $a, b, c$ are not coprime, case to reject. In the following, we suppose that $c > \rad(abc)$ and $a, b$ and $c$ are not prime numbers.

$$c = a + b = \mu_a \rad(a) + \mu_b \rad(b) < \rad^{1.63}(abc) \tag{7}$$

2.1. $\mu_a \neq 1, \mu_a \leq \rad^{0.63}(a)$. We obtain:

$$c = a + b < 2a \leq 2\rad^{1.63}(a) < \rad^{1.63}(abc) \implies c < \rad^{1.63}(abc) \implies c < R^{1.63}$$

Then (7) is satisfied.

2.2. $\mu_c \neq 1, \mu_c \leq \rad^{0.63}(c)$. We obtain:

$$c = \mu_c \rad(c) \leq \rad^{1.63}(c) < \rad^{1.63}(abc) \implies c < R^{1.63}$$

and the condition (7) is satisfied.

2.3. $\mu_a > \rad^{0.63}(a)$ and $\mu_c > \rad^{0.63}(c)$.

2.3.1. **Case: $\rad^{0.63}(c) < \mu_c \leq \rad^{1.63}(c)$ and $\rad^{0.63}(a) < \mu_a \leq \rad^{1.63}(a)$:** We can write:

$$\mu_c \leq \rad^{1.63}(c) \implies c \leq \rad^{2.63}(c) \implies ac \leq \rad^{2.63}(ac) \implies a^2 < ac \leq \rad^{2.63}(ac) \implies a < \rad^{1.315}(ac) \implies c < 2a < 2\rad^{1.315}(ac) < \rad^{1.63}(abc) \implies c = a + b < R^{1.63}$$

2.3.2. **Case: $\mu_c > \rad^{1.63}(c)$ or $\mu_a > \rad^{1.63}(a)$:**

1- We suppose that $\mu_c > \rad^{1.63}(c)$ and $\mu_a \leq \rad^2(a)$:

1-1. Case $\rad(a) < \rad(c)$: In this case $a = \mu_a \rad(a) \leq \rad^3(a) \leq \rad^{1.63}(a) \rad^{1.37}(a) < \rad^{1.63}(a) \rad^{1.37}(c) \implies c < 2a < 2\rad^{1.63}(a) \rad^{1.37}(c) < \rad^{1.63}(abc) \implies c < R^{1.63}$

1-2. Case $\rad(c) < \rad(a) < \rad^{1.63}(c)$: As $a \leq \rad^{1.63}(a) \rad^{1.37}(a) < \rad^{1.63}(a) \rad^{1.63}(c) \implies c < 2a < 2\rad^{1.63}(a) \rad^{1.63}(c) < R^{1.63} \implies c < R^{1.63}$

1-3. Case $\rad^{1.63}(c) < \rad(a)$:
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I-3-1- We suppose \( c \leq \text{rad}^{3.26}(c) \), we obtain:

\[
c \leq \text{rad}^{3.26}(c) \implies c \leq \text{rad}^{1.63}(c).\text{rad}^{1.63}(c) \implies
c < \text{rad}^{1.63}(c).\text{rad}(a)^{1.37} < \text{rad}^{1.63}(c).\text{rad}(a)^{1.63}.\text{rad}^{1.63}(b) = R^{1.63} \implies c < R^{1.63}
\]

I-3-2- We suppose \( c > \text{rad}^{3.26}(c) \implies \mu_c > \text{rad}^{2.26}(c) \). We consider the case \( \mu_a = \text{rad}^2(a) \implies a = \text{rad}^3(a) \). Then, we obtain that \( X = \text{rad}(a) \) is a solution in positive integers of the equation:

\[
X^3 + 1 = c - b + 1 = c'
\]

But it is the case \( c' = 1 + a \). If \( c' = \text{rad}^m(c') \) with \( n \geq 4 \), we obtain the equation:

\[
\text{rad}^m(c') - \text{rad}^3(a) = 1
\]

But the solutions of the equation (9) are [2]: \( (\text{rad}(c') = 3, n = 2, \text{rad}(a) = +2) \), it follows the contradiction with \( n \geq 4 \) and the case \( c' = \text{rad}^n(c'), n \geq 4 \) is to reject.

In the following, we will study the cases \( \mu_c = A.\text{rad}^n(c') \) with \( \text{rad}(c') \nmid A, n \geq 0 \). The above equation (8) can be written as :

\[
(X + 1)(X^2 - X + 1) = c'
\]

Let \( \delta \) any divisor of \( c' \), then:

\[
X + 1 = \delta
\]

\[
X^2 - X + 1 = \frac{c'}{\delta} = c'' = \delta^2 - 3X
\]

We recall that \( \text{rad}(a) > \text{rad}^{1.63}(c) \).

I-3-2-1- We suppose \( \delta = l.\text{rad}(c') \). We have \( \delta = l.\text{rad}(c') < c' = \mu_c.\text{rad}(c') \implies l < \mu_c \). As \( \delta \) is a divisor of \( c' \), then \( l \) is a divisor of \( \mu_c \), we write \( \mu_c = l.m \). From \( \mu_c = l(\delta^2 - 3X) \), we obtain:

\[
m = l^2\text{rad}^2(c') - 3\text{rad}(a) \implies 3\text{rad}(a) = l^2\text{rad}^2(c') - m
\]

A- Case \( 3|m \implies m = 3m', m' > 1 \): As \( \mu_c = ml = 3m'l \implies 3|\text{rad}(c') \) and \( (\text{rad}(c'),m') \) not coprime. We obtain:

\[
\text{rad}(a) = \frac{l^2\text{rad}(c').\text{rad}(c')}{3} - m'
\]

It follows that \( a,c' \) are not coprime, then the contradiction.

B - Case \( m = 3 \implies \mu_c = 3l \implies c' = 3l\text{rad}(c') = 3\delta = \delta(\delta^2 - 3X) \implies \delta^2 = 3(1 + X) = 3\delta \implies \delta = l\text{rad}(c') = 3 \), then the contradiction.

I-3-2-2- We suppose \( \delta = l.\text{rad}^2(c'), l \geq 2 \). If \( l\text{rad}(c') \nmid \mu_c \) then the case is to reject. We suppose \( l\text{rad}(c')|\mu_c \implies \mu_c = m.l\text{rad}(c') \), then \( \frac{c'}{\delta} = m = \delta^2 - 3\text{rad}(a) \).

C - Case \( m = 1 = c' \div \delta \implies \delta^2 - 3\text{rad}(a) = 1 \implies (\delta - 1)(\delta + 1) = 3\text{rad}(a) = \text{rad}(a)(\delta + 1) \implies \delta = 2 = l.\text{rad}^2(c') \), then the contradiction.

D - Case \( m = 3 \), we obtain \( 3(1 + \text{rad}(a)) = \delta^2 = 3\delta \implies \delta = 3 = l\text{rad}^2(c') \). Then the contradiction.
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E - Case \( m \neq 1,3 \), we obtain: \( 3 \text{rad}(a) = l^2 \text{rad}^4(c') - m \implies \text{rad}(a) \) and \( \text{rad}(c') \) are not coprime. Then the contradiction.

I-3-2-3 - We suppose \( \delta = l \text{rad}^n(c'), l \geq 2 \) with \( n \geq 3 \). From \( c' = \mu'_c \text{rad}(c') = l \text{rad}^n(c') (\delta^2 - 3 \text{rad}(a)) \), we denote \( m = \delta^2 - 3 \text{rad}(a) = \delta^2 - 3X \).

F - As seen above (paragraphs C,D), the cases \( m = 1 \) and \( m = 3 \) give contradictions, it follows the reject of these cases.

G - Case \( m \neq 1,3 \). Let \( q \) be a prime that divides \( m \), it follows \( q | \mu'_c \implies q = c'_{j_0} \implies c'_j | \delta^2 \implies c'_{j_0} | 3 \text{rad}(a) \). Then \( \text{rad}(a) \) and \( \text{rad}(c') \) are not coprime. It follows the contradiction.

I-3-2-4 - We suppose \( \delta = \prod_{j \in J} c_j^{(j)} \), \( \beta_j \geq 1 \) with at least one \( j_0 \in J_1 \) with \( \beta_{j_0} \geq 2 \), \( \text{rad}(c') \delta \). We can write:
\[
\delta = \mu_\delta \text{rad}(\delta), \quad \text{rad}(c') = m \text{rad}(\delta), \quad m > 1, \quad (m, \mu_\delta) = 1
\]

Then, we obtain:
\[
c' = \mu'_c \text{rad}(c') = \mu'_c m \text{rad}(\delta) = \delta (\delta^2 - 3X) = \mu_\delta \text{rad}(\delta) (\delta^2 - 3X) \implies m \mu'_c = \mu_\delta (\delta^2 - 3X)
\]

(14) - If \( \mu'_c = \mu_\delta \implies m = \delta^2 - 3X = (\mu'_c \text{rad}(\delta)^2 - 3X \). As \( \delta < \delta^2 - 3X \implies m > \delta \implies \text{rad}(c') > m > \mu'_c \text{rad}(\delta) > \text{rad}^3(c') \) because \( \mu'_c > \text{rad}^2 2c'(c') \), it follows \( \text{rad}(c') > \text{rad}^2(c') \). Then the contradiction.

- We suppose \( \mu'_c < \mu_\delta \). As \( \text{rad}(a) = \mu_\delta \text{rad}(\delta) - 1 \), we obtain:
\[
\text{rad}(a) > \mu'_c \text{rad}(\delta) - 1 > 0 \implies \text{rad}(ac') > c' \text{rad}(\delta) - \text{rad}(c') > 0 \implies c' > \text{rad}(ac') > c' \text{rad}(\delta) - \text{rad}(c') > 0 \implies 1 > \text{rad}(\delta) - \frac{\text{rad}(c')}{c'} > 0, \quad \text{rad}(\delta) \geq 2
\]

The contradiction

(15) - We suppose \( \mu_\delta < \mu'_c \). In this case, from the equation (14) and as \( (m, \mu_\delta) = 1 \), it follows we can write:
\[
\mu'_c = \mu_1, \mu_2, \quad \mu_1, \mu_2 > 1
\]

(16) so that:
\[
m \mu_1 = \delta^2 - 3X, \quad \mu_2 = \mu_\delta \implies \delta = \mu_2 \text{rad}(\delta)
\]

(17) ** We suppose \( (\mu_1, \mu_2) \neq 1 \), then \( \exists c'_{j_0} \) so that \( j_0 | \mu_1 \) and \( j_0 | \mu_2 \). But \( \mu_\delta = \mu_2 \implies c'_{j_0} | \delta \). From \( 3X = \delta^2 - m\mu_1 \implies c'_{j_0} | 3X \implies c'_{j_0} | X \) or \( c'_{j_0} = 3 \).

- If \( c'_{j_0} | X \), it follows the contradiction with \( (c', a) = 1 \).
- If \( c'_{j_0} = 3 \). We have \( m\mu_1 = \delta^2 - 3X = \delta^2 - 3(\delta - 1) \implies \delta^2 - 3\delta + 3 - m\mu_1 = 0 \). As \( 3 | \mu_1 \implies \mu_1 = 3^k \mu'_1, 3 \nmid \mu'_1, k \geq 1 \), we obtain:
\[
\delta^2 - 3\delta + 3(1 - 3^{k-1}m\mu'_1) = 0
\]

(19) - We consider the case \( k > 1 \implies 3 \nmid (1 - 3^{k-1}m\mu'_1) \). Let us recall the Eisenstein criterion [7]:

**Theorem 2.1. (Eisenstein Criterion)** Let \( f = a_0 + \cdots + a_nX^n \) be a polynomial \( \in \mathbb{Z}[X] \). We suppose that \( \exists p \) a prime number so that \( p \nmid a_i, p | a_i, (0 \leq i \leq n - 1) \), and \( p^2 \nmid a_0 \), then \( f \) is irreducible in \( \mathbb{Q} \).

We apply Eisenstein criterion to the polynomial \( R(Z) \) given by:
\[
R(Z) = Z^2 - 3Z + 3(1 - 3^{k-1}m\mu'_1)
\]

(20)
then:
- \(3 \nmid 1, - 3 \mid (-3), - 3 \mid 3(1 - 3k^{-1}m\mu'_1),\) and \(- 3^2 \nmid 3(1 - 3k^{-1}m\mu'_1)\).

It follows that the polynomial \(R(Z)\) is irreducible in \(\mathbb{Q}\), then, the contradiction with \(R(\delta) = 0\).

- We consider the case \(k = 1\), then \(\mu_1 = 3\mu'_1\) and \((\mu'_1, 3) = 1\), we obtain:
  \[
  \delta^2 - 3\delta + 3(1 - m\mu'_1) = 0
  \]
  \[\text{expression of } 3 \]

\[\text{(21)}\]

* If \(3 \nmid (1 - m\mu'_1)\), we apply the same Eisenstein criterion to the polynomial \(R'(Z)\) given by:
  \[
  R'(Z) = Z^2 - 3Z + 3(1 - m\mu'_1)
  \]
  and we find a contradiction with \(R'(\delta) = 0\).

* We consider that \(3|(1 - m\mu'_1) \implies m\mu'_1 - 1 = 3^i, i \geq 1, 3 \nmid h, h \in \mathbb{N}^*\). \(\delta\) is an integer root of the polynomial \(R'(Z)\):
  \[
  R'(Z) = Z^2 - 3Z + 3(1 - m\mu'_1) = 0 \Rightarrow \text{the discriminant of } R'(Z) \text{ is } \Delta = 3^2 + 3^{i+1} \times 4h
  \]
  As the root \(\delta\) is an integer, it follows that \(\Delta = l^2 > 0\) with \(l\) a positive integer. We obtain:
  \[
  \Delta = 3^2(1 + 3^{i-1} \times 4h) = l^2
  
  \Rightarrow 1 + 3^{i-1} \times 4h = q^2 > 1, q \in \mathbb{N}^*
  \]

We can write the equation (21) as :
  \[
  \delta(\delta - 3) = 3^{i+1}.h \implies 3^3\mu'_1 \frac{\operatorname{rad}(\delta)}{3}.(\mu'_1 \operatorname{rad}(\delta) - 1) = 3^{i+1}.h \implies
  
  \mu'_1 \frac{\operatorname{rad}(\delta)}{3}.(\mu'_1 \operatorname{rad}(\delta) - 1) = h
  \]

We obtain \(i = 2\) and \(q^2 = 1 + 12h = 1 + 4\mu'_1 \operatorname{rad}(\delta)(\mu'_1 \operatorname{rad}(\delta) - 1).\) Then, \(q\) satisfies :
  \[
  q^2 - 1 = 12h \Rightarrow \frac{(q-1)(q+1)}{2} = 3h \Rightarrow (\mu'_1 \operatorname{rad}(\delta) - 1) \Rightarrow
  
  q - 1 = 2\mu'_1 \operatorname{rad}(\delta) - 2
  
  q + 1 = 2\mu'_1 \operatorname{rad}(\delta)
  \]

It follows that \((q = x, 1 = y)\) is a solution of the Diophantine equation:
  \[
  x^2 - y^2 = N
  \]

with \(N = 12h > 0\). Let \(Q(N)\) be the number of the solutions of (30) and \(\tau(N)\) is the number of suitable factorization of \(N\), then we announce the following result concerning the solutions of the Diophantine equation (30) (see theorem 27.3 in [6]):
- If \(N \equiv 2(\text{mod } 4)\), then \(Q(N) = 0\).
- If \(N \equiv 1\) or \(N \equiv 3(\text{mod } 4)\), then \(Q(N) = [\tau(N)/2]\).
- If \(N \equiv 0(\text{mod } 4)\), then \(Q(N) = [\tau(N/4)/2]\).

\([x]\) is the integral part of \(x\) for which \([x] \leq x < [x] + 1\).

Let \((\alpha', m')\), \(\alpha', m' \in \mathbb{N}^*\) be another pair, solution of the equation (30), then \(\alpha'^2 - m'^2 = x^2 - y^2 = N = 12h\), but \(q = x\) and \(1 = y\) satisfy the equation (29) given by \(x + y = 2\mu'_1 \operatorname{rad}(\delta)\), it follows \(\alpha', m'\) verify also \(\alpha' + m' = 2\mu'_1 \operatorname{rad}(\delta)\), that gives \(\alpha' - m' = 2(\mu'_1 \operatorname{rad}(\delta) - 1)\), then \(\alpha' = x = q = 2\mu'_1 \operatorname{rad}(\delta)\) and \(m' = y = 1\). So, we have given the proof of the uniqueness of the solutions of the equation (30) with the condition \(x + y = 2\mu'_1 \operatorname{rad}(\delta)\). As \(N = 12h \equiv 0(\text{mod } 4) \implies Q(N) = [\tau(N/4)/2] = [\tau(3h)/2]\), the expression of \(3h = \mu'_1 \operatorname{rad}(\delta).(\mu'_1 \operatorname{rad}(\delta) - 1)\), then \(Q(N) = [\tau(3h)/2] > 1.\) But \(Q(N) = 1\), then the
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contradiction and the case 3|$(1 - m.\mu_1')$ is to reject.

** We suppose that $(\mu_1, \mu_2) = 1$.

From the equation $m \mu_1 = \delta^2 - 3X = \delta^2 - 3(\delta - 1)$, we obtain that $\delta$ is a root of the following polynomial:

$$ R(Z) = Z^2 - 3Z + 3 - m.\mu_1 = 0 $$

The discriminant of $R(Z)$ is:

$$ \Delta = 9 - 4(3 - m.\mu_1) = 4m.\mu_1 - 3 = q^2 \quad \text{with} \quad q \in \mathbb{N}^* \quad \text{as} \quad \delta \in \mathbb{N}^* $$

- We suppose that $2|m\mu_1 \implies c'$ is even. Then $q^2 \equiv 5 \pmod{8}$, it gives a contradiction because a square is $\equiv 0, 1$ or $4 \pmod{8}$.

- We suppose $c'$ an odd integer, then $a$ is even. It follows $a = rad^3(a) \equiv 0 \pmod{8} \implies c' \equiv 1 \pmod{8}$. As $c' = \delta^2 - 3X.\delta$, we obtain $\delta^2 - 3X.\delta \equiv 1 \pmod{8}$. If $\delta^2 \equiv 1 \pmod{8} \implies -3X.\delta \equiv 0 \pmod{8} \implies 8|X.\delta \implies 4|\delta \implies c'$ is even. Then, the contradiction. If $\delta^2 \equiv 4 \pmod{8} \implies \delta \equiv 2 \pmod{8} \text{ or } \delta \equiv 6 \pmod{8}$. In the two cases, we obtain $2|\delta$. Then, the contradiction with $c'$ an odd integer.

It follows that the case $c > rad^3(26)(c)$ and $a = rad^3(a)$ is impossible.

1-3-3- We suppose $c > rad^3(26)(c)$ and large, then $c = rad^3(c) + h, h > rad^3(c)$, $h$ a positive integer and $\mu_\alpha < rad^2(\alpha) \implies a + l = rad^3(\alpha), l > 0$. Then we obtain:

$$ rad^3(c) + h = rad^3(a) - l + b \implies rad^3(a) - rad^3(c) = h + l - b > 0 $$

as $rad(a) > rad^{(26)}(c)$. We obtain the equation:

$$ rad^3(a) - rad^3(c) = h + l - b = m > 0 $$

Let $X = rad(a) - rad(c)$, then $X$ is an integer root of the polynomial $H(X)$ defined as:

$$ H(X) = X^3 + 3rad(ac)X - m = 0 $$

To resolve the above equation, we denote $X = u + v$. It follows that $u^3, v^3$ are the roots of the polynomial $G(t)$ given by:

$$ G(t) = t^2 - mt - rad^3(ac) = 0 $$

The discriminant of $G(t)$ is $\Delta = m^2 + 4rad^3(ac) = \alpha^2$, $\alpha > 0$. The two real roots of (36) are:

$$ t_1 = u^3 = \frac{m + \alpha}{2}, \quad t_2 = v^3 = \frac{m - \alpha}{2} $$

As $m = rad^3(a) - rad^3(c) > 0$, we obtain that $\alpha = rad^3(a) + rad^3(c) > 0$, then from the expression of the discriminant $\Delta$, it follows that $(\alpha = x, m = y)$ is a solution of the Diophantine equation:

$$ x^2 - y^2 = N $$

with $N = 4rad^3(ac) > 0$. From the expression of $\Delta$ above, we remark that $\alpha$ and $m$ verify the following equations:

$$ x + y = 2u^3 = 2rad^3(a) $$

$$ x - y = -2v^3 = 2rad^3(c) $$

then $x^2 - y^2 = N = 4rad^3(a).rad^3(c)$
Let $Q(N)$ be the number of the solutions of (38) and $\tau(N)$ is the number of suitable factorization of $N$, and using the same method as in the paragraph I-3-2-4- (case $3|1-m\mu^0$), we obtain a contradiction.

It follows that the cases $\mu_3 \leq \text{rad}^2(a)$ and $c > \text{rad}^{3.26}(c)$ are impossible.

II- We suppose that $\text{rad}^{1.63}(c) < \mu_3 \leq \text{rad}^2(c)$ and $\mu_3 > \text{rad}^{1.63}(a)$:

II-1- Case $\text{rad}(c) < \text{rad}(a)$: As $c \leq \text{rad}^3(c) = \text{rad}^{1.63}(c) \cdot \text{rad}^{1.37}(c) \Rightarrow c < \text{rad}^{1.63}(c) \cdot \text{rad}^{1.37}(a) < \text{rad}^{1.63}(ac) < \text{rad}^{1.63}(abc) \Rightarrow c < \text{R}^{1.63}$.

II-2- Case $\text{rad}(a) < \text{rad}(c) < \text{rad}^{1.63}(a)$: As $c \leq \text{rad}^3(c) \leq \text{rad}^{1.63}(c) \cdot \text{rad}^{1.37}(c) \Rightarrow c < \text{rad}^{1.63}(c) \cdot \text{rad}^{1.63}(a) < \text{rad}^{1.63}(abc) \Rightarrow c < \text{R}^{1.63}$.

II-3- Case $\text{rad}^{1.63}(a) < \text{rad}(c)$:

II-3-1- We suppose $\text{rad}^{2.63}(a) < a \leq \text{rad}^{3.26}(a) \Rightarrow a \leq \text{rad}^{1.63}(a) \cdot \text{rad}^{1.63}(a) \Rightarrow a < \text{rad}^{1.63}(a) \cdot \text{rad}^{1.37}(c) \Rightarrow c = a + b < 2a < 2\text{rad}^{1.63}(a). \text{rad}^{1.63}(c) < \text{rad}^{1.63}(abc) \Rightarrow c < \text{R}^{1.63} \Rightarrow c < \text{R}^{1.63}$.

II-3-2- We suppose $a > \text{rad}^{2.63}(a)$ and $\mu_3 \leq \text{rad}^2(c)$. Using the same method as it was explicated in the paragraphs I-3-2, I-3-3 (permuting $a,c$), we arrive at a contradiction. It follows that the case $\mu_3 \leq \text{rad}^2(c)$ and $a > \text{rad}^{2.63}(a)$ is impossible.

Finally, we have finished the study of the case $\text{rad}^{1.63}(c) < \mu_3 \leq \text{rad}^2(c)$ and $\mu_3 > \text{rad}^{1.63}(a)$.

2.3.3. Case $\mu_3 > \text{rad}^{1.63}(c)$ and $\mu_3 > \text{rad}^{1.63}(a)$. Taking into account the cases studied above, it remains to see the following two cases:
- $\mu_3 > \text{rad}^2(c)$ and $\mu_3 > \text{rad}^{1.63}(a)$,
- $\mu_3 > \text{rad}^2(a)$ and $\mu_3 > \text{rad}^{1.63}(c)$.

III-1- We suppose $\mu_3 > \text{rad}^2(c)$ and $\mu_3 > \text{rad}^{1.63}(a) \Rightarrow c > \text{rad}^3(c)$ and $a > \text{rad}^{2.63}(a)$. We can write $c = \text{rad}^3(c) + h$ and $a = \text{rad}^3(a) + l$ with $h$ a positive integer and $l \in \mathbb{Z}.$

III-1-1- We suppose $\text{rad}(c) < \text{rad}(a)$. We obtain the equation:

$$\text{rad}^3(a) - \text{rad}^3(c) = h - l - b = m > 0 \quad (42)$$

Let $X = \text{rad}(a) - \text{rad}(c)$, from the above equation, $X$ is a real root of the polynomial:

$$H(X) = X^3 + 3\text{rad}(ac)X - m = 0 \quad (43)$$

As above, to resolve (43), we denote $X = u + v$. It follows that $u^3, v^3$ are the roots of the polynomial $G(t)$ given by:

$$G(t) = t^2 - mt - \text{rad}^3(ac) = 0 \quad (44)$$

The discriminant of $G(t)$ is:

$$\Delta = m^2 + 4\text{rad}^3(ac) = \alpha^2, \quad \alpha > 0 \quad (45)$$

The two real roots of (44) are:

$$t_1 = u^3 = \frac{m + \alpha}{2}, \quad t_2 = v^3 = \frac{m - \alpha}{2} \quad (46)$$
As \( m = \text{rad}^3(a) - \text{rad}^3(c) > 0 \), we obtain that \( \alpha = \text{rad}^3(a) + \text{rad}^3(c) > 0 \), then from the equation (45), it follows that \( \alpha = x, m = y \) is a solution of the Diophantine equation:

\[
x^2 - y^2 = N
\]  
(47)

with \( N = 4\text{rad}^3(ac) > 0 \). From the equations (46), we remark that \( \alpha \) and \( m \) verify the following equations:

\[
x + y = 2u^3 = 2\text{rad}^3(a)
\]  
(48)

\[
x - y = -2v^3 = 2\text{rad}^3(c)
\]  
(49)

then 

\[
x^2 - y^2 = N = 4\text{rad}^3(a)\text{rad}^3(c)
\]  
(50)

Let \( Q(N) \) be the number of the solutions of (47) and \( \tau(N) \) is the number of suitable factorization of \( N \), and using the same method as in the paragraph I-3-2-4- (case \( 3|1 - m, \mu' \)), we obtain a contradiction.

III-1-2- We suppose \( \text{rad}(a) < \text{rad}(c) \). We obtain the equation:

\[
\text{rad}^3(c) - \text{rad}^3(a) = b + l - h = m > 0
\]  
(51)

Using the same calculations as in III-1-1-, we find a contradiction. It follows that the case \( \mu_c > \text{rad}^2(c) \) and \( \mu_a > \text{rad}^{1.63}(a) \) is impossible.

III-2- We suppose \( \mu_a > \text{rad}^2(a) \) and \( \mu_c > \text{rad}^{1.63}(c) \implies a > \text{rad}^3(a) \) and \( c > \text{rad}^{2.63}(c) \). We can write \( a = \text{rad}^3(a) + h \) and \( c = \text{rad}^3(c) + l \) with \( h \) a positive integer and \( l \in \mathbb{Z} \).

The calculations are similar to those in case III-1-. We obtain the same results namely the cases of III-2- to be rejected.

It follows that the case \( \mu_c > \text{rad}^{1.63}(c) \) and \( \mu_a > \text{rad}^2(a) \) is impossible. \( \square \)

We can state the following important theorem:

**Theorem 2.2.** Let \( a, b, c \) positive integers relatively prime with \( c = a + b \), then \( c < \text{rad}^{1.63}(abc) \).

3. The Proof of the abc conjecture

We note \( R = \text{rad}(abc) \) in the case \( c = a + b \) or \( R = \text{rad}(ac) \) in the case \( c = a + 1 \). We recall the following proposition [4]:

**Proposition 3.1.** Let \( \varepsilon \longrightarrow K(\varepsilon) \) the application verifying the abc conjecture, then:

\[
\lim_{\varepsilon \rightarrow 0} K(\varepsilon) = +\infty
\]  
(52)

3.1. Case: \( \varepsilon \geq 0.63 \). As \( c < R^{1.63} \) is true, we have \( \forall \varepsilon \geq 0.63: \)

\[
c < R^{1.63} \leq R^{1+\varepsilon} < K(\varepsilon)R^{1+\varepsilon}, \quad \text{with} \quad K(\varepsilon) = e^{0.63^2}, \quad \varepsilon \geq 0.63
\]  
(53)

Then the abc conjecture is true.

3.2. Case: \( \varepsilon < 0.63 \).
3.2.1. Case: $c < R$. In this case, we can write:

$$c < R < R^{1+\varepsilon} < K(\varepsilon).R^{1+\varepsilon}, \quad \text{with } K(\varepsilon) = e^{0.63^{27}} > 1, \; \varepsilon < 0.63 \quad (54)$$

Then the $abc$ conjecture is true.

3.2.2. Case: $c > R$. From the statement of the $abc$ conjecture 1.1, we want to give a proof that $c < K(\varepsilon)R^{1+\varepsilon} \iff \log c < \log K(\varepsilon) + (1 + \varepsilon)\log R \iff \log K(\varepsilon) + (1 + \varepsilon)\log R - \log c > 0$. For our proof, we proceed by contradiction of the $abc$ conjecture, so we assume that the conjecture is false:

$$\exists \varepsilon_0 \in ]0, 0.63[, \forall K(\varepsilon) > 0, \quad \exists c_0 = a_0 + b_0 \quad \text{so that } c_0 > K(\varepsilon_0)R_0^{1+\varepsilon_0} \implies c_0 \text{ not a prime } \quad (55)$$

We choose the constant $K(\varepsilon) = e^{0.63^{27}}$. Let $Y_{c_0}(\varepsilon) = \frac{1}{\varepsilon^2} + (1 + \varepsilon)\log R_0 - \log c_0, \varepsilon \in ]0, 0.63[$. From the above explications, if we will obtain $\forall \varepsilon \in ]0, 0.63[, Y_{c_0}(\varepsilon) > 0 \implies Y_{c_0}(\varepsilon_0) > 0$, then the contradiction with (55).

About the function $Y_{c_0}$, we have $\lim_{\varepsilon \to 0} Y_{c_0}(\varepsilon) = 1/0.63^2 + \log(R_0^{1/0.63}/c_0) > 0$ and $\lim_{\varepsilon \to 0} Y_{c_0}(\varepsilon) = +\infty$. The function $Y_{c_0}(\varepsilon)$ has a derivative for $\forall \varepsilon \in ]0, 0.63[$, we obtain with $R_0 > 2977$:

$$Y'_{c_0}(\varepsilon) = -\frac{2}{\varepsilon^3} + \log R_0 = \frac{\varepsilon^3\log R_0 - 2}{\varepsilon^3} \implies Y'_{c_0}(\varepsilon) = 0 \implies \varepsilon = \varepsilon' = \sqrt[3]{\frac{2}{\log R_0}} \in ]0, 0.63[ \quad (56)$$

Discussion:
- If $Y_{c_0}(\varepsilon') \geq 0$, it follows that $\forall \varepsilon \in ]0, 0.63[, Y_{c_0}(\varepsilon) \geq 0$, then the contradiction with $Y_{c_0}(\varepsilon_0) < 0 \implies c_0 > K(\varepsilon_0)R_0^{1+\varepsilon_0}$. Hence the $abc$ conjecture is true for $\varepsilon \in ]0, 0.63[$.

- If $Y_{c_0}(\varepsilon') < 0 \implies \exists 0 < \varepsilon_1 < \varepsilon' < \varepsilon_2 < 0.63$, so that $Y_{c_0}(\varepsilon_1) = Y_{c_0}(\varepsilon_2) = 0$. Then we obtain $c_0 = K(\varepsilon_1)R_0^{1+\varepsilon_1} = K(\varepsilon_2)R_0^{1+\varepsilon_2}$. We recall the following definition:

**Definition 3.1.** The number $\xi$ is called algebraic number if there is at least one polynomial:

$$l(x) = l_0 + l_1x + \cdots + a_mx^m, \quad a_m \neq 0 \quad (57)$$

with integral coefficients such that $l(\xi) = 0$, and it is called transcendental if no such polynomial exists.

We consider the equality $c_0 = K(\varepsilon_1)R_0^{1+\varepsilon_1} \implies c_0 = \frac{\mu_c}{\text{rad}(ab)} = e^{0.63^{27}}R_0^{\varepsilon_1}.$

i) - We suppose that $\varepsilon_1 = \beta_1$ is an algebraic number then $\beta_0 = 1/\varepsilon^2$ and $R_0 = \alpha_1$ are also algebraic numbers. We obtain:

$$\frac{\mu_c}{\text{rad}(ab)} = e^{0.63^{27}}R_0^{\varepsilon_1} = e^{\beta_0}.\alpha_1^{\beta_1} \quad (58)$$

From the theorem (see theorem 3, page 196 in [9]):

**Theorem 3.1.** $e^{\beta_0}\alpha_1^{\beta_1} \cdots \alpha_n^{\beta_n}$ is transcendental for any nonzero algebraic numbers $\alpha_1, \ldots, \alpha_n, \beta_0, \ldots, \beta_n$.

We deduce that the right member $e^{\beta_0}.\alpha_1^{\beta_1}$ of (58) is transcendental, but the term $\frac{\mu_c}{\text{rad}(ab)}$ is an algebraic number, then the contradiction and the $abc$ conjecture is true.

ii) - We suppose that $\varepsilon_1$ is transcendental, in this case there is also a contradiction, and the $abc$ conjecture is true.

Then the proof of the $abc$ conjecture is finished.
4. Conclusion

We have given an elementary proof of the abc conjecture. We can announce the important theorem:

**Theorem 4.1.** For each \( \varepsilon > 0 \), there exists \( K(\varepsilon) > 0 \) such that if \( a, b, c \) positive integers relatively prime with \( c = a + b \), then:

\[
c < K(\varepsilon) \cdot \text{rad}^{1+\varepsilon}(abc)
\]

where \( K \) is a constant depending of \( \varepsilon \).

REFERENCES


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