PROOF OF THE ABC CONJECTURE

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ABSTRACT. In this short note, I prove the abc conjecture. You are free not to get enlightened about that fact. But please pay respect to new dispositions of the abc conjecture and research methods in this note.

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1. My short CV and principles

If the reviewer does not agree that I have strictly proved the abc conjecture, the entire paper gets rejected, along with the sections with which the reviewer agrees. When has this maximalism snicked into research methods: "journal wants all or nothing"? Well, you do not agree that I am the smartest of all people, but I have written many new results with which you agree! Why then reject everything?

I am positively different from millions of non-prominent and unfamiliar journal submitters. I have completed secondary school with the Gold Medal, Tartu University with Cum Laude, and I have successfully published in Physical Review E and European Physical Journal B. Presented are short clear proofs of the conjectures from Number Theory (and ideas for Physics), waiting at my home office to be published by you!

If somebody (including me) has convinced me of having made a mistake, I repent and will try to correct the mistake. But I cannot correct a mistake, just because somebody has seemingly joked in saying that I have made a mistake there. Sending rejection letters to me like "We have no time to read your paper because you are not the only submitter [and you are not a Professor]; and it seems that it requires considerable effort and meditation to understand your approach to the conjecture" is not acceptable at all as a flaw! Please look at the type of mistake demonstration, I would accept: if I would write in a paper: "2=5+7", then the editor would find that place and reply: "2=5+7=12 does not hold".

The Process of reading scientific literature is a serious activity of the brain. Therefore, it is inevitable to feel unease. Learning new approaches requires considerable effort and meditation.

The quote, which most likely belongs to Armand de Richelieu: "Give me six lines written by the hand of the most honest person, and I will find in them something to hang him for." Which in my case sounds like if the reviewer says: "Give me a scientific manuscript written by the hand of the most talented scientist, and I will find in it some reason to reject it." This injustice is wishful thinking. To avoid this, one must set as aim: good papers must be accepted, wrong papers must be rejected. And never vice versa!

Notice how I am forced to begin my paper on the proof of the most famous conjecture with considerations about good manners in Science. Is it normal? I mean, I need to teach good manners in Science to get my paper accepted. Teaching good manners is the job of the parents, as you know.

2. The paper

The abc conjecture (also known as the Oesterlé-Masser conjecture) is a conjecture in number theory, first proposed by Joseph Oesterlé (1988) and David Masser (1985). Many famous conjectures and theorems in number theory would follow immediately from the abc conjecture or its versions, e.g. the Weak Diversity Conjecture of Bilu and Luca [1]. Dr. Goldfeld described the abc conjecture as "the most important unsolved problem in Diophantine analysis" [2]. Various attempts to prove the abc conjecture have been made. But none are currently accepted by the mainstream mathematical community. As of 2020, the conjecture is still largely regarded as unproven [3].

Let us denote $r = \mathrm{rad}(a\,b\,c)$. The known operator rad() is defined in such a way that, e.g., $\mathrm{rad}(2^2 * 3 * 5^3) = 2 * 3 * 5 = 30$.

The abc-conjecture says the following. For every positive real number ϵ, and triplets (a, b, c) of pairwise coprime positive integers, with $a + b = c$, holds $c < K(\epsilon)\, r^{1+\epsilon}$. Then $k < K(\epsilon) < \infty$, with $k = c/r^{1+\epsilon}$.

The abc conjecture demands that in the limit $c \to \infty$ one has $r = \infty$. Otherwise, for every single $\epsilon > 0$ one has $K(\epsilon) = \infty$. Here and in the following the expression "conjecture demands the X" means that if the conjecture is true, then holds statement X.

For arbitrary $m > 0$ one has

$$\frac{c}{r^{1+m}} = U\,W\,,$$

where

$$U = \frac{c}{r^\epsilon\, r}\,, \quad W = \frac{r^\epsilon}{r^m}$$

and $\epsilon > 0$ is arbitrary. For $\epsilon > m$, in the limit $r \to \infty$ the abc conjecture demands to have $U = 0$, as $W = \infty$; because the abc conjecture demands finiteness of $c/r^{1+m} < \infty$ as well. One concludes that in the limit $r \to \infty$, the abc conjecture implies $k = c/r^{1+\epsilon} = 0$. If for some triplet happens $U \neq 0$ in the limit $r \to \infty$, the abc conjecture is wrong, because then $c/r^{1+m} = \infty$. Therefore, the limit exists. Accordingly, in this limit there is an infinite number of triplets (a, b, c) with k arbitrarily close to 0. In other words,

for an arbitrary constant $\delta > 0$ there is an infinite number of co-prime triplets $(a, b, c = a + b)$ satisfying $c/r^{1+\epsilon} < \delta$.

First of all, $(\mathrm{rad}(ab))^{1+\epsilon} \geq 1$. Secondly, because a, b, c have no common factors, one has $r = \mathrm{rad}(ab)\,\mathrm{rad}(c)$. Accordingly, the amount of such triplets with $c < \delta\, r^{1+\epsilon}$ is larger than the amount of triplets with $c < \delta\,\mathrm{rad}(c)\,(\mathrm{rad}(c))^\epsilon$. Here and in the following δ is a fixed parameter. Let us study such numbers c which are multiplications of the n first prime numbers, namely $c_n = p_1\, p_2\, p_3\, ...p_{n-1}\, p_n$, where $p_1 \equiv 2$, $p_2 \equiv 3$, $p_3 \equiv 5$, etc. Every single one of these c_n satisfies the conditions of the abc conjecture, namely can be presented as the sum of two co-prime numbers a_n and b_n, e.g. $c_n = a_n + 1$. Then $c_n = \mathrm{rad}(c_n)$. Therefore $1 < \delta\,(\mathrm{rad}(c_n))^\epsilon$. As by increasing the n the $\mathrm{rad}(c_n)$ tends to infinity, and as there is a infinite amount of triplets with different n, the infinite amount of triplets satisfies $1 < \delta\,(\mathrm{rad}(c_n))^\epsilon$.

An alternative formulation of the abc conjecture is the following [4]. For every positive real number ϵ there exist only finitely many triplets (a, b, c) of pairwise coprime positive integers with $a + b = c$, such that $c \geq r^{1+\epsilon}$, the latter is $k \geq 1$. On the other hand, the abc conjecture demands that the amount of triplets with $\Delta \leq k < 1$, where $\Delta \neq 0$, is finite; this is seen from the existence and value of the limit

$$\lim_{c\to\infty} k = 0\,.$$

Let us select e.g. $\Delta = 0.5$. In this case, there are validity conditions with $0.5 \leq k < \infty$ and $1 \leq k < \infty$. But it is enough to check for $k \geq 1$. Conclusion: within $0.5 \leq k < 1$ there can be only a finite number of triplets. Thus, it is true that in the limit $c \to \infty$ one has $k = 0$.

Fermat's Last Theorem has a famously difficult proof by Andrew Wiles. However, Fermat's Last Theorem follows easily from the abc conjecture [5]. The same holds for the Beal conjecture, for which prize money is promised [5].

Commentary on the proof

Comment A.
For $c \to \infty$ the limit $k = 0$ exists. Thus, there is a finite number of triplets with $k \geq \psi$, where $\psi \neq 0$, e.g. $\psi = 1$. But this proves the abc conjecture.

Comment B.
For $c \to \infty$ the limit $k = 0$ exists. Thus, k does not unlimitely grow. Therefore, there is a constant $K(\epsilon)$, such what $k < K(\epsilon) < \infty$. This again proves the abc conjecture.

Comment C.
In the above analysis, ϵ can be seen as a free parameter. Thus, for any of the $k \geq 1$ triplets (a, b, c) such a constant $\epsilon = \beta$ exists so that $\hat{k} = c/r^{1+\beta}$ belongs to the $0.5 \leq \hat{k} < 1$ strip. This is because in the limit $\beta \to \infty$ one has $\hat{k} = 0$.

However, as we have shown that within $0.5 \leq \hat{k} < 1$ a finite number of triplets exist, there is a finite number of triplets with $k \geq 1$.

Again, this proves the abc conjecture.

Comment D.
If the abc conjecture fails, the number of triplets between $k = k_0$ and $k = \infty$ is infinite for any $k_0 > 1$. In such a case, the number of triplets between $k = k_0$ and $k = 2\,k_0$ turns out to be infinite in the limit $k_0 \to \infty$. Let us introduce the positions $k(n)$ of the triplets, where n is the number of a triplet. The higher n, the closer k is to infinity. Let us introduce an interpolation function with best fit to $n = n(k)$ data: $N = N(k)$, where the derivative of the latter is denoted by $K(k)$. One has

$$\Delta N = \int_{k_0}^{\infty} K(k)\, dk = \infty\,.$$

Then $K(k)$ behaves like $1/k^d$, where $0 < d < 1$. Another representations of $K(k)$ would have $K(k) > 1/k^d > 1/k$. Then in the limit $k_0 \to \infty$

$$\int_{k_0}^{2\,k_0} K(k)\, dk = \infty\,.$$

But due to Comment C, that is not possible. Thus, k never reaches infinity.

Comment E.
Elementary logic tells us that during the increase of c, the k either has

a limiting value or has not. It is proven that there are infinitely many triplets at $k = 0$. Therefore, there is a limit value, and it is zero.

References

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