Remarks on the circle arising from Laurent expansion

HIROSHI OKUMURA
Takahanaadai Maebashi Gunma 371-0123, Japan
e-mail: hokmr@yandex.com

Abstract. We consider the notable circle for the arbelos arising from Laurent expansion appeared in [1], [2, 3] in detail.

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In [1], [2, 3], we have considered a notable circle, which is denoted by $\delta$ in Figure 1, arising from Laurent expansion under the definition of division by zero calculus [4]. In this remark we consider the properties of the circle in detail.

For a point $O$ on the segment $AB$, we consider an arbelos configuration consisting of three circles $\alpha$, $\beta$ and $\gamma$ of diameters $AO$, $BO$ and $AB$, respectively, where $|AO| = 2a$ and $|BO| = 2b$. The radical axis of $\alpha$ and $\beta$ is called the axis. We use a rectangular coordinate system with origin $O$ such that the farthest point on $\alpha$ from $AB$ has coordinates $(a, a)$. The point of coordinates $(k\sqrt{ab}, 0)$ is denoted by $I_k$, where $I_0 = O$. The external common tangents of $\alpha$ and $\beta$ meet in
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the point of coordinates \((2ab/(b - a), 0)\), which is denoted by \(E\). The axis meets
the two common tangents in the points \(I_{\pm 1}\) and the circle \(\gamma\) in the points \(I_{\pm 2}\).

Let the line \(EI_1\) touch \(\alpha\) and \(\beta\) at \(P\) and \(Q\), respectively, and let the lines
\(PI_{-1}\) and \(QI_{-1}\) meet \(\alpha\) and \(\beta\) again in \(S\) and \(R\), respectively. The four points
have coordinates

\[
P : \left(2r_A, 2r_A \sqrt{\frac{a}{b}}\right), \quad Q : \left(-2r_A, 2r_A \sqrt{\frac{b}{a}}\right),
\]

\[
R : \left(-\frac{2ab}{a + 9b}, \frac{-6b\sqrt{ab}}{a + 9b}\right), \quad S : \left(\frac{2ab}{9a + b}, \frac{-6\sqrt{ab}}{9a + b}\right),
\]

where \(r_A = ab/(a + b)\), and lie on the circle of center \(D\) and radius given by

\[
D : \left(\frac{a - b}{4}, \frac{\sqrt{ab}}{2}\right), \quad \frac{\sqrt{a^2 + 18ab + b^2}}{4}.
\]

It seems that this circle has never been considered in the long history of studying
the arbelos. However the circle has recently been discovered by using Laurent
expansion under the definition of division by zero calculus \([1], [2, 3], [4]\). We
denote the circle by \(\delta\).

The circle \(\delta\) makes the same angle with the circles \(\alpha\) and \(\beta\). The line \(DI_1\) is
the perpendicular bisector of \(PQ\), and \(DI_1\) and the two tangents of \(\delta\) at \(P\) and \(Q\)
meet in a point on the circle of diameter \(I_1I_3\), whose coordinates equal

\[
\left(\frac{4ab(b - a)}{(a + b)^2}, \frac{\sqrt{ab}(a^2 + 10ab + b^2)}{(a + b)^2}\right).
\]

The line \(DI_1\) passes through the midpoint of the segment joining \(O\) and the center
of \(\gamma\), and this point and \(O\) and \(I_1\) lie on the circle of radius \((a + b)/4\) and center
\(D\). The line \(DO\) is perpendicular to the line \(EI_{-1}\).

The line \(EI_2\) has an equation \(3(a - b)x - 2\sqrt{ab}y + 6ab = 0\) and is the radical
axis of the circles \(\gamma\) and \(\delta\). The perpendicular from \(D\) to this line passes through
the center of \(\gamma\). Let \(\varepsilon\) be the circle of center \(E\) passing through \(O\). It is orthogonal
to any circle touching \(\alpha\) and \(\beta\) at points different from \(O\), and is also orthogonal
to \(\delta\) and the circle of diameter \(OI_2\). The circle of diameter \(OI_2\) and \(\gamma\) and the line
\(EI_2\) meet in the point of coordinates

\[
\left(\frac{2ab(b - a)}{a^2 - ab + b^2}, \frac{2ab\sqrt{ab}}{a^2 - ab + b^2}\right).
\]

REFERENCES

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