A relationship between exponential growing and a basic asymptotic function

by

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1.- Abstract

In this paper we study a simple exponential growing problem which leads to a basic asymptotic function. It shows a hidden property of asymptotic functions.

2.- Introduction

Nowadays basic asymptotic functions are largely studied at high-school levels, in order to present the notion of limit of a function.

Some centuries ago, a relationship between exponential functions and a rectangular hyperbola, which has an asymptote, has been found indirectly through the inverse function of an exponential function. It lead to the definition of the natural logarithm function:

$$\ln(x) = \int_{1}^{x} \frac{1}{t} dt$$

This relationship is the result of researches that began during the XVIIth century thanks to the mathematician Grégoire de Saint-Vincent. He was working on the quadrature of the hyperbola $y = \frac{1}{x}$. He was applying Fermat’s method and he noticed that « when the bases form a geometric progression, the rectangles have equal areas ; thus the area is proportional to the logarithm of the horizontal distance », as E. Maor wrote in [1]. There is also interesting information in Hairer-Wanner [2].

In this article, we will neither work with areas nor with logarithms. Instead, we will find a direct relationship between exponential growing and a basic asymptotic function.

3.- Simple problems leading to exponential functions

In simple problems of that kind, there is growing at a certain rate per unit of time. For example, we begin with an amount $A_0$, and, say, this amount grows at the rate of 2 % per month. So the amount we get in time is :

$$A(t) = A_0 \cdot \left(\frac{102}{100}\right)^t$$

where $t$ is the number of months.

1 Independent researcher
In that kind of problem, after every fixed period of time, the amount grows at a fixed rate.

Let’s call \( p \) that fixed period of time.

4.- What happens if the period of time \( p \) decreases as the rate remains unchanged?

One simple problem of this kind could be the following one:

We begin with an amount \( A_0 \) and, after a month, it doubles, and after 15 days it doubles again, and after 7.5 days it doubles again, and so on…

So we get the following graph:
5.- Finding a function which describes that kind of growing

In order to solve this problem, we will write a parametric equation, where $n$ is the number of decreasing periods of time and $t$ is the number of months.

For $n=0$ we have:

\[
\begin{align*}
& t = 0 \\
& A = A_0
\end{align*}
\]

And for $n \geq 1, n \in \mathbb{N}$:

\[
\begin{align*}
& t = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \ldots + \left(\frac{1}{2}\right)^{n-1} \\
& A = A_0 \cdot 2^n
\end{align*}
\]

We can check that:

If $n=0$ we have $t=0$ and $A=A_0$

If $n=1$ we have $t=1$ and $A=A_0 \cdot 2$

If $n=2$ we have $t=1,5$ and $A=A_0 \cdot 2^2$

If $n=3$ we have $t=1,75$ and $A=A_0 \cdot 2^3$

And so on…

We can rewrite the parametric equation:

For $n \geq 1, n \in \mathbb{N}$:

\[
\begin{align*}
& t=(\frac{1}{2})^0 + (\frac{1}{2})^1 + (\frac{1}{2})^2 + (\frac{1}{2})^3 + \ldots + (\frac{1}{2})^{n-1} \\
& A = A_0 \cdot 2^n
\end{align*}
\]

Or:

\[
\begin{align*}
& t = \sum_{k=0}^{n-1} \left(\frac{1}{2}\right)^k \\
& A = A_0 \cdot 2^n
\end{align*}
\]
And we remark that $t$ is given by a geometric series whose common ratio is $\frac{1}{2}$. So we can use the formula for the sum of the first $n$ terms as we find it in [3]:

$$
\begin{align*}
\frac{1 - (\frac{1}{2})^n}{1 - (\frac{1}{2})} &= t \\
A &= A_0 \cdot 2^n
\end{align*}
$$

So we get:

$$
\begin{align*}
t &= 2 - \frac{2}{2^n} \\
A &= A_0 \cdot 2^n
\end{align*}
$$

And now, if we look for a relation between the variables $t$ and $A$, we must eliminate the variable $n$:

$$
\begin{align*}
2 - t &= \frac{2}{2^n} \\
A &= A_0 \cdot 2^n
\end{align*}
$$

$$
\begin{align*}
\frac{2 - t}{2} &= \frac{1}{2^n} \\
A &= A_0 \cdot 2^n
\end{align*}
$$

$$
\begin{align*}
\frac{2}{2 - t} &= 2^n \\
A &= A_0 \cdot 2^n
\end{align*}
$$

So finally we get:

$$
A = \frac{2 \cdot A_0}{2 - t}
$$

which means that $A$ is given by a basic asymptotic function depending on the variable $t$…

Here is the graph of the function:
6.- Conclusions

So we got directly a relationship between exponential growing and an hyperbola. This fact will contribute to increase our knowledge about exponential functions and asymptotic functions.

References