Proof of Riemann hypothesis

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Abstract

This paper is a trial to prove Riemann hypothesis according to the following process.

1   We create the infinite number of infinite series from one equation that gives $\zeta(s)$ analytic continuation to $\mathrm{Re}(s)>0$ and 2 formulas (1/2+a+bi, 1/2-a-bi) which show zero points of $\zeta(s)$.

(Proof 1)

2   We find that the value of F(a) (which is infinite series of a) must be zero from the above infinite number of infinite series.

3   We find that F(a)=0 has only one solution of a=0. Therefore zero point of $\zeta(s)$ must be 1/2±bi.

(Proof 2)

4   We find a=0 from the above infinite number of infinite series directly.

1 Introduction

The following (1) gives Riemann zeta function $\zeta(s)$ analytic continuation to $\mathrm{Re}(s)>0$. "+ -----" means infinite series in all equations in this paper.

$$1-2^{-s}+3^{-s}-4^{-s}+5^{-s}-6^{-s}+ ----- = (1-2^{1-s})\,\zeta(s) \tag{1}$$

The following (2) shows non-trivial zero point of $\zeta(s)$. S_0 is the zero points of the left side of (1) and also zero points of $\zeta(s)$.

$$S_0 = 1/2+a+bi \tag{2}$$

The range of a is $0 \leqq a < 1/2$ by the critical strip of $\zeta(s)$. The range of b is $b>0$ due to the following reasons. And i is $\sqrt{-1}$.

1.1 There is no zero point on the real axis of the critical strip.

1.2 [Conjugate complex number of S_0] = 1/2+a-bi is also zero point of $\zeta(s)$. Therefore $b>0$ is necessary and sufficient range for investigation.

The following (3) also shows zero points of $\zeta(s)$ by the functional equation of $\zeta(s)$.

$$S_1 = 1-S_0 = 1/2-a-bi \tag{3}$$

We have the following (4) and (5) by substituting S_0 for s in the left side of (1) and putting both the real part and the imaginary part of the left side of (1) at zero respectively.

$$1 = \frac{\cos(b\log2)}{2^{1/2+a}} - \frac{\cos(b\log3)}{3^{1/2+a}} + \frac{\cos(b\log4)}{4^{1/2+a}} - \frac{\cos(b\log5)}{5^{1/2+a}} + \frac{\cos(b\log6)}{6^{1/2+a}} - \text{-----} \tag{4}$$

$$0 = \frac{\sin(b\log2)}{2^{1/2+a}} - \frac{\sin(b\log3)}{3^{1/2+a}} + \frac{\sin(b\log4)}{4^{1/2+a}} - \frac{\sin(b\log5)}{5^{1/2+a}} + \frac{\sin(b\log6)}{6^{1/2+a}} - \text{-----} \tag{5}$$

We also have the following (6) and (7) by substituting S_1 for s in the left side of (1) and putting both the real part and the imaginary part of the left side of (1) at zero respectively.

$$1 = \frac{\cos(b\log2)}{2^{1/2-a}} - \frac{\cos(b\log3)}{3^{1/2-a}} + \frac{\cos(b\log4)}{4^{1/2-a}} - \frac{\cos(b\log5)}{5^{1/2-a}} + \frac{\cos(b\log6)}{6^{1/2-a}} - \text{------} \tag{6}$$

$$0 = \frac{\sin(b\log2)}{2^{1/2-a}} - \frac{\sin(b\log3)}{3^{1/2-a}} + \frac{\sin(b\log4)}{4^{1/2-a}} - \frac{\sin(b\log5)}{5^{1/2-a}} + \frac{\sin(b\log6)}{6^{1/2-a}} - \text{------} \tag{7}$$

2 Infinite number of infinite series

We define $f(n)$ as follows.

$$f(n) = \frac{1}{n^{1/2-a}} - \frac{1}{n^{1/2+a}} \geqq 0 \qquad (n=2,3,4,5,6,\text{------}) \tag{8}$$

We have the following (9) from (4) and (6) with the method shown in item 1 of [Appendix 1: Equation construction].

$$0 = f(2)\cos(b\log2) - f(3)\cos(b\log3) + f(4)\cos(b\log4) - f(5)\cos(b\log5) + \text{-----} \tag{9}$$

We also have the following (10) from (5) and (7) with the method shown in item 2 of [Appendix 1: Equation construction].

$$0 = f(2)\sin(b\log2) - f(3)\sin(b\log3) + f(4)\sin(b\log4) - f(5)\sin(b\log5) + \text{-----} \tag{10}$$

We can have the following (11) (which is the function of real number x) from the above (9) and (10) with the method shown in item 3 of [Appendix 1: Equation construction]. And the value of (11) is always zero at any value of x.

0 ≡ cosx{right side of (9)}+sinx{right side of (10)}

  = cosx{f(2)cos(blog2)−f(3)cos(blog3)+f(4)cos(blog4)−f(5)cos(blog5)+ −−−−−}

    + sinx{f(2)sin(blog2)−f(3)sin(blog3)+f(4)sin(blog4)−f(5)sin(blog5)+ −−−−−}

  = f(2)cos(blog2−x)−f(3)cos(blog3−x)+f(4)cos(blog4−x)−f(5)cos(blog5−x)+ −−−−   (11)

We have (12−1) by substituting blog1 for x in (11).

  0 = f(2)cos(blog2−blog1)−f(3)cos(blog3−blog1)+f(4)cos(blog4−blog1)

    −f(5)cos(blog5−blog1)+f(6)cos(blog6−blog1)+ −−−−−   (12−1)

We have (12−2) by substituting blog2 for x in (11).

  0 = f(2)cos(blog2−blog2)−f(3)cos(blog3−blog2)+f(4)cos(blog4−blog2)

    −f(5)cos(blog5−blog2)+f(6)cos(blog6−blog2)+ −−−−−   (12−2)

We have (12−3) by substituting blog3 for x in (11).

  0 = f(2)cos(blog2−blog3)−f(3)cos(blog3−blog3)+f(4)cos(blog4−blog3)

    −f(5)cos(blog5−blog3)+f(6)cos(blog6−blog3)+ −−−−−   (12−3)

In the same way as above we can have (12−n) by substituting blogn for x in (11). (n = 4, 5, 6, 7, 8, −−−−−)

  0 = f(2)cos(blog2−blogn)−f(3)cos(blog3−blogn)+ f(4)cos(blog4−blogn)

    −f(5)cos(blog5−blogn)+ −−−−−   (12−n)

We define g(k) as follows. (k = 2, 3, 4, 5, 6 −−−−−)

  g(k) = cos(blogk−blog1)+cos(blogk−blog2)+cos(blogk−blog3)+cos(blogk−blog4)+ −−−−

    = cos(blog1−blogk)+cos(blog2−blogk)+cos(blog3−blogk)+cos(blog4−blogk)+ −−−−

  = cos(blog1/k)+cos(blog2/k)+cos(blog3/k)+cos(blog4/k)+cos(blog5/k)+ −−−−   (13)

We can have the following (14) from infinite equations of (12−1),(12−2),(12−3), −−−−−−−−−−,(12−n),−−−−−−−−−−−− with the method shown in item 4 of [Appendix 1: Equation construction].

  0 = f(2){cos(blog2−blog1)+cos(blog2−blog2)+cos(blog2−blog3)+cos(blog2−blog4)+ −−−}

    −f(3){cos(blog3−blog1)+cos(blog3−blog2)+cos(blog3−blog3)+cos(blog3−blog4)+ −−−}

    +f(4){cos(blog4−blog1)+cos(blog4−blog2)+cos(blog4−blog3)+cos(blog4−blog4)+ −−−}

    −f(5){cos(blog5−blog1)+cos(blog5−blog2)+cos(blog5−blog3)+cos(blog5−blog4)+ −−−}

    +f(6){cos(blog6−blog1)+cos(blog6−blog2)+cos(blog6−blog3)+cos(blog6−blog4)+ −−−}

    − −−−−−−−−−−

$$= f(2)g(2) - f(3)g(3) + f(4)g(4) - f(5)g(5) + f(6)g(6) - f(7)g(7) + \text{-------} \tag{14}$$

3 Proof 1

Here we define F(a) as follows.

$$F(a) = f(2) - f(3) + f(4) - f(5) + f(6) - \text{-----} \tag{15}$$

We have F(a)=0 as shown in the following (16) because of the following reasons.

   3.1   g(2) fluctuates between $-\infty$ and $+\infty$ and g(2) does not have the value of zero as shown in [Appendix 2: Proof of g(2)≠0]. Therefore we can divide (14) by g(2) because of g(2)≠0 .

   3.2   g(k)/g(2)=1 (k=3,4,5,6,7 ----) is true as shown in [Appendix 3: Proof of g(k)/g(2)=1].

$$0 = f(2) - \frac{f(3)g(3)}{g(2)} + \frac{f(4)g(4)}{g(2)} - \frac{f(5)g(5)}{g(2)} + \frac{f(6)g(6)}{g(2)} - \frac{f(7)g(7)}{g(2)} + \text{------}$$

$$= f(2) - f(3) + f(4) - f(5) + f(6) - \text{-------} = F(a) \tag{16}$$

   F(a)=0 has the only one solution of a=0 as shown in [Appendix 4: Solution for F(a)=0 (1)] or [Appendix 5: Solution for F(a)=0 (2)]. a has the range of 0≦a< 1/2 by the critical strip of ζ(s). But a cannot have any value but zero because a is the solution for F(a)=0.

$$S_0 = 1/2 + a + bi \tag{2}$$
$$S_1 = 1 - S_0 = 1/2 - a - bi \tag{3}$$

   Due to a=0 non-trivial zero point of Riemann zeta function ζ(s) shown by the above 2 equations must be 1/2±bi and other zero point does not exist. Therefore Riemann hypothesis which says "All non-trivial zero points of Riemann zeta function ζ(s) exist on the line of Re(s)=1/2." is true.

   From (16) F(a)=0 must have solution and F(a) is a monotonically increasing function as shown in [Appendix 5: Solution for F(a)=0 (2)]. So F(a)=0 has the only one solution. If the solution were not a=0, there would not be any zero points on the line of Re(s)=1/2. This assumption is contrary to the following (Fact 1) or (Fact 2). Therefore the only one solution for F(a)=0 must be a=0 and Riemann hypothesis must be true.

   Fact 1: In 1914 G.H.Hardy proved that there are infinite zero points on the line of Re(s)=1/2.

   Fact 2: All zero points found until now exist on the line of Re(s)=1/2.

4 Proof 2

$$g(k) = \frac{\lim_{N\to\infty} N\sin\{b\log N/k + \tan^{-1}(1/b)\}}{\sqrt{1+b^2}} \quad (k=2,3,4,5 \; ----) \tag{31}$$

From (14) and the above (31) in [Appendix 3: Proof of g(k)/g(2)=1] we can have the following (17).

$$0 = f(2)g(2) - f(3)g(3) + f(4)g(4) - f(5)g(5) + f(6)g(6) - f(7)g(7) + -------$$

$$= f(2)\frac{\lim_{N\to\infty} N\sin\{b\log N/2 + \tan^{-1}(1/b)\}}{\sqrt{1+b^2}} - f(3)\frac{\lim_{N\to\infty} N\sin\{b\log N/3 + \tan^{-1}(1/b)\}}{\sqrt{1+b^2}}$$

$$+ f(4)\frac{\lim_{N\to\infty} N\sin\{b\log N/4 + \tan^{-1}(1/b)\}}{\sqrt{1+b^2}} - f(5)\frac{\lim_{N\to\infty} N\sin\{b\log N/5 + \tan^{-1}(1/b)\}}{\sqrt{1+b^2}} + -----$$

$$= \frac{\lim_{N\to\infty} N[f(2)\sin\{b\log N/2 + \tan^{-1}(1/b)\} - f(3)\sin\{b\log N/3 + \tan^{-1}(1/b)\} + f(4)\sin\{b\log N/4 + \tan^{-1}(1/b)\} - -----]}{\sqrt{1+b^2}}$$

$$\tag{17}$$

As shown in [Appendix 6: Sum of infinite series of sine waves] sum of infinite series of sine waves in (17) converges as follows.

$$f(2)\sin\{b\log N/2 + \tan^{-1}(1/b)\} - f(3)\sin\{b\log N/3 + \tan^{-1}(1/b)\} + f(4)\sin\{b\log N/3 + \tan^{-1}(1/b)\} - ---$$
$$= f(2)\sin\{b\log N - b\log 2 + \tan^{-1}(1/b)\} - f(3)\sin\{b\log N - b\log 3 + \tan^{-1}(1/b)\}$$
$$+ f(4)\sin\{b\log N - b\log 4 + \tan^{-1}(1/b)\} - f(5)\sin\{b\log N - b\log 5 + \tan^{-1}(1/b)\} + ---$$
$$= A(a,b)\sin\{b\log N + B(a,b) + \tan^{-1}(1/b)\} \tag{18}$$

In the above (18) A(a,b) and B(a,b) are constant which depends on a and b. If A(a,b)=0 is true, a=0 holds true and if A(a,b)≠0 is true, 0<a<1/2 holds true as shown in [Appendix 6: Sum of infinite series of sine waves]. From (17) and (18) we have the following (19).

$$0 = \lim_{N\to\infty} N[A(a,b)\sin\{b\log N + B(a,b) + \tan^{-1}(1/b)\}] \tag{19}$$

If A(a,b)≠0 is true, the right side of (19) diverges to ±∞. Therefore A(a,b)=0 must be true for (19) to hold. Due to A(a,b)=0 a cannot have any value but zero. And non-trivial zero point of Riemann zeta function $\zeta(s)$ must be 1/2±bi and other zero point does not exist. Riemann hypothesis is true.

Appendix 1: Equation construction

We can construct (9),(10),(11) and (14) by applying the following Theorem 1[1].

Theorem 1:　　On condition that the following (Series 1) and (Series 2) converge,
　　　　　　the following (Series 3) and (Series 4) are true.

(Series 1) = $a_1+a_2+a_3+a_4+a_5+$ ----- = A

(Series 2) = $b_1+b_2+b_3+b_4+b_5+$ ----- = B

(Series 3) = $(a_1+b_1)+(a_2+b_2)+(a_3+b_3)+(a_4+b_4)+(a_5+b_5)+$ ----- = A+B

(Series 4) = $(a_1-b_1)+(a_2-b_2)+(a_3-b_3)+(a_4-b_4)+(a_5-b_5)+$ ----- = A-B

1　Construction of (9)

We can have the following (9) as (Series 4) by regarding (6) and (4) as (Series 1) and (Series 2) respectively.

$$(\text{Series 1}) = \frac{\cos(b\log 2)}{2^{1/2-a}} - \frac{\cos(b\log 3)}{3^{1/2-a}} + \frac{\cos(b\log 4)}{4^{1/2-a}} - \frac{\cos(b\log 5)}{5^{1/2-a}} + \frac{\cos(b\log 6)}{6^{1/2-a}} - \text{-----} = 1 \quad (6)$$

$$(\text{Series 2}) = \frac{\cos(b\log 2)}{2^{1/2+a}} - \frac{\cos(b\log 3)}{3^{1/2+a}} + \frac{\cos(b\log 4)}{4^{1/2+a}} - \frac{\cos(b\log 5)}{5^{1/2+a}} + \frac{\cos(b\log 6)}{6^{1/2+a}} - \text{-----} = 1 \quad (4)$$

$$(\text{Series 4}) = f(2)\cos(b\log 2) - f(3)\cos(b\log 3) + f(4)\cos(b\log 4) - f(5)\cos(b\log 5) + \text{-----}$$
$$= 1-1 = 0 \quad (9)$$

$$\text{Here}\qquad f(n) = \frac{1}{n^{1/2-a}} - \frac{1}{n^{1/2+a}} \geqq 0 \qquad (n=2, 3, 4, 5, 6, \text{------}) \quad (8)$$

2　Construction of (10)

We can have the following (10) as (Series 4) by regarding (7) and (5) as (Series 1) and (Series 2) respectively.

$$(\text{Series 1}) = \frac{\sin(b\log 2)}{2^{1/2-a}} - \frac{\sin(b\log 3)}{3^{1/2-a}} + \frac{\sin(b\log 4)}{4^{1/2-a}} - \frac{\sin(b\log 5)}{5^{1/2-a}} + \frac{\sin(b\log 6)}{6^{1/2-a}} - \text{-----} = 0 \quad (7)$$

$$(\text{Series 2}) = \frac{\sin(b\log 2)}{2^{1/2+a}} - \frac{\sin(b\log 3)}{3^{1/2+a}} + \frac{\sin(b\log 4)}{4^{1/2+a}} - \frac{\sin(b\log 5)}{5^{1/2+a}} + \frac{\sin(b\log 6)}{6^{1/2+a}} - \text{-----} = 0 \quad (5)$$

$$(\text{Series 4}) = f(2)\sin(b\log 2) - f(3)\sin(b\log 3) + f(4)\sin(b\log 4) - f(5)\sin(b\log 5) + \text{-----}$$
$$= 0-0 = 0 \quad (10)$$

3 Construction of (11)

We can have the following (11) as (Series 3) by regarding the following equations as (Series 1) and (Series 2).

(Series 1) = cosx{right side of (9)}
  = cosx{f(2)cos(blog2)-f(3)cos(blog3)+f(4)cos(blog4)-f(5)cos(blog5)+ -----} = 0

(Series 2) = sinx{right side of (10)}
  = sinx{f(2)sin(blog2)-f(3)sin(blog3)+f(4)sin(blog4)-f(5)sin(blog5)+ -----} = 0

(Series 3) = f(2)cos(blog2-x)-f(3)cos(blog3-x)+f(4)cos(blog4-x)-f(5)cos(blog5-x)+
  ---- = 0+0   (11)

4 Construction of (14)

4.1   We can have the following (12-1*2) as (Series 3) by regarding (12-1) and (12-2) as (Series 1) and (Series 2) respectively.

(Series 1) = f(2)cos(blog2-blog1)-f(3)cos(blog3-blog1)+f(4)cos(blog4-blog1)
  -f(5)cos(blog5-blog1)+f(6)cos(blog6-blog1)+ ----- = 0   (12-1)

(Series 2) = f(2)cos(blog2-blog2)-f(3)cos(blog3-blog2)+f(4)cos(blog4-blog2)
  -f(5)cos(blog5-blog2)+f(6)cos(blog6-blog2)+ ----- = 0   (12-2)

(Series 3) = f(2){cos(blog2-blog1)+cos(blog2-blog2)}
  -f(3){cos(blog3-blog1)+cos(blog3-blog2)}
  +f(4){cos(blog4-blog1)+cos(blog4-blog2)}
  -f(5){cos(blog5-blog1)+cos(blog5-blog2)}
  +f(6){cos(blog6-blog1)+cos(blog6-blog2)}- ----- = 0+0   (12-1*2)

4.2   We can have the following (12-1*3) as (Series 3) by regarding (12-1*2) and (12-3) as (Series 1) and (Series 2) respectively.

(Series 2) = f(2)cos(blog2-blog3)-f(3)cos(blog3-blog3)+f(4)cos(blog4-blog3)
  -f(5)cos(blog5-blog3)+f(6)cos(blog6-blog3)+ ----- = 0   (12-3)

(Series 3) = f(2){cos(blog2-blog1)+cos(blog2-blog2)+cos(blog2-blog3)}
  -f(3){cos(blog3-blog1)+cos(blog3-blog2)+cos(blog3-blog3)}
  +f(4){cos(blog4-blog1)+cos(blog4-blog2)+cos(blog4-blog3)}
  -f(5){cos(blog5-blog1)+cos(blog5-blog2)+cos(blog5-blog3)}
  +f(6){cos(blog6-blog1)+cos(blog6-blog2)+cos(blog6-blog3)}
  - ------- = 0+0   (12-1*3)

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4.3 　　We can have the following (12-1*4) as (Series 3) by regarding (12-1*3) and (12-4) as (Series 1) and (Series 2) respectively.

(Series 2) = f(2)cos(blog2-blog4)-f(3)cos(blog3-blog4)+f(4)cos(blog4-blog4)
　　　　　-f(5)cos(blog5-blog4)+f(6)cos(blog6-blog4)+ ----- = 0 　　　　　　(12-4)

(Series 3) = f(2){cos(blog2-blog1)+cos(blog2-blog2)+cos(blog2- blog3)+cos(blog2-blog4)}
　　　-f(3){cos(blog3-blog1)+cos(blog3-blog2)+cos(blog3-blog3)+cos(blog3-blog4)}
　　　+f(4){cos(blog4-blog1)+cos(blog4-blog2)+cos(blog4-blog3)+cos(blog4-blog4)}
　　　-f(5){cos(blog5-blog1)+cos(blog5-blog2)+cos(blog5-blog3)+cos(blog5-blog4)}
　　　+f(6){cos(blog6-blog1)+cos(blog6-blog2)+cos(blog6-blog3)+cos(blog6-blog4)}
　　　- ------ = 0+0 　　　　　　　　　　　　　　　　　　　　　　　　　　(12-1*4)

4.4 　　In the same way as above we can have (12-1*n) as (Series 3) by regarding (12-1*n-1) and (12-n) as (Series 1) and (Series 2) respectively. If we repeat this operation infinitely i.e. we do n→∞, we can have (12-1*∞)=(14).

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Appendix 2: Proof of g(2)≠0

1 Proof (1)
1.1 Investigation of g(2)

We define g(2,N) as the partial sum from the first term of g(2) to the N-th term of g(2). (N=1,2,3,4,5,------)　　From (13) g(2,N) is as follows.

$$g(2,N) = \cos(b\log1/2) + \cos(b\log2/2) + \cos(b\log3/2) + \cos(b\log4/2) + \cos(b\log5/2)$$
$$+ ---- + \cos(b\log N/2)$$

$$= N\left(\frac{1}{N}\right)\left[\cos\left\{b\log\left(\frac{1}{N}\right)\left(\frac{N}{2}\right)\right\} + \cos\left\{b\log\left(\frac{2}{N}\right)\left(\frac{N}{2}\right)\right\} + \cos\left\{b\log\left(\frac{3}{N}\right)\left(\frac{N}{2}\right)\right\} + \cos\left\{b\log\left(\frac{4}{N}\right)\left(\frac{N}{2}\right)\right\}\right.$$

$$\left. + \cos\left\{b\log\left(\frac{5}{N}\right)\left(\frac{N}{2}\right)\right\} + ----- + \cos\left\{b\log\left(\frac{N}{N}\right)\left(\frac{N}{2}\right)\right\}\right]$$

$$= N(1/N)\{\cos(b\log1/N+b\log N/2) + \cos(b\log2/N+b\log N/2) + \cos(b\log3/N+b\log N/2)$$
$$+ \cos(b\log4/N+b\log N/2) + \cos(b\log5/N+b\log N/2) + ---- + \cos(b\log N/N+b\log N/2)\}$$

$$= N(1/N)\{\cos(b\log N/2)\}\{\cos(b\log1/N) + \cos(b\log2/N) + \cos(b\log3/N) + ---- + \cos(b\log N/N)\}$$
$$- N(1/N)\{\sin(b\log N/2)\}\{\sin(b\log1/N) + \sin(b\log2/N) + \sin(b\log3/N) + ---- + \sin(b\log N/N)\}$$

Here we do　N→∞　as follows. $\lim\limits_{N\to\infty} g(2,N)$ means g(2).

$$\lim_{N\to\infty} g(2,N) = g(2)$$

$$= \lim_{N\to\infty}\{N\cos(b\log N/2)\}\lim_{N\to\infty}(1/N)\{\cos(b\log1/N) + \cos(b\log2/N) + \cos(b\log3/N) + ---- + \cos(b\log N/N)\}$$
$$- \lim_{N\to\infty}\{N\sin(b\log N/2)\}\lim_{N\to\infty}(1/N)\{\sin(b\log1/N) + \sin(b\log2/N) + \sin(b\log3/N) + ---- + \sin(b\log N/N)\}$$

$$= \lim_{N\to\infty}\{N\cos(b\log N/2)\}\int_0^1 \cos(b\log x)\,dx - \lim_{N\to\infty}\{N\sin(b\log N/2)\}\int_0^1 \sin(b\log x)\,dx \qquad (21)$$

We define A and B as follows.

$$A = \int_0^1 \cos(b\log x)\,dx \qquad B = \int_0^1 \sin(b\log x)\,dx$$

We calculate A and B.

$$A = [x\cos(b\log x)]_0^1 + bB = 1 + bB$$

$$B = [x\sin(b\log x)]_0^1 - bA = -bA$$

Then we can have the values of A and B from the above equations as follows.

$$A = 1/(1+b^2) \qquad B = -b/(1+b^2)$$

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We have the following (22) by substituting the above values of A and B for $\int_0^1 \cos(b\log x)dx$ and $\int_0^1 \sin(b\log x)dx$ in (21).

$$g(2) = \lim_{N\to\infty}\{N\cos(b\log N/2)\}\{1/(1+b^2)\} - \lim_{N\to\infty}\{N\sin(b\log N/2)\}\{-b/(1+b^2)\}$$

$$= \frac{\lim_{N\to\infty}N\{\cos(b\log N/2)+b\sin(b\log N/2)\}}{1+b^2} = \frac{\lim_{N\to\infty}N\sin\{b\log N/2+\tan^{-1}(1/b)\}}{\sqrt{1+b^2}} \qquad (22)$$

(Graph 1) shows the value of $[N\sin\{b\log N/2+\tan^{-1}(1/b)\}/\sqrt{1+b^2}$ at b=1]. The scale of horizontal axis is $\log_{10}N$ and the scale of vertical axis is $\pm\log_{10}|N\sin(\log N/2+\pi/4)/\sqrt{2}|$. $\pm$ is subject to the sign of $\sin(\log N/2+\pi/4)$.

Graph 1 : $N\sin(\log N/2+\pi/4)/\sqrt{2}$

$\pm\log_{10}|N\sin(\log N/2+\pi/4)/\sqrt{2}|$

$\log_{10}N$
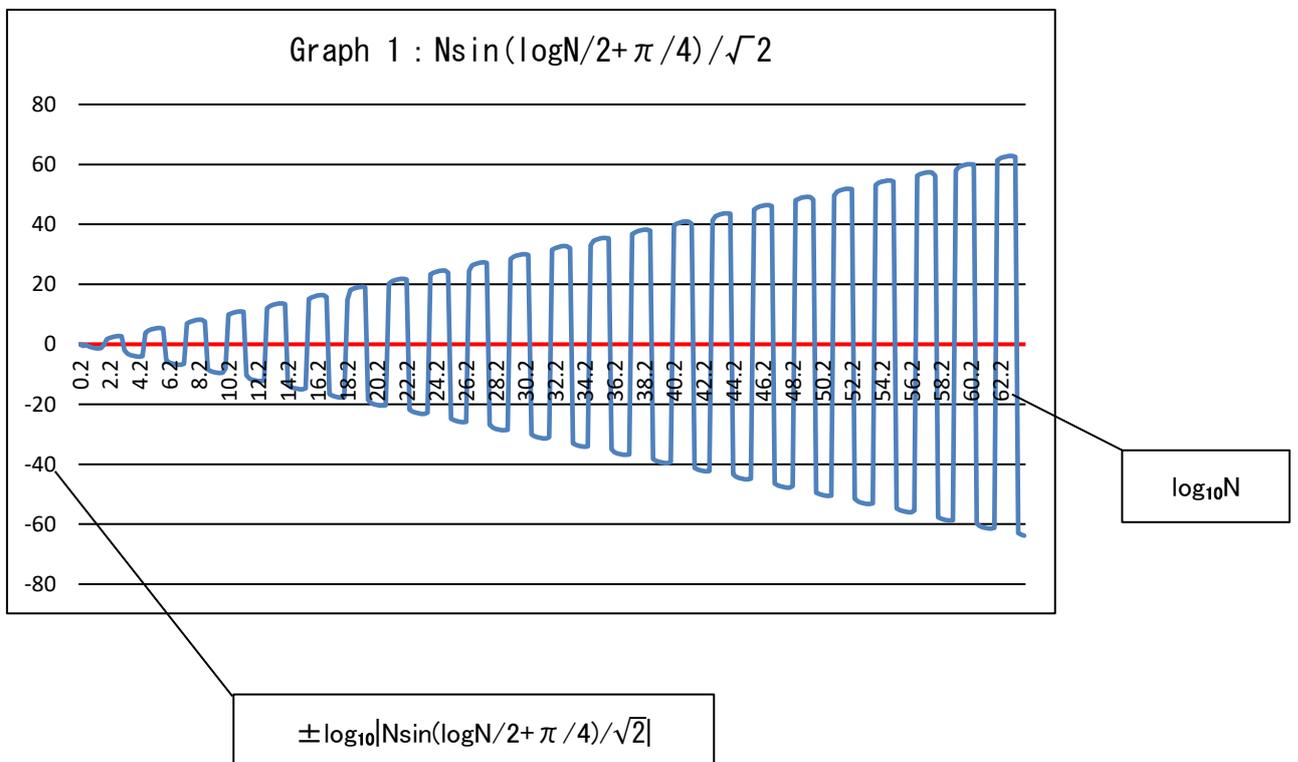
1.2 Verification of $\sin\{b\log N/2+\tan^{-1}(1/b)\}\neq 0$

If we assume $\sin\{b\log N/2+\tan^{-1}(1/b)\}=0$ (N=3,4,5,6,7,————),the following (23) is supposed to be true.

$$b\log N/2+\tan^{-1}(1/b) = k\pi \qquad (k=1,2,3,4,————) \qquad (23)$$

In (23) k is natural number because of 0<{left side of (23)} that is due to 0<b, 0<logN/2 and 0<$\tan^{-1}(1/b)$<$\pi/2$ as shown in item 1.2.1.

1.2.1 $\tan^{-1}(1/b)$ has the value of $L\pi$ as shown in (Table 1) and the range of L is $0<L<1/2$.

Table 1 : Value of $\tan^{-1}(1/b)$

b	0	$1/\sqrt{3}$	1	$\sqrt{3}$	∞
$\tan^{-1}(1/b)$	$\pi/2$	$\pi/3$	$\pi/4$	$\pi/6$	0

1.2.2   From (23)

$$b\log N/2 + L\pi = k\pi$$

$$\log N/2 = \frac{(k-L)\pi}{b} = M\pi$$

$k-L>1/2$ is true due to $1\leqq k$ and $0<L<1/2$. $(k-L)\pi/b=M>0$ is true due to $0<b$ and $k-L>1/2$. From the above equation

$$N/2 = e^{M\pi}$$
$$N = 2e^{M\pi} \tag{24}$$

1.2.3   N is natural number. (24) has impossible formation like (natural number) = (irrational number). Therefore (24) is false and (23) (which is the original formula of (24)) is also false. Now we can have the following (25).

$$\sin\{b\log N/2+\tan^{-1}(1/b)\} \neq 0 \quad (N=3,4,5,6,7,-----) \tag{25}$$

1.3 Verification of $g(2)\neq 0$

$$g(2) = \frac{\lim_{N\to\infty} N\sin\{b\log N/2+\tan^{-1}(1/b)\}}{\sqrt{1+b^2}} \neq 0$$

The above inequality is true due to the following reasons.

1.3.1 $\lim_{N\to\infty} \sin\{b\log N/2+\tan^{-1}(1/b)\}$ fluctuates between -1 and 1 during $N\to\infty$.

So $\lim_{N\to\infty} N\sin\{b\log N/2+\tan^{-1}(1/b)\}$ fluctuates between $-\infty$ and $+\infty$ as shown in (Graph 1) in the previous page. Therefore $g(2)$ does not converge to zero.

1.3.2   The value of $g(2)$ cannot be zero during $N\to\infty$ due to the above (25) in item 1.2.

2 Proof (2)

If we assume g(2)=0, the following (26) is supposed to be true from (22).

$$g(2) = \lim_{N \to \infty} \{N\cos(b\log N/2)\}\{1/(1+b^2)\} - \lim_{N \to \infty} \{N\sin(b\log N/2)\}\{-b/(1+b^2)\} = 0 \qquad (26)$$

The following (27) and (28) are true because of the following reasons.

2.1 $\lim_{N \to \infty} \{N\cos(b\log N/2)\}$ and $\lim_{N \to \infty} \{N\sin(b\log N/2)\}$ fluctuate between $-\infty$ and $+\infty$ and does not converge to zero.

2.2 In (N=3,4,5,6,7,-----) we can confirm $\sin(b\log N/2) \neq 0$ by putting L=0 in item 1.2. Hence $\lim_{N \to \infty} \{N\sin(b\log N/2)\}$ cannot be zero during N→∞. In (N=3,4,5,6,7,-----) we can confirm $\cos(b\log N/2) = \sin(b\log N/2+\pi/2) \neq 0$ by putting L=1/2 in item 1.2. Hence $\lim_{N \to \infty} \{N\cos(b\log N/2)\}$ cannot be zero during N→∞.

$$(N=3,4,5,6,7,-----) \qquad \lim_{N \to \infty} \{N\cos(b\log N/2)\}\{1/(1+b^2)\} \neq 0 \qquad (27)$$

$$\lim_{N \to \infty} \{N\sin(b\log N/2)\}\{-b/(1+b^2)\} \neq 0 \qquad (28)$$

From (26), (27) and (28) we have the following (29).

$$\frac{\lim_{N \to \infty} \{N\sin(b\log N/2)\}\{-b/(1+b^2)\}}{\lim_{N \to \infty} \{N\cos(b\log N/2)\}\{1/(1+b^2)\}} = 1 \qquad (29)$$

From (29) we have the following (30).

$$\frac{\lim_{N \to \infty} \{N\sin(b\log N/2)\}}{\lim_{N \to \infty} \{N\cos(b\log N/2)\}} = \frac{\lim_{N \to \infty} \{\sin(b\log N/2)\}}{\lim_{N \to \infty} \{\cos(b\log N/2)\}} = \lim_{N \to \infty} \tan(b\log N/2) = \frac{-1}{b} \qquad (30)$$

But tangent function fluctuates between $-\infty$ and $+\infty$ during N→∞ and does not converge to the fixed value. So (30) is false and (26) (which is the original formula of (30)) is also false. Therefore we can confirm g(2)≠0.

Appendix 3: Proof of g(k)/g(2)=1

1. Proof (1)

We can have the following (31) for g(k) by calculating in the same way as for g(2) in item 1.1 of Appendix 2.

$$g(k) = \frac{\lim\limits_{N\to\infty} N\sin\{b\log N/k + \tan^{-1}(1/b)\}}{\sqrt{1+b^2}} \quad (k=3, 4, 5, 6, 7 \text{ ----}) \quad (31)$$

We define $h(2,N)$ and $h(k,N)$ as follows.

$$h(2,N) = b\log N/2 + \tan^{-1}(1/b)$$
$$h(k,N) = b\log N/k + \tan^{-1}(1/b)$$

We have the following equation from the above definition.

$$\lim_{N\to\infty}\frac{h(k,N)}{h(2,N)} = \lim_{N\to\infty}\frac{b\log N/k + \tan^{-1}(1/b)}{b\log N/2 + \tan^{-1}(1/b)} = \lim_{N\to\infty}\frac{1-\log k/\log N + \tan^{-1}(1/b)/b\log N}{1-\log 2/\log N + \tan^{-1}(1/b)/b\log N} = 1 \quad (32)$$

We have the following (33) from the above (32).

$$\frac{\lim\limits_{n\to\infty}\lim\limits_{N\to\infty}h(k,N)^{2n-1}}{\lim\limits_{n\to\infty}\lim\limits_{N\to\infty}h(2,N)^{2n-1}} = \lim_{n\to\infty}\lim_{N\to\infty}\frac{h(k,N)^{2n-1}}{h(2,N)^{2n-1}} = \lim_{n\to\infty}\lim_{N\to\infty}\left\{\frac{h(k,N)}{h(2,N)}\right\}^{2n-1} = \lim_{n\to\infty}\{1\}^{2n-1} = \lim_{n\to\infty}1 = 1 \quad (33)$$

We have the following (34) by performing Mclaughlin expansion for $\sin\{h(2,N)\}$. A is natural number.

$$\lim_{n\to\infty}\lim_{N\to A}\left\{h(2,N) - \frac{h(2,N)^3}{3!} + \frac{h(2,N)^5}{5!} - \frac{h(2,N)^7}{7!} + \text{----} + \frac{(-1)^{n-2}h(2,N)^{2n-3}}{(2n-3)!} + \frac{(-1)^{n-1}h(2,N)^{2n-1}}{(2n-1)!}\right\}$$

$$= \lim_{n\to\infty}\left\{h(2,A) - \frac{h(2,A)^3}{3!} + \frac{h(2,A)^5}{5!} - \frac{h(2,A)^7}{7!} + \text{----} + \frac{(-1)^{n-2}h(2,A)^{2n-3}}{(2n-3)!} + \frac{(-1)^{n-1}h(2,A)^{2n-1}}{(2n-1)!}\right\}$$

$$= \sin\{h(2,A)\} = \lim_{N\to A}\sin\{h(2,N)\} \quad (34)$$

(34) is the identity regarding A. In other words (34) holds at any value of A, even though through A→∞. From (34) we can have the following (35).

$$\lim_{n\to\infty}\lim_{N\to\infty}\left\{h(2,N) - \frac{h(2,N)^3}{3!} + \frac{h(2,N)^5}{5!} - \frac{h(2,N)^7}{7!} + \text{----} + \frac{(-1)^{n-2}h(2,N)^{2n-3}}{(2n-3)!} + \frac{(-1)^{n-1}h(2,N)^{2n-1}}{(2n-1)!}\right\}$$

$$= \lim_{A\to\infty}\left[\lim_{n\to\infty}\lim_{N\to A}\left\{h(2,N) - \frac{h(2,N)^3}{3!} + \frac{h(2,N)^5}{5!} - \frac{h(2,N)^7}{7!} + \text{----} + \frac{(-1)^{n-2}h(2,N)^{2n-3}}{(2n-3)!} + \frac{(-1)^{n-1}h(2,N)^{2n-1}}{(2n-1)!}\right\}\right]$$

$$= \lim_{A\to\infty}\lim_{N\to A}\sin\{h(2,N)\} = \lim_{N\to\infty}\sin\{h(2,N)\} \tag{35}$$

Similarly we can have the following (36).

$$\lim_{n\to\infty}\lim_{N\to\infty}\{h(k,N)-\frac{h(k,N)^3}{3!}+\frac{h(k,N)^5}{5!}-\frac{h(k,N)^7}{7!}+----+\frac{(-1)^{n-2}h(k,N)^{2n-3}}{(2n-3)!}+\frac{(-1)^{n-1}h(k,N)^{2n-1}}{(2n-1)!}\}$$

$$= \lim_{N\to\infty}\sin\{h(k,N)\} \tag{36}$$

From (22),(31),(33),(35) and (36) we can have $g(k)/g(2)=1$ as follows.

$$\frac{g(k)}{g(2)} = \frac{\lim\limits_{N\to\infty}N\sin\{b\log N/k+\tan^{-1}(1/b)\}}{\lim\limits_{N\to\infty}N\sin\{b\log N/2+\tan^{-1}(1/b)\}} = \frac{\lim\limits_{N\to\infty}\sin\{b\log N/k+\tan^{-1}(1/b)\}}{\lim\limits_{N\to\infty}\sin\{b\log N/2+\tan^{-1}(1/b)\}} = \frac{\lim\limits_{N\to\infty}\sin\{h(k,N)\}}{\lim\limits_{N\to\infty}\sin\{h(2,N)\}}$$

$$= \frac{\lim\limits_{n\to\infty}\lim\limits_{N\to\infty}\{h(k,N)-\frac{h(k,N)^3}{3!}+\frac{h(k,N)^5}{5!}-\frac{h(k,N)^7}{7!}+----+\frac{(-1)^{n-2}h(k,N)^{2n-3}}{(2n-3)!}+\frac{(-1)^{n-1}h(k,N)^{2n-1}}{(2n-1)!}\}}{\lim\limits_{n\to\infty}\lim\limits_{N\to\infty}\{h(2,N)-\frac{h(2,N)^3}{3!}+\frac{h(2,N)^5}{5!}-\frac{h(2,N)^7}{7!}+----+\frac{(-1)^{n-2}h(2,N)^{2n-3}}{(2n-3)!}+\frac{(-1)^{n-1}h(2,N)^{2n-1}}{(2n-1)!}\}}$$

$$= \frac{\lim\limits_{n\to\infty}\lim\limits_{N\to\infty}\{h(k,N)^{2n-1}\}\{h(k,N)^{2-2n}-\frac{h(k,N)^{4-2n}}{3!}+\frac{h(k,N)^{6-2n}}{5!}-----+\frac{(-1)^{n-2}h(k,N)^{-2}}{(2n-3)!}+\frac{(-1)^{n-1}}{(2n-1)!}\}}{\lim\limits_{n\to\infty}\lim\limits_{N\to\infty}\{h(2,N)^{2n-1}\}\{h(2,N)^{2-2n}-\frac{h(2,N)^{4-2n}}{3!}+\frac{h(2,N)^{6-2n}}{5!}-----+\frac{(-1)^{n-2}h(2,N)^{-2}}{(2n-3)!}+\frac{(-1)^{n-1}}{(2n-1)!}\}}$$

$$= \frac{\lim\limits_{n\to\infty}\lim\limits_{N\to\infty}\{h(k,N)^{2n-1}\}\lim\limits_{n\to\infty}\lim\limits_{N\to\infty}\{h(k,N)^{2-2n}-\frac{h(k,N)^{4-2n}}{3!}+\frac{h(k,N)^{6-2n}}{5!}-----+\frac{(-1)^{n-2}h(k,N)^{-2}}{(2n-3)!}+\frac{(-1)^{n-1}}{(2n-1)!}\}}{\lim\limits_{n\to\infty}\lim\limits_{N\to\infty}\{h(2,N)^{2n-1}\}\lim\limits_{n\to\infty}\lim\limits_{N\to\infty}\{h(2,N)^{2-2n}-\frac{h(2,N)^{4-2n}}{3!}+\frac{h(2,N)^{6-2n}}{5!}-----+\frac{(-1)^{n-2}h(2,N)^{-2}}{(2n-3)!}+\frac{(-1)^{n-1}}{(2n-1)!}\}}$$

$$= \frac{\lim\limits_{n\to\infty}\lim\limits_{N\to\infty}\{h(k,N)^{2-2n}-\frac{h(k,N)^{4-2n}}{3!}+\frac{h(k,N)^{6-2n}}{5!}-\frac{h(k,N)^{8-2n}}{7!}+----+\frac{(-1)^{n-2}h(k,N)^{-2}}{(2n-3)!}+\frac{(-1)^{n-1}}{(2n-1)!}\}}{\lim\limits_{n\to\infty}\lim\limits_{N\to\infty}\{h(2,N)^{2-2n}-\frac{h(2,N)^{4-2n}}{3!}+\frac{h(2,N)^{6-2n}}{5!}-\frac{h(2,N)^{8-2n}}{7!}+----+\frac{(-1)^{n-2}h(2,N)^{-2}}{(2n-3)!}+\frac{(-1)^{n-1}}{(2n-1)!}\}}$$

$$= \frac{\lim\limits_{n\to\infty}\frac{(-1)^{n-1}}{(2n-1)!}}{\lim\limits_{n\to\infty}\frac{(-1)^{n-1}}{(2n-1)!}} = \lim_{n\to\infty}\frac{\frac{(-1)^{n-1}}{(2n-1)!}}{\frac{(-1)^{n-1}}{(2n-1)!}} = \lim_{n\to\infty}1 = 1 \tag{37}$$

The 4th equal sine (=) in (37) is true due to (35) and (36).

The 7th equal sine (=) in (37) is true due to (33).

The 8th equal sine (=) in (37) is true due to $\lim\limits_{N\to\infty}h(2,N)=\infty$ and $\lim\limits_{N\to\infty}h(k,N)=\infty$.

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2. Proof (2)

We have the following (41) from (32).

$$\frac{\lim_{N\to\infty}\lim_{n\to\infty}h(k,N)^{2n-1}}{\lim_{N\to\infty}\lim_{n\to\infty}h(2,N)^{2n-1}} = \lim_{N\to\infty}\lim_{n\to\infty}\frac{h(k,N)^{2n-1}}{h(2,N)^{2n-1}} = \lim_{N\to\infty}\lim_{n\to\infty}\left\{\frac{h(k,N)}{h(2,N)}\right\}^{2n-1} = \lim_{N\to\infty}\left\{\frac{h(k,N)}{h(2,N)}\right\}^{\infty} = 1^{\infty} = 1 \qquad (41)$$

We have the following (42) and (43) for (46) in the next page.

$$\lim_{N\to\infty}\lim_{n\to\infty}\{h(2,N)^{2-2n} - \frac{h(2,N)^{4-2n}}{3!} + \frac{h(2,N)^{6-2n}}{5!} - \frac{h(2,N)^{8-2n}}{7!} - \text{-----} + \frac{(-1)^{n-2}h(2,N)^{-2}}{(2n-3)!} + \frac{(-1)^{n-1}}{(2n-1)!}\}$$

$$= \lim_{N\to\infty}\{h(2,N)^{-\infty} - \frac{h(2,N)^{-\infty}}{3!} + \frac{h(2,N)^{-\infty}}{5!} - \frac{h(2,N)^{-\infty}}{7!} + \text{--------}\}$$

$$= 0 \qquad (42)$$

The 2nd equal sign (=) of (42) is true due to $\lim_{N\to\infty}h(2,N) = \infty$. Here we exchange

$\lim_{N\to\infty}$ with $\lim_{n\to\infty}$ each other in the above (42) as follows.

$$\lim_{n\to\infty}\lim_{N\to\infty}\{h(2,N)^{2-2n} - \frac{h(2,N)^{4-2n}}{3!} + \frac{h(2,N)^{6-2n}}{5!} - \text{------} + \frac{(-1)^{n-4}h(2,N)^{-4}}{(2n-5)!} + \frac{(-1)^{n-2}h(2,N)^{-2}}{(2n-3)!} + \frac{(-1)^{n-1}}{(2n-1)!}\}$$

$$= \lim_{n\to\infty}\frac{(-1)^{n-1}}{(2n-1)!}$$

$$= 0 \qquad (43)$$

The 1st equal sign (=) of (43) is true due to $\lim_{N\to\infty}h(2,N) = \infty$. We can have the

following (44) from (42) and (43).

$$\lim_{N\to\infty}\lim_{n\to\infty}\{h(2,N)^{2-2n} - \frac{h(2,N)^{4-2n}}{3!} + \frac{h(2,N)^{6-2n}}{5!} - \frac{h(2,N)^{8-2n}}{7!} + \text{--------} + \frac{(-1)^{n-2}h(2,N)^{-2}}{(2n-3)!} + \frac{(-1)^{n-1}}{(2n-1)!}\}$$

$$= \lim_{n\to\infty}\frac{(-1)^{n-1}}{(2n-1)!} \qquad (44)$$

By calculating in the same way as (42),(43) and (44) we can have the following (45).

$$\lim_{N\to\infty}\lim_{n\to\infty}\{h(k,N)^{2-2n} - \frac{h(k,N)^{4-2n}}{3!} + \frac{h(k,N)^{6-2n}}{5!} - \frac{h(k,N)^{8-2n}}{7!} + \text{--------} + \frac{(-1)^{n-2}h(k,N)^{-2}}{(2n-3)!} + \frac{(-1)^{n-1}}{(2n-1)!}\}$$

$$= \lim_{n\to\infty}\frac{(-1)^{n-1}}{(2n-1)!} \qquad (45)$$

From (22),(31),(41),(44) and (45) we can have $g(k)/g(2)=1$ as follows by performing Mclaughlin expansion for $\sin\{h(2,N)\}$ and $\sin\{h(k,N)\}$.

$$\frac{g(k)}{g(2)} = \frac{\lim\limits_{N\to\infty} N\sin\{b\log N/k + \tan^{-1}(1/b)\}}{\lim\limits_{N\to\infty} N\sin\{b\log N/2 + \tan^{-1}(1/b)\}} = \frac{\lim\limits_{N\to\infty}\sin\{b\log N/k + \tan^{-1}(1/b)\}}{\lim\limits_{N\to\infty}\sin\{b\log N/2 + \tan^{-1}(1/b)\}} = \frac{\lim\limits_{N\to\infty}\sin\{h(k,N)\}}{\lim\limits_{N\to\infty}\sin\{h(2,N)\}}$$

$$= \frac{\lim\limits_{N\to\infty}\lim\limits_{n\to\infty}\{h(k,N) - \frac{h(k,N)^3}{3!} + \frac{h(k,N)^5}{5!} - \frac{h(k,N)^7}{7!} + ---- + \frac{(-1)^{n-2}h(k,N)^{2n-3}}{(2n-3)!} + \frac{(-1)^{n-1}h(k,N)^{2n-1}}{(2n-1)!}\}}{\lim\limits_{N\to\infty}\lim\limits_{n\to\infty}\{h(2,N) - \frac{h(2,N)^3}{3!} + \frac{h(2,N)^5}{5!} - \frac{h(2,N)^7}{7!} + ---- + \frac{(-1)^{n-2}h(2,N)^{2n-3}}{(2n-3)!} + \frac{(-1)^{n-1}h(2,N)^{2n-1}}{(2n-1)!}\}}$$

$$= \frac{\lim\limits_{N\to\infty}\lim\limits_{n\to\infty}\{h(k,N)^{2n-1}\}\{h(k,N)^{2-2n} - \frac{h(k,N)^{4-2n}}{3!} + \frac{h(k,N)^{6-2n}}{5!} - ---- + \frac{(-1)^{n-2}h(k,N)^{-2}}{(2n-3)!} + \frac{(-1)^{n-1}}{(2n-1)!}\}}{\lim\limits_{N\to\infty}\lim\limits_{n\to\infty}\{h(2,N)^{2n-1}\}\{h(2,N)^{2-2n} - \frac{h(2,N)^{4-2n}}{3!} + \frac{h(2,N)^{6-2n}}{5!} - ---- + \frac{(-1)^{n-2}h(2,N)^{-2}}{(2n-3)!} + \frac{(-1)^{n-1}}{(2n-1)!}\}}$$

$$= \frac{\lim\limits_{N\to\infty}\lim\limits_{n\to\infty}\{h(k,N)^{2n-1}\}\lim\limits_{N\to\infty}\lim\limits_{n\to\infty}\{h(k,N)^{2-2n} - \frac{h(k,N)^{4-2n}}{3!} + \frac{h(k,N)^{6-2n}}{5!} - ---- + \frac{(-1)^{n-2}h(k,N)^{-2}}{(2n-3)!} + \frac{(-1)^{n-1}}{(2n-1)!}\}}{\lim\limits_{N\to\infty}\lim\limits_{n\to\infty}\{h(2,N)^{2n-1}\}\lim\limits_{N\to\infty}\lim\limits_{n\to\infty}\{h(2,N)^{2-2n} - \frac{h(2,N)^{4-2n}}{3!} + \frac{h(2,N)^{6-2n}}{5!} - ---- + \frac{(-1)^{n-2}h(2,N)^{-2}}{(2n-3)!} + \frac{(-1)^{n-1}}{(2n-1)!}\}}$$

$$= \frac{\lim\limits_{N\to\infty}\lim\limits_{n\to\infty}\{h(k,N)^{2-2n} - \frac{h(k,N)^{4-2n}}{3!} + \frac{h(k,N)^{6-2n}}{5!} - \frac{h(k,N)^7}{7!} + ---- + \frac{(-1)^{n-2}h(k,N)^{-2}}{(2n-3)!} + \frac{(-1)^{n-1}}{(2n-1)!}\}}{\lim\limits_{N\to\infty}\lim\limits_{n\to\infty}\{h(2,N)^{2-2n} - \frac{h(2,N)^{4-2n}}{3!} + \frac{h(2,N)^{6-2n}}{5!} - \frac{h(2,N)^7}{7!} + ---- + \frac{(-1)^{n-2}h(2,N)^{-2}}{(2n-3)!} + \frac{(-1)^{n-1}}{(2n-1)!}\}}$$

$$= \frac{\lim\limits_{n\to\infty}\frac{(-1)^{n-1}}{(2n-1)!}}{\lim\limits_{n\to\infty}\frac{(-1)^{n-1}}{(2n-1)!}} = \lim\limits_{n\to\infty}\frac{\frac{(-1)^{n-1}}{(2n-1)!}}{\frac{(-1)^{n-1}}{(2n-1)!}} = \lim\limits_{n\to\infty}1 = 1 \qquad (46)$$

The 7th equal sine (=) in (46) is true due to (41).

The 8th equal sine (=) in (46) is true due to (44) and (45).

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Appendix 4 : Solution for F(a)=0　　(1)

1　Preparation for verification of F(a)＞0
1.1　Investigation of f(n)

$$f(n) = \frac{1}{n^{1/2-a}} - \frac{1}{n^{1/2+a}} \geqq 0 \qquad (n=2,3,4,5,\text{--------}) \qquad (8)$$

$$F(a) = f(2)-f(3)+f(4)-f(5)+f(6)- \text{-----} \qquad (15)$$

　　a=0 is the solution for F(a)=0 due to f(n)≡0 at a=0. Hereafter we define the range of a as 0＜a＜1/2 to verify F(a)＞0. The alternating series F(a) converges due to $\lim_{n\to\infty} f(n)=0$.

　　We have the following equation by differentiating f(n) regarding n.

$$\frac{df(n)}{dn} = \frac{1/2+a}{n^{a+3/2}} - \frac{1/2-a}{n^{3/2-a}} = \frac{1/2+a}{n^{a+3/2}}\left\{1 - \left(\frac{1/2-a}{1/2+a}\right)n^{2a}\right\}$$

　　The value of f(n) increases with the increase of n and reaches the maximum value f(n_{max}) at n=n_{max}. Afterward f(n) decreases to zero through n→∞. n_{max} is the nearest natural number to $\left(\frac{1/2+a}{1/2-a}\right)^{1/2a}$. (Graph 1) shows f(n) in various value of a. At a=1/2 f(n) does not have f(n_{max}) and increases to 1 through n→∞ due to n_{max} =∞.

Graph 1: f(n) in various a
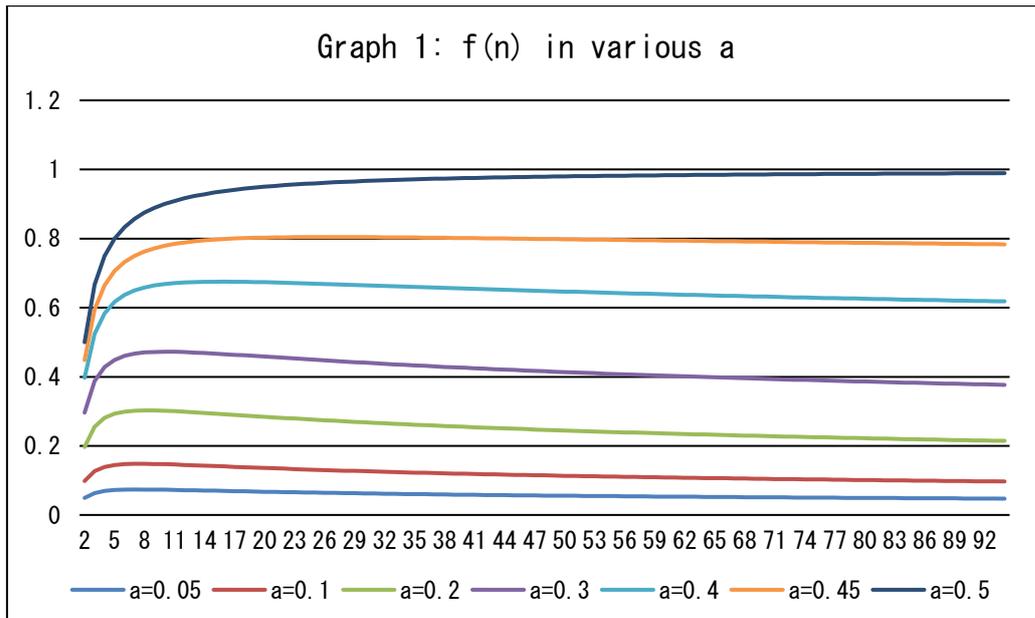
1.2 The method for verifying F(a)＞0

　　We define F(a,N) as the partial sum from the first term of F(a) to the N-th term of F(a). (N=1,2,3,4,5,------)　F(a,N) repeats increase and decrease by f(n) with increase of N as shown in (Graph 2), because F(a) is the alternating series. In (Graph 2) upper points mean F(a,2N-1) and lower points mean F(a,2N). F(a,2N-1) decreases and converges to F(a). F(a,2N) increases and also converges to F(a) due to $\lim_{n \to \infty} f(n)=0$.

Graph 2 : F(0.1,N) from 1st to 100th term

F(a,2N-1)

f(n)
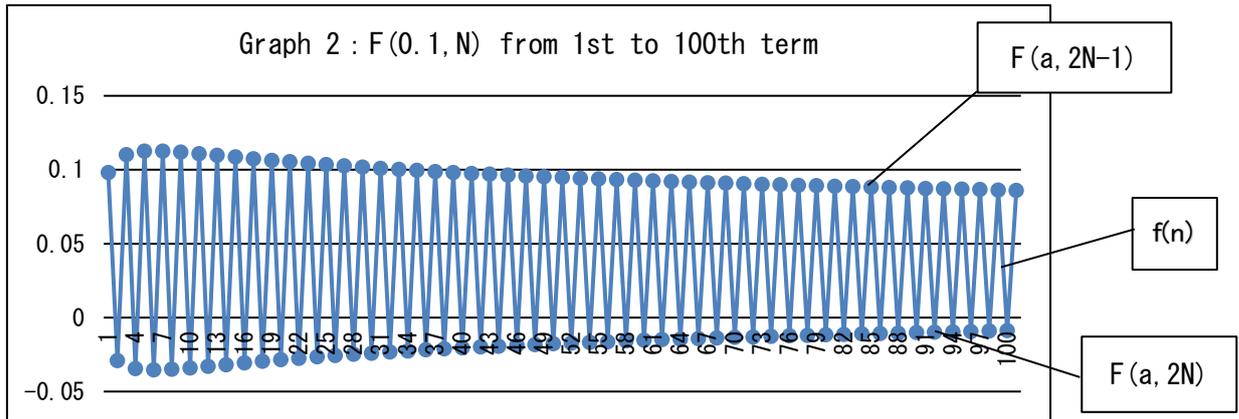
F(a,2N)

　　F1(a,2N) which is the partial sum from the first term of the following F1(a) to the 2N-th term of F1(a) is equal to F(a,2N).

$$F1(a) = \{f(2)-f(3)\}+\{f(4)-f(5)\}+\{f(6)-f(7)\}+\{f(8)-f(9)\}+ -----$$

　　Therefore $\lim_{N \to \infty} F1(a,2N)$ also converges to F(a). That means F(a)=F1(a). We use F1(a) instead of F(a) for verifying F(a)＞0. On the condition of $n_{max}=k$ or $n_{max}=k+1$ (k:odd number) ,after enclosing 2 terms of F(a) each from the first term with { } as follows, the inside sum of { } from f(2) to f(k) is negative value and the inside sum of { } after f(k+1) is positive value.

$$F(a) = f(2)-f(3)+f(4)-f(5)+f(6)-f(7)+ ------$$
$$=\{f(2)-f(3)\}+\{f(4)-f(5)\}+ --- +\{f(k-1)-f(k)\}+\{f(k+1)-f(k+2)\} + ----$$

　　　　　　　　(inside sum of { })＜0 ← I → (inside sum of { })＞0
　　　　　　　　(total sum of { }) = -B ← I → (total sum of { }) = A

　We define as follows.
　　　[the partial sum from f(2) to f(k)] = -B ＜0
　　　[the partial sum from f(k+1) to f(∞)] = A ＞0
　　　F(a) = A-B
　So we can verify F(a)＞0 by verifying A＞B.

1.3 Investigation of f(n)-f(n+1)

We have the following equation by differentiating [f(n)-f(n+1)] regarding n.

$$\frac{df(n)}{dn} - \frac{df(n+1)}{dn} = \frac{1/2+a}{n^{3/2+a}}\left\{1-\left(\frac{n}{n+1}\right)^{3/2+a}\right\} - \frac{1/2-a}{n^{3/2-a}}\left\{1-\left(\frac{n}{n+1}\right)^{3/2-a}\right\}$$

$$= C(n) - D(n)$$

"Convergence velocity to zero" of $n^{-a-3/2}$ is larger than that of $n^{a-3/2}$. When n is small number the value of [f(n)-f(n+1)] increases with increase of n due to [C(n)>D(n)]. As n increases the value reaches the maximum value $\{k_{max}\}$ at C(n)≒ D(n). (n is natural number. The situation cannot be C(n)=D(n).) After that the situation changes to C(n)<D(n) and the value decreases to zero through n→∞. (Graph 3) shows the value of [f(n)-f(n+1)] in various value of a. (Graph 4) shows the value of [f(n)-f(n+1)] at a=0.1.

Graph 3: [f(n)-f(n+1)] in various a

Graph 4: [f(n)-f(n+1)] at a=0.1
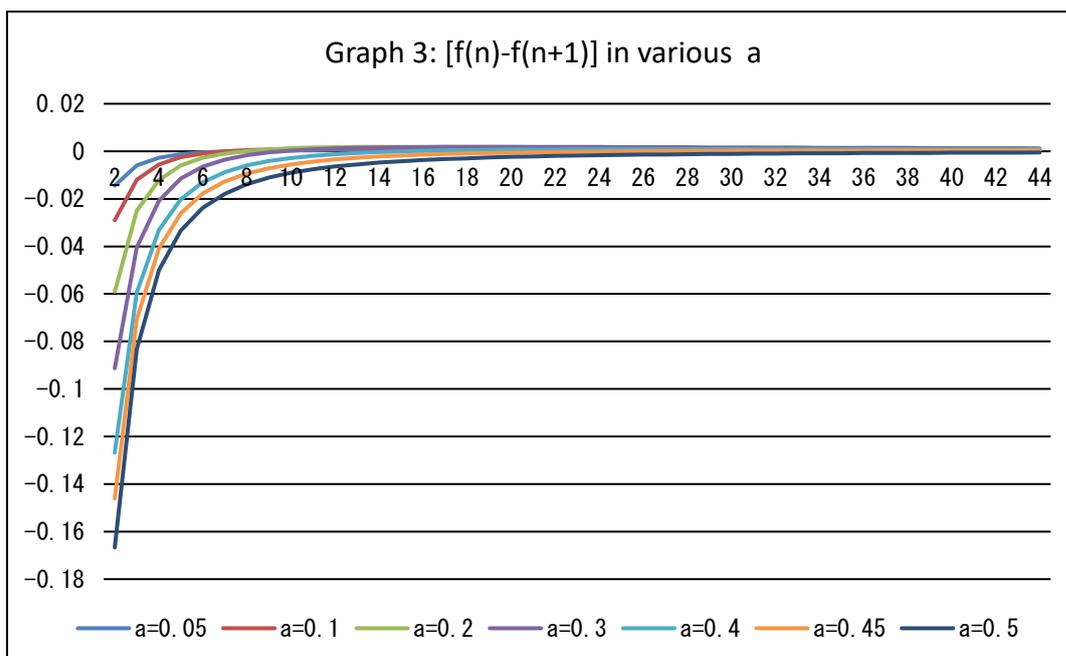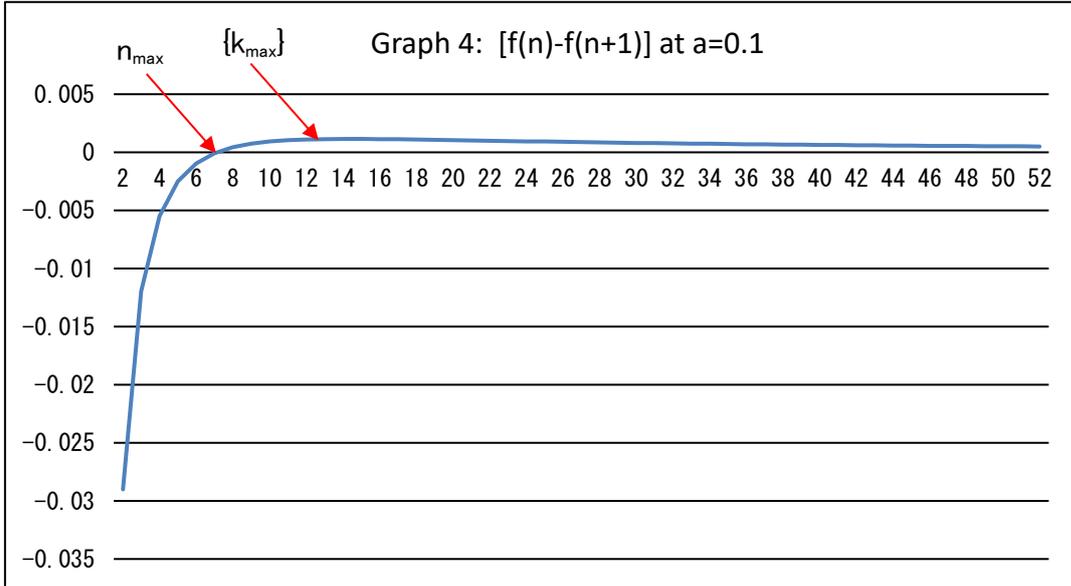
We can find the following from (Graph 3) and (Graph 4).

1.3.1　The maximum value of $|f(n)-f(n+1)|$ is $f(3)-f(2)$ at same value of a.

1.3.2　 With increase of n the sign of $[f(n)-f(n+1)]$ changes from "−" to "+" at $n=n_{max}$ ($n=n_{max}+1$) when n_{max} is even(odd) number.

1.3.3　After that the value reaches the maximum value $\{k_{max}\}$ and the value decreases to zero through $n\rightarrow\infty$.

2　Verification of A>B ($f(n_{max})$ is even-numbered term.)

Hereafter a is fixed within $0<a<1/2$ to find the condition for A>B. $f(n_{max})$ is even-numbered term as follows.

$F(a)$ = $f(2)-f(3)+f(4)-f(5)+f(6)-$ ------
= $\{f(2)-f(3)\}+\{f(4)-f(5)\}+$ --- $+\{f(n_{max}-3)-f(n_{max}-2)\}+\{f(n_{max}-1)-f(n_{max})\}$
　 $+\{f(n_{max}+1)-f(n_{max}+2)\}+\{f(n_{max}+3)-f(n_{max}+4)\}+\{f(n_{max}+5)-f(n_{max}+6)\}+$ -----

We can have A and B as follows.

B = $\{f(3)-f(2)\}+\{f(5)-f(4)\}+\{f(7)-f(6)\}+$ --- $+\{f(n_{max}-2)-f(n_{max}-3)\}+\{f(n_{max})-f(n_{max}-1)\}$
A = $\{f(n_{max}+1)-f(n_{max}+2)\}+\{f(n_{max}+3)-f(n_{max}+4)\}+\{f(n_{max}+5)-f(n_{max}+6)\}+$ -----

2.1　Condition for B

We define as follows.

$\{\ \}$ is included within B.

$\{\ \}$ is not included within B.

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We have the following equation.

$$f(n_{max})-f(2) = \{f(n_{max})-f(n_{max}-1)\}+\{f(n_{max}-1)-f(n_{max}-2)\}+\{f(n_{max}-2)-f(n_{max}-3)\}+ ---$$
$$+\{f(7)-f(6)\}+\{f(6)-f(5)\}+\{f(5)-f(4)\}+\{f(4)-f(3)\}+\{f(3)-f(2)\}$$

And we have the following inequalities from (graph 3) and (graph 4).

$$\{f(3)-f(2)\} > \{f(4)-f(3)\} > \{f(5)-f(4)\} > \{f(6)-f(5)\} > \{f(7)-f(6)\} >-----$$
$$> \{f(n_{max}-2)-f(n_{max}-3)\} > \{f(n_{max}-1)-f(n_{max}-2)\} > \{f(n_{max})-f(n_{max}-1)\} > 0$$

From above equation and inequality we have the following (41).

$$f(n_{max})-f(2)+\{f(3)-f(2)\}$$
$$= \{f(3)-f(2)\}+\{f(5)-f(4)\}+\{f(7)-f(6)\}+ --- +\{f(n_{max}-2)-f(n_{max}-3)\}+\{f(n_{max})-f(n_{max}-1)\}$$

‖　　　　∧　　　　∧　　　　　　　　∧　　　　　　　∧　　　←Value comparison

$$+ \{f(3)-f(2)\}+\{f(4)-f(3)\}+\{f(6)-f(5)\}+ --- +\{f(n_{max}-3)-f(n_{max}-4)\}+\{f(n_{max}-1)-f(n_{max}-2)\}$$
$$> 2B \tag{51}$$

Due to [Total sum of upper row of (51) = B < Total sum of lower row of (51)], we have the following inequality.

$$f(n_{max})-f(2)+\{f(3)-f(2)\} > 2B \tag{52}$$

2.2 Condition for A ({k_{max}} is included within A.)

We abbreviate $\{f(n_{max}+k)-f(n_{max}+k+1)\}$ to $\{k\}$ for easy description. (k=0,1,2,3----) All $\{k\}$ have positive value as shown in item 1.2. We define as follows.

$\{\boxed{}\}$ is included within A.

$\{\boxed{}\}$ is not included within A.

$\{k_{max}\}$ has the maximum value in all $\{k\}$.

$\{k_{max}\}$ is included within A. Then value comparison of $\{k\}$ is as follows from item 1.3.

$$\{1\} < \{2\} < \{3\} <-----< \{k_{max}-3\} < \{k_{max}-2\} < \{k_{max}-1\} < \{k_{max}\} > \{k_{max}+1\} > \{k_{max}+2\} > \{k_{max}+3\} >----$$

We have the following equation.

$$f(n_{max}+1) = \{f(n_{max}+1)-f(n_{max}+2)\}+\{f(n_{max}+2)-f(n_{max}+3)\}+\{f(n_{max}+3)-f(n_{max}+4)\}$$
$$+\{f(n_{max}+4)-f(n_{max}+5)\}+ -----$$
$$= \{1\}+\{2\}+\{3\}+\{4\}+ --- +\{k_{max}-3\}+\{k_{max}-2\}+\{k_{max}-1\}+\{k_{max}\}+\{k_{max}+1\}+\{k_{max}+2\}+\{k_{max}+3\}+ -----$$

From the above equation

$f(n_{max}+1)-\{k_{max}-1\}$

$= \{1\}+\{2\}+\{3\}+\{4\}+ --- +\{k_{max}-3\}+\{k_{max}-2\}+\{k_{max}\}+\{k_{max}+1\}+\{k_{max}+2\}+\{k_{max}+3\}+ -----$

←------------ Range 1 -----------→ | ←------------- Range 2 --------------

(Range 1) and (Range 2) are determined as above.

In (Range 1) value comparison is as follows.

$\{1\} < \{2\} < \{3\} < \{4\} ----- \{k_{max}-4\} < \{k_{max}-3\} < \{k_{max}-2\}$

And

Total sum of $\{\ \}$ = $\{1\}$ + $\{3\}$ + $\{5\}$ + $\{7\}$ + --------- + $\{k_{max}-4\}$ + $\{k_{max}-2\}$

           V   V   V   V   V ←value comparison

Total sum of $\{\ \}$ = $\{2\}$ + $\{4\}$ + $\{6\}$ + -------- + $\{k_{max}-5\}$ + $\{k_{max}-3\}$

Therefore Total sum of $\{\ \}$ > Total sum of $\{\ \}$ holds.

In (Range 2) value comparison is as follows.

$\{k_{max}\} > \{k_{max}+1\} > \{k_{max}+2\} > \{k_{max}+3\} > \{k_{max}+4\} > \{k_{max}+5\} -----$

And

Total sum of $\{\ \}$ = $\{k_{max}\}$ + $\{k_{max}+2\}$ + $\{k_{max}+4\}$ + $\{k_{max}+6\}$ + ----------

          V   V   V   V   ←value comparison

Total sum of $\{\ \}$ = $\{k_{max}+1\}$ + $\{k_{max}+3\}$ + $\{k_{max}+5\}$ + $\{k_{max}+7\}$ + --------

Therefore Total sum of $\{\ \}$ > Total sum of $\{\ \}$ holds.

In (Range 1)+(Range 2) we have [A=Total sum of $\{\ \}$ >Total sum of $\{\ \}$]. So we have the following inequality.

$$f(n_{max}+1) - \{k_{max}-1\} < 2A \tag{53}$$

2.3 Condition for A ($\{k_{max}\}$ is not included within A.)

We have the following equations. $\{k_{max}\}$ is not included within A.

$f(n_{max}+1) = \{f(n_{max}+1)-f(n_{max}+2)\}+\{f(n_{max}+2)-f(n_{max}+3)\}+\{f(n_{max}+3)-f(n_{max}+4)\}$

    $+\{f(n_{max}+4)-f(n_{max}+5)\}$ + -----

$= \{1\}+\{2\}+\{3\}+\{4\}+ --- +\{k_{max}-3\}+\{k_{max}-2\}+\{k_{max}-1\}+\{k_{max}\}+\{k_{max}+1\}+\{k_{max}+2\}+\{k_{max}+3\}+ -----$

$f(n_{max}+1) - \{k_{max}\}$

$= \{1\}+\{2\}+\{3\}+\{4\}+ --- +\{k_{max}-3\}+\{k_{max}-2\}+\{k_{max}-1\}+\{k_{max}+1\}+\{k_{max}+2\}+\{k_{max}+3\}+\{k_{max}+4\}+ -----$

←------------------ Range 1 ------------------→ | ←------------- Range 2 --------------

(Range 1) and (Range 2) are determined as above.

In (Range 1) value comparison is as follows.

$$\{1\} < \{2\} < \{3\} < \{4\} < ----- < \{k_{max}-3\} < \{k_{max}-2\} < \{k_{max}-1\}$$

And

Total sum of $\{\;\}$ = $\{1\}$ + $\{3\}$ + $\{5\}$ + $\{7\}$ + --------- + $\{k_{max}-3\}$ + $\{k_{max}-1\}$
                                V   V   V   V   V ←value comparison
Total sum of $\{\;\}$ = $\{2\}$ + $\{4\}$ + $\{6\}$ + -------- + $\{k_{max}-4\}$ + $\{k_{max}-2\}$

Therefore   Total sum of $\{\;\}$ > Total sum of $\{\;\}$ holds.

In (Range 2) value comparison is as follows.

$$\{k_{max}+1\} > \{k_{max}+2\} > \{k_{max}+3\} > \{k_{max}+4\} > \{k_{max}+5\} > \{k_{max}+6\} -----$$

And

Total sum of $\{\;\}$ = $\{k_{max}+1\}$ + $\{k_{max}+3\}$ + $\{k_{max}+5\}$ + $\{k_{max}+7\}$ + ---------
                                V   V   V   V   ←value comparison
Total sum of $\{\;\}$ = $\{k_{max}+2\}$ + $\{k_{max}+4\}$ + $\{k_{max}+6\}$ + $\{k_{max}+8\}$ + --------

Therefore   Total sum of $\{\;\}$ > Total sum of $\{\;\}$ holds.

In (Range 1)+(Range 2) we have [A=total sum of $\{\;\}$ >Total sum of $\{\;\}$]. So we have the following inequality.

$$f(n_{max}+1) - \{k_{max}\} < 2A \qquad\qquad (54)$$

2.4 Condition for A>B

From (53) and (54) we have the following inequality.

$$f(n_{max}+1) - [\{k_{max}\} \text{ or } \{k_{max}-1\}] < 2A$$

As shown in item 1.3.1 $\{f(3)-f(2)\}$ has the maximum value in all $|\{\;\}|$. The following inequalities holds.

$$\{f(3)-f(2)\} > [\{k_{max}\} \text{ or } \{k_{max}-1\}]$$
$$\{f(3)-f(2)\} > f(n_{max}) - f(n_{max}+1)$$

We have the following inequality from the above conditions.

$$2A > f(n_{max}+1) - [\{k_{max}\} \text{ or } \{k_{max}-1\}] > f(n_{max}+1) - \{f(3)-f(2)\}$$
$$> f(n_{max}) - \{f(3)-f(2)\} - \{f(3)-f(2)\} = f(n_{max}) - 2\{f(3)-f(2)\} \qquad (55)$$

We have the following condition for A>B from (52) and (55).

$$2A > f(n_{max}) - 2\{f(3) - f(2)\} > f(n_{max}) - f(2) + \{f(3) - f(2)\} > 2B \qquad (56)$$

From (56) we can have the final condition for A>B as follows.

$$(4/3)f(2) > f(3) \qquad (57)$$

(Graph 6) shows $(4/3)f(2) - f(3) = (4/3)(2^{a-1/2} - 2^{-a-1/2}) - (3^{a-1/2} - 3^{-a-1/2})$.

Graph 6:(4/3)f(2)-f(3)
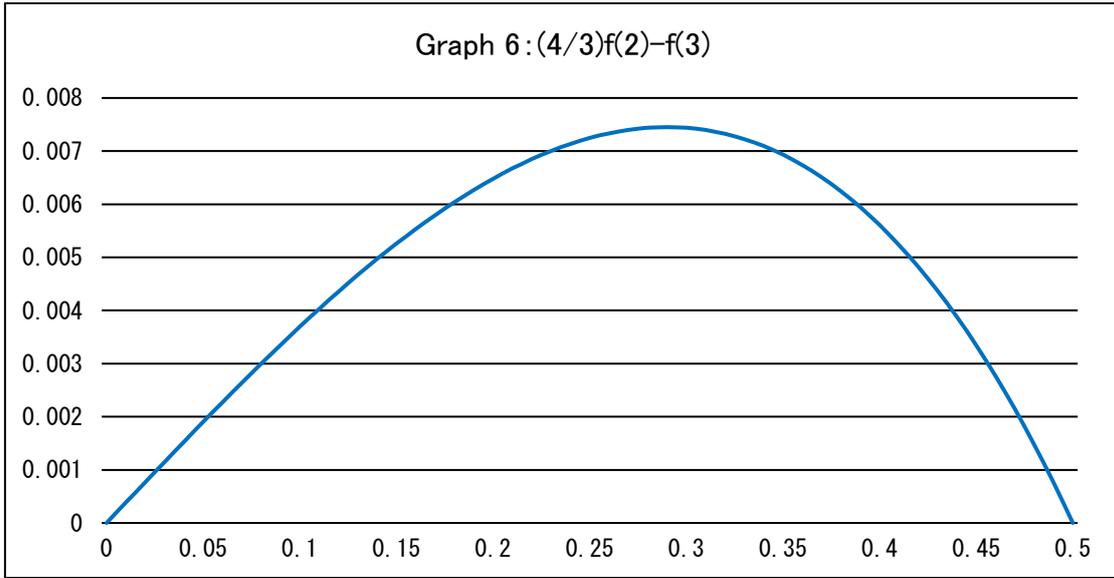
Table 1 : The values of (4/3)f(2)-f(3)

a=	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
(4/3)f(2)-f(3)	0	0.001903	0.003694	0.005257	0.00648	0.007246	0.007437	0.006933	0.005611	0.003343	0

(Graph 7) shows [differentiated (4/3)f(2)-f(3) regarding a] i.e.

(4/3)f`(2)-f`(3) = (4/3){log2(2^{a-1/2}+2^{-a-1/2})}-{log3(3^{a-1/2}+3^{-a-1/2})}.
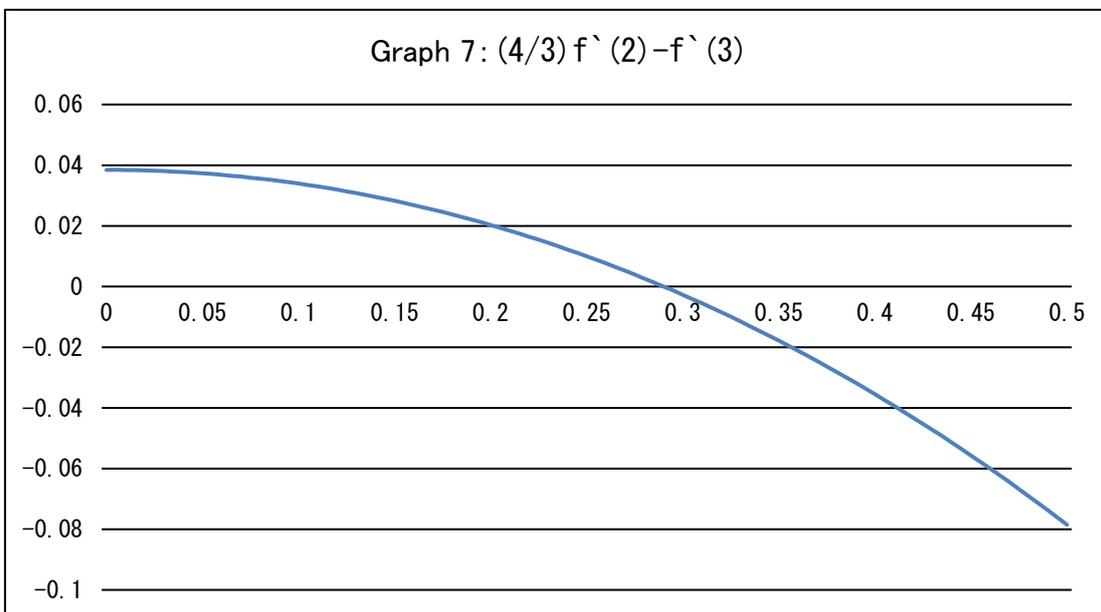
Graph 7: (4/3)f`(2)-f`(3)

Table 2 : The values of (4/3)f`(2)-f`(3)

a=	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
(4/3)f'(2)-f'(3)	0.038443	0.037313	0.033921	0.02825	0.020277	0.009967	−0.00272	−0.01785	−0.03547	−0.05567	−0.07852

From (Graph 6) and (Graph 7) we can find [(4/3)f(2)-f(3)>0 in 0<a<1/2] that means A>B i.e. F(a)>0 in 0<a<1/2.

3 Verification of A>B (f(n_max) is odd-numbered term.)

f(n_max) is odd-numbered term as follows.

F(a) = f(2)-f(3)+f(4)-f(5)+f(6)- ------
     = {f(2)-f(3)}+{f(4)-f(5)}+ --- +{f(n_max-4)-f(n_max-3)}+{f(n_max-2)-f(n_max-1)}
   +{f(n_max)-f(n_max+1)}+{f(n_max+2)-f(n_max+3)} + -----

     And

B = {f(3)-f(2)}+{f(5)-f(4)}+ --- +{f(n_max-3)-f(n_max-4)}+{f(n_max-1)-f(n_max-2)}
A = {f(n_max)-f(n_max+1)}+{f(n_max+2)-f(n_max+3)}+{f(n_max+4)-f(n_max+5)}+ -----
f(n_max) = {f(n_max)-f(n_max+1)}+{f(n_max+1)-f(n_max+2)}+{f(n_max+2)-f(n_max+3)}+{f(n_max+3)-f(n_max+4)} +

     = {0}+{1}+{2}+{3}+ --- +{k_max-3}+{k_max-2}+{k_max-1}+{k_max}+{k_max+1}+{k_max+2}+{k_max+3}+ -----

     After the same process as in item 2 we can have the following condition.

$$f(n_{max}-1)-f(2)+\{f(3)-f(2)\} > 2B \tag{58}$$

As shown in item 1.3.1 $\{f(3)-f(2)\}$ has the maximum value in all $|\{\ \}|$. The following inequalities hold.

$$\{f(3)-f(2)\} > [\{k_{max}\} \text{ or } \{k_{max}-1\}]$$
$$f(n_{max}) > f(n_{max}-1)$$

We have the following inequality from the same process as in item 2 and the above conditions.

$$2A > f(n_{max}) - [\{k_{max}\} \text{ or } \{k_{max}-1\}] > f(n_{max}) - \{f(3)-f(2)\} > f(n_{max}-1) - \{f(3)-f(2)\} \tag{59}$$

We have the following condition for A>B from (58) and (59).

$$2A > f(n_{max}-1)-\{f(3)-f(2)\} > f(n_{max}-1)-f(2)+\{f(3)-f(2)\} > 2B \tag{60}$$

From (60) we can have the final condition for A>B as follows.

$$(3/2)f(2) > f(3) \tag{61}$$

In the inequality of $[(3/2)f(2) > (4/3)f(2) > f(3) > 0]$, $(3/2)f(2) > (4/3)f(2)$ is true self-evidently and in item 2.4 we already confirmed that the following (57) is true in $0 < a < 1/2$.

$$(4/3)f(2) > f(3) \tag{57}$$

Therefore (61) is true in $0 < a < 1/2$.

4 Conclusion

$F(a)=0$ has the only one solution of a=0 due to $[0 \leqq a < 1/2]$, $[F(0)=0]$ and $[F(a) > 0$ in $0 < a < 1/2]$.

Appendix 5 : Solution for F(a)=0　(2)

1　Investigation of $F(a)_N$

$$f(n) = \frac{1}{n^{1/2-a}} - \frac{1}{n^{1/2+a}} \geqq 0 \qquad (n=2,3,4,5,\text{-------}) \qquad (8)$$

$$F(a) = f(2)-f(3)+f(4)-f(5)+f(6)- \text{-----} \qquad (15)$$

F(a,N): the partial sum from the first term of F(a) to the N-th term of F(a)

a=0 is the solution for F(a)=0 because of f(n)≡0 at a=0.　F(a) is the alternating series. So F(a,N) repeats increase and decrease by f(n) with increase of N. $\lim_{N \to \infty} F(a,N)$ converges to F(a) due to $\lim_{n \to \infty} f(n)=0$.

(Graph 1) shows F(0.1,N) from N=1 to N=5,000. The upper edge of blue area shows F(0.1,2N-1) and lower edge of blue area shows F(0.1,2N). ((Graph 1) is line graph. Graph has so many data points that the area surrounded by data points becomes blue.)

Graph 1：F(0.1,N) from N=1 to N=5,000
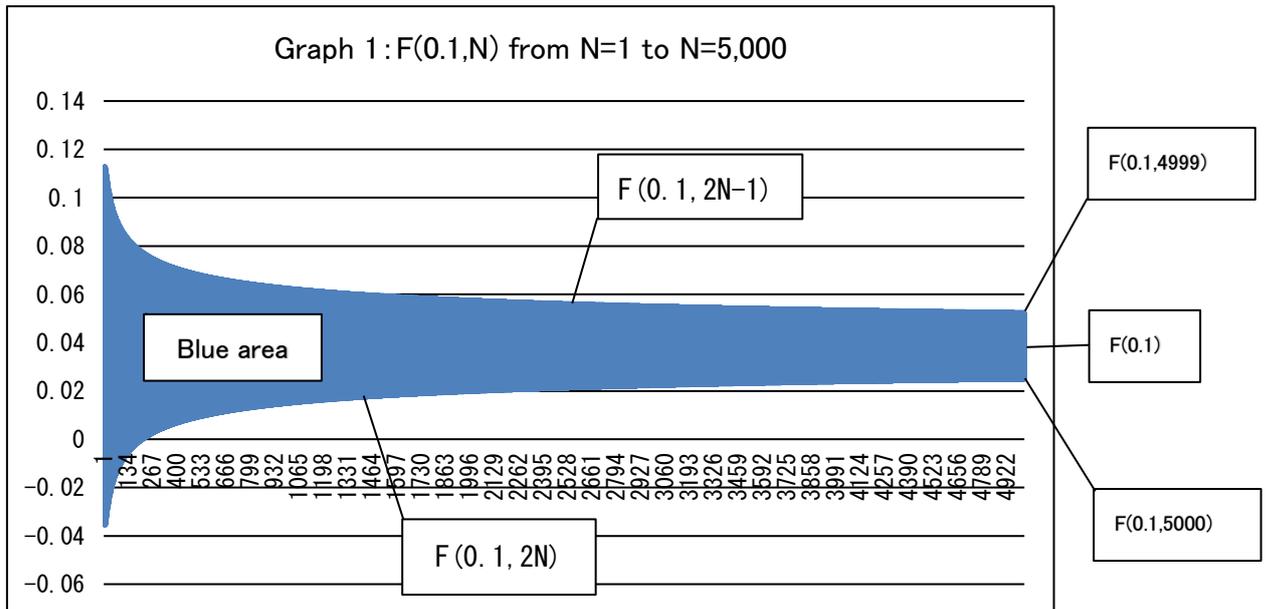
Upper-right point of blue area, F(0.1,4999) decreases to F(0.1) through N→∞ and lower-right point of blue area, F(0.1,5000) increases to F(0.1) through N→∞. F(0.1) can be approximated with {F(0.1,4999)+ F(0.1,5000)}/2.

But {F(a,N-1)+F(a,N)}/2 is also the partial sum of alternating series. It repeats increase(decrease) of {f(n)-f(n-1)}/2 and decrease(increase) of {f(n+1)-f(n)}/2 when n is even(odd) number. So we approximate F(a) with the average of {F(a,N-1)+F(a,N)}/2 i.e. $F(a)_N$ for better accuracy according to the following

(71).

$$\frac{\dfrac{F(a,N)+F(a,N-1)}{2} + \dfrac{F(a,N+1)+F(a,N)}{2}}{2} = F(a)_N \qquad (71)$$

Left side of (71) converges to $F(a)$ through $N\to\infty$. If N is large number, we can have the accurate $F(a)_N$ from $F(a,N)$. (Graph 2) shows $F(a)_N$ calculated at 3 cases of N=500, 1000, 5000.

Graph 2: $F(a)_N$ at 3 cases
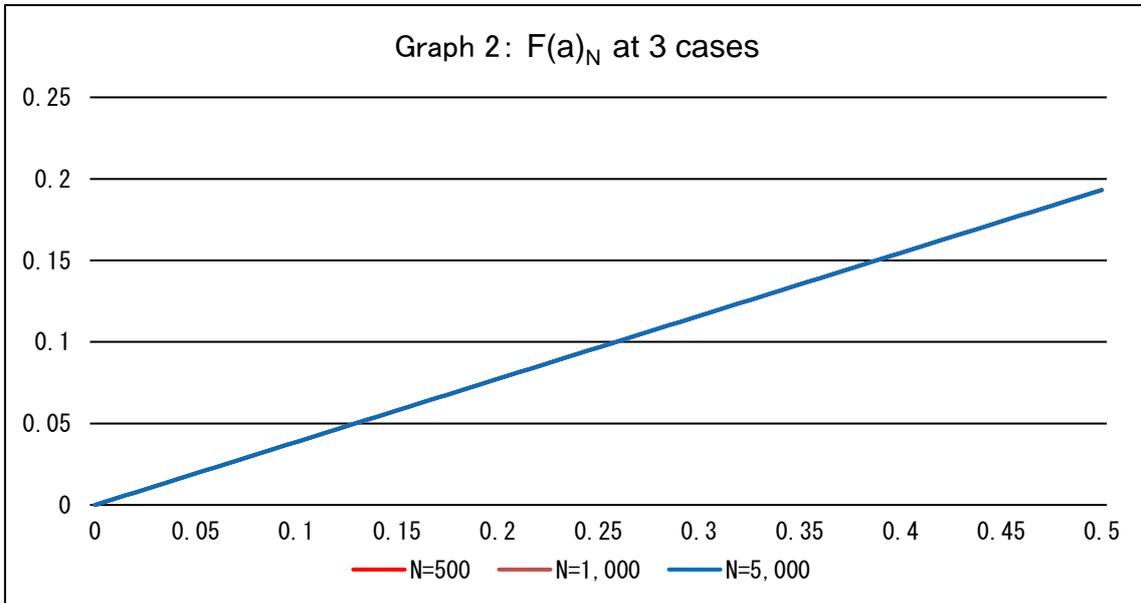
Table 1 : The values of $F(a)_N$

a	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
N=500	0	0.01932876	0.03865677	0.05798326	0.0773074	0.09662832	0.11594507	0.13525658	0.15456168	0.17385904	0.19314718
N=1,000	0	0.01932681	0.03865282	0.05797725	0.0772993	0.09661821	0.11593325	0.13524382	0.15454955	0.17385049	0.19314743
N=5,000	0	0.01932876	0.03865676	0.05798324	0.07730738	0.09662829	0.11594504	0.13525655	0.15456165	0.17385902	0.19314718

3 line graphs overlapped. Because $F(a)_N$ calculated at 3 cases of N=500, 1000, 5000 are equal to 4 digits after the decimal point.

The range of a is $0 \leq a < 1/2$. a=1/2 is not included in the range. But we added $F(1/2)_N$ to calculation according to the following reason.

[f(n) at a=1/2] is $(1-1/n)$ and $\lim_{n\to\infty}(1-1/n)$ does not converge to zero. Therefore $F(1/2)$ fluctuates due to $\lim_{n\to\infty}f(n)=1$. But $\{F(a,N)+F(a,N-1)\}/2$ is partial sum of alternating series with the term of $\{f(n+1)-f(n)\}/2$ and it can converge to the fixed value on the condition of $\lim_{n\to\infty}\{f(n+1)-f(n)\}=0$. $\lim_{n\to\infty}\{f(n+1)-f(n)\}$ converges to zero due to $f(n+1)-f(n)=1/(n+n^2)$.

2 Investigation of F`(a)$_N$

　　　We define as follows.

$$f`(n) = df(n)/da = n^{a-1/2}\log n + n^{-a-1/2}\log n = n^{a-1/2}\log n\ (1 + n^{-2a}) > 0$$

$$F`(a) = f`(2)-f`(3)+f`(4)-f`(5) + ----$$

F`(a,N): the partial sum from the first term of F`(a) to the N-th term of F`(a)

　　F`(a) converges due to $\lim_{n\to\infty} f`(n)=0$. We can calculate approximation of F`(a) i.e. F`(a)$_N$ according to the following (72). $\lim_{N\to\infty} F`(a)_N$ converges to F`(a).

$$\frac{\dfrac{F`(a,N)+F`(a,N-1)}{2} + \dfrac{F`(a,N+1)+F`(a,N)}{2}}{2} = F`(a)_N \tag{72}$$

　　　(Graph 3) shows F`(a)$_N$ calculated by (72) at 5 cases of N=500, 1000, 2000, 5000, 10000. 5 line graphs overlapped. Because F`(a)$_N$ of 5 cases are equal to 6 digits after the decimal point.

Graph 3 : F`(a)$_N$ at 5 cases
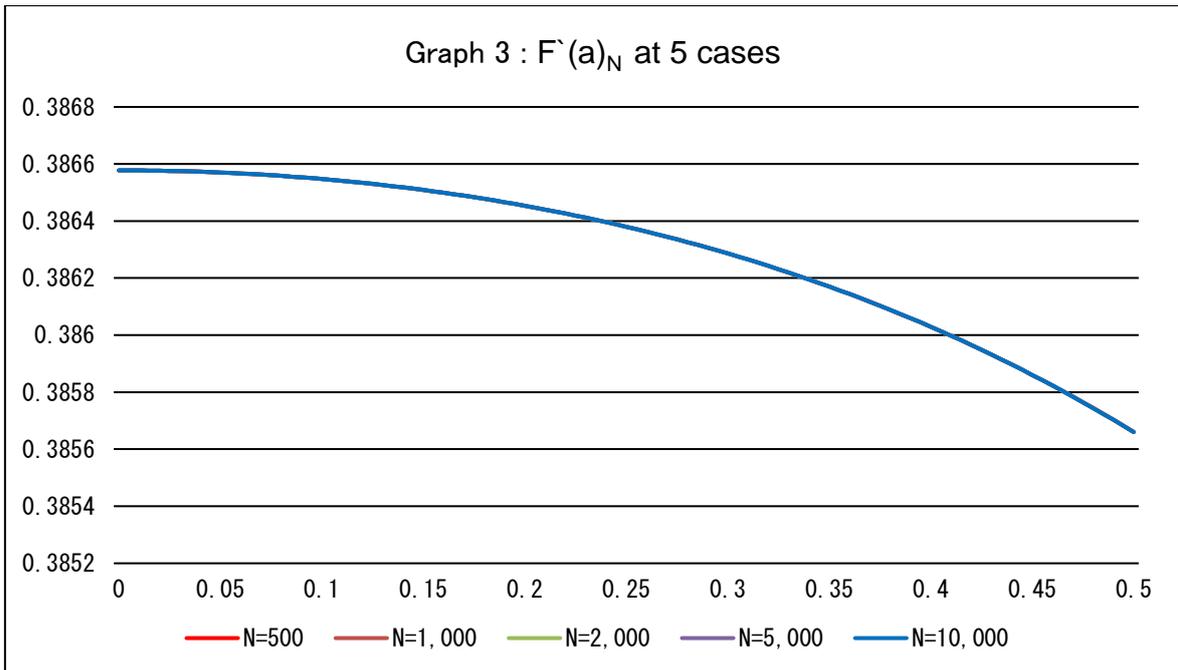
Table 2 : The values of F`(a)$_N$

a	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
N=500	0.38657754	0.38657004	0.38654734	0.38650882	0.38645348	0.3863799	0.38628625	0.38617032	0.3860295	0.38586078	0.38566075
N=1,000	0.38657764	0.38657014	0.38654743	0.38650891	0.38645355	0.38637995	0.38628627	0.3861703	0.3860294	0.38586057	0.38566038
N=2,000	0.38657766	0.38657016	0.38654745	0.38650893	0.38645357	0.38637996	0.38628628	0.3861703	0.38602938	0.38586052	0.38566029
N=5,000	0.38657766	0.38657016	0.38654745	0.38650893	0.38645358	0.38637997	0.38628628	0.3861703	0.38602938	0.38586051	0.38566026
N=10,000	0.38657766	0.38657016	0.38654745	0.38650893	0.38645358	0.38637997	0.38628629	0.3861703	0.38602938	0.3858605	0.38566026

The range of a is $0 \leqq a < 1/2$. $a=1/2$ is not included in the range. But we added $F`(1/2)_N$ to calculation according to the following reason.

[$f`(n)$ at $a=1/2$] is $(1+1/n)\log n$ and $\lim_{n \to \infty}(1+1/n)\log n$ does not converge to zero. $F(1/2)$ diverges to $\pm\infty$ due to $\lim_{n \to \infty}f`(n)=\infty$. But $\{F`(a,N)+F`(a,N-1)\}/2$ is partial sum of alternating series with the term of $\{f`(n+1)-f`(n)\}/2$ and it can converge to the fixed value on the condition of $\lim_{n \to \infty}\{f`(n+1)-f`(n)\}=0$. $\lim_{n \to \infty}\{f`(n+1)-f`(n)\}=0$ is true as follows.

$f`(n)$ is the increasing function regarding n due to $[\dfrac{df`(n)}{dn}=\dfrac{1+n-\log n}{n^2}>0]$.

It means $[0 < f`(n+1)-f`(n)]$.

$$0 < f`(n+1)-f`(n) = \{1+1/(n+1)\}\log(n+1) - (1+1/n)\log n$$
$$< (1+1/n)\log(n+1) - (1+1/n)\log n = (1+1/n)\log(1+1/n)$$

From the above inequality we can have $\lim_{n \to \infty}\{f`(n+1)-f`(n)\}=0$ due to $\lim_{n \to \infty}\{(1+1/n)\log(1+1/n)\}=0$.

3 Approximation of F`(a)

$F`(a)_N$ calculated by (72) converges to $F`(a)$ through $N \to \infty$. To confirm how large N we need to approximate $F`(a)$ accurately, we calculated $F`(a)_N$ with N from $N=500$ to $N=100,000$. (Graph 4) shows $F`(a)_N/F`(a)_{500}$ from $N=500$ to $N=100,000$ in various a.


グラフ4：F`(a)N/F`(a)$_{500}$ in various a
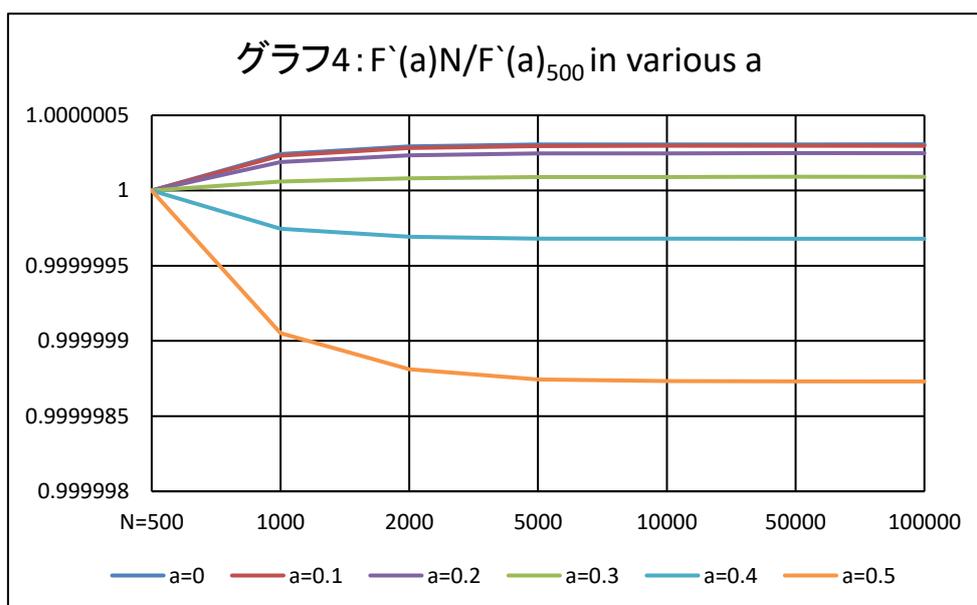
Table 3 : The values of $F`(a)_N/F`(a)_{500}$

a	0	0.1	0.2	0.3	0.4	0.5
N=500	1	1	1	1	1	1
1,000	1.000000242	1.000000232	1.000000189	1.000000061	0.999999745	0.999999051
2,000	1.000000294	1.000000284	1.000000234	1.000000082	0.999999692	0.999998811
5,000	1.000000306	1.000000296	1.000000246	1.000000089	0.999999681	0.999998743
10,000	1.000000307	1.000000297	1.000000248	1.000000091	0.999999679	0.999998734
50,000	1.000000307	1.000000297	1.000000248	1.000000091	0.999999679	0.999998731
100,000	1.000000307	1.000000297	1.000000248	1.000000091	0.999999679	0.999998731

We can find the following from (Graph 4) and (Table 3).

3.1 $F`(a)_{50,000}/F`(a)_{500}$ and $F`(a)_{100,000}/F`(a)_{500}$ have the same values. When N is larger than N=50,000 the values are same as at N=50,000. So we can consider $F`(a)_{50,000} = F`(a)$.

3.2 The differences between $F`(a)_{500}$ and $F`(a)_{50,000}$ have the maximum value at a=1/2. The maximum difference is [1−0.999998731 = 0.00013%] as shown in (Table 3). Therefore $F`(a)_{500}$ is almost equal to $F`(a)_{50,000}$ i.e. $F`(a)$. N=500 is enough to obtain the accurate $F`(a)$.

From item 3.2 we can consider that (Graph 3) shows $F`(a)$ accurately. (Graph 3) illustrates [0.3866 > $F`(a)$ > 0.3856 in 0≦a<1/2]. Therefore F(a) is the monotonically increasing function in 0≦a<1/2.

4 Conclusion

F(a)=0 has the only one solution of a=0 due to

[0≦a<1/2], [F(0)=0] and

[F(a) is the monotonically increasing function in 0≦a<1/2.].

Appendix 6: Sum of infinite series of sine waves

$f(2)\sin\{b\log N/2+\tan^{-1}(1/b)\}-f(3)\sin\{b\log N/3+\tan^{-1}(1/b)\}+f(4)\sin\{b\log N/4+\tan^{-1}(1/b)\}-$ ---

$= f(2)\sin\{b\log N-b\log 2+\tan^{-1}(1/b)\}-f(3)\sin\{b\log N-b\log 3+\tan^{-1}(1/b)\}$

$+f(4)\sin\{b\log N-b\log 4+\tan^{-1}(1/b)\}-f(5)\sin\{b\log N-b\log 5+\tan^{-1}(1/b)\}+$ ---

$= A(a,b)\sin\{b\log N+B(a,b)+\tan^{-1}(1/b)\}$ (18)

Sum of infinite series of sine waves in (18) converges to one sine wave like the rightmost side of (18) due to the following reasons.

1　　The general term of the infinite series in (18) is

$(-1)^n f(n)\sin\{b\log N-b\log n+\tan^{-1}(1/b)\}$ $(n=2,3,4,5,-----)$. If n is large natural number, the value of $b\log n$ increases very slowly with the increase of n and the sign of $\sin\{b\log N-b\log n+\tan^{-1}(1/b)\}$ does not change often. Therefore +term and −term appear alternately and 2 +terms or 2 −terms appear in succession only when the sign of $\sin\{b\log N-b\log n+\tan^{-1}(1/b)\}$ changes. +term and −term in the above explanation are defined as follows. （ $b\log N-b\log n+\tan^{-1}(1/b)=\alpha$ ）

　　　　+term ：　$+f(n)\sin\alpha$ when the sign of $\sin\alpha$ is "+".

　　　　　　　　　$-f(n)\sin\alpha$ when the sign of $\sin\alpha$ is "−".

　　　　−term ：　$+f(n)\sin\alpha$ when the sign of $\sin\alpha$ is "−".

　　　　　　　　　$-f(n)\sin\alpha$ when the sign of $\sin\alpha$ is "+".

On the condition of $[\sin\{b\log N-b\log X_0+\tan^{-1}(1/b)\}=0]$ and $[n_0+1>X_0>n_0]$ $[\log(n_0+1)-\log n_0>\log X_0-\log n_0>0]$ and $[\log(n_0+1)-\log n_0>\log(n_0+1)-\log X_0>0]$ are true.　Due to $\lim_{n_0\to\infty}\{\log(n_0+1)-\log n_0\}=0$ the following equations hold.

$\lim_{n_0\to\infty}\log(n_0+1) = \log X_0 = \lim_{n_0\to\infty}\log n_0$

$\lim_{n_0\to\infty}\sin\{b\log N-b\log(n_0+1)+\tan^{-1}(1/b)\} = \sin\{b\log N-b\log X_0+\tan^{-1}(1/b)\}$

$= \lim_{n_0\to\infty}\sin\{b\log N-b\log n_0+\tan^{-1}(1/b)\} = 0$

Therefore 2 +terms or 2 −terms （ $\pm f(n_0)\sin\{b\log N-b\log n_0+\tan^{-1}(1/b)\}$ and $\mp f(n_0+1)\sin\{b\log N-b\log(n_0+1)+\tan^{-1}(1/b)\}$ ） which appear in succession only when the sign of $\sin\{b\log N-b\log n+\tan^{-1}(1/b)\}$ changes have almost the values of zero, if n_0 is large number. If we regard the sum of these 2 +terms or 2 −terms that exist in succession as one +terms or one −terms, we can consider this infinite series as alternating series and this alternating series converges due to $\lim_{n\to\infty} f(n)=0$.

$$f(n) = \frac{1}{n^{1/2-a}} - \frac{1}{n^{1/2+a}} \geqq 0 \qquad (n=2,3,4,5,6,\ \text{------}) \qquad (8)$$

2   If we calculate sum or difference of 2 sine waves which have the same period, the result becomes another sine wave which has the same period as shown in (Figure 1) and (Figure 2).

Figure1: Sum of 2 sine waves

$\theta(2+3)$ $\theta(2)$

$blogN+tan^{-1}(1/b)$

$\theta(2)=-blog2$

$\theta(3)=-blog3$

$f(2)e^{i\theta(2)}$

$f(2)\sin\theta(2)$

$f(2+3)\sin\theta(2+3)$

$f(2+3)e^{i\theta(2+3)}$

$\pi+\theta(2)-\theta(3)$ $\theta(2)$ $\theta(3)$

$f(3)\sin\theta(3)$

$f(3)e^{i\theta(3)}$

Figure2: Difference of 2 sine waves

$\theta(2-3)$ $\theta(2)$ $f(2-3)e^{i\theta(2-3)}$

$blogN+tan^{-1}(1/b)$

$f(2-3)\sin\theta(2-3)$

$\theta(2)=-blog2$

$\theta(3)=-blog3$

$f(2)e^{i\theta(2)}$

$-f(3)e^{i\theta(3)}$

$f(3)\sin\theta(3)$

$f(2)\sin\theta(2)$

$\theta(3)-\theta(2)$

$\theta(2)$

$\theta(3)$

$f(3)e^{i\theta(3)}$

$f(2)\sin\{b\log N+\theta(2)+\tan^{-1}(1/b)\}\pm f(3)\sin\{b\log N+\theta(3)+\tan^{-1}(1/b)\}$
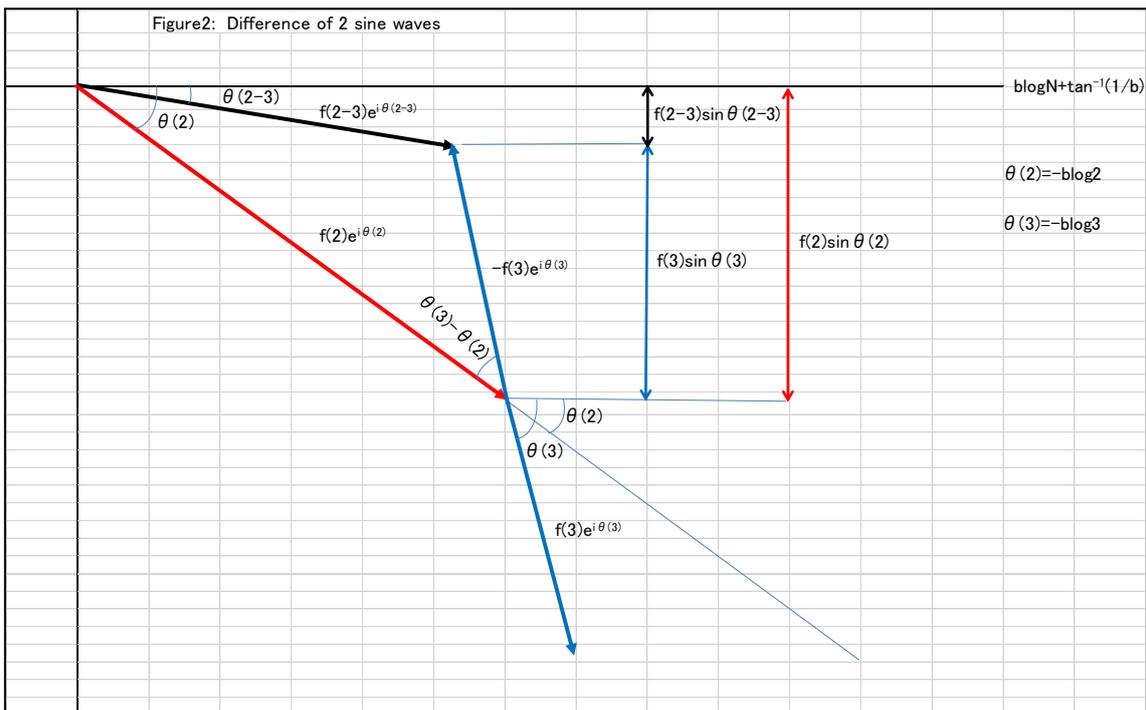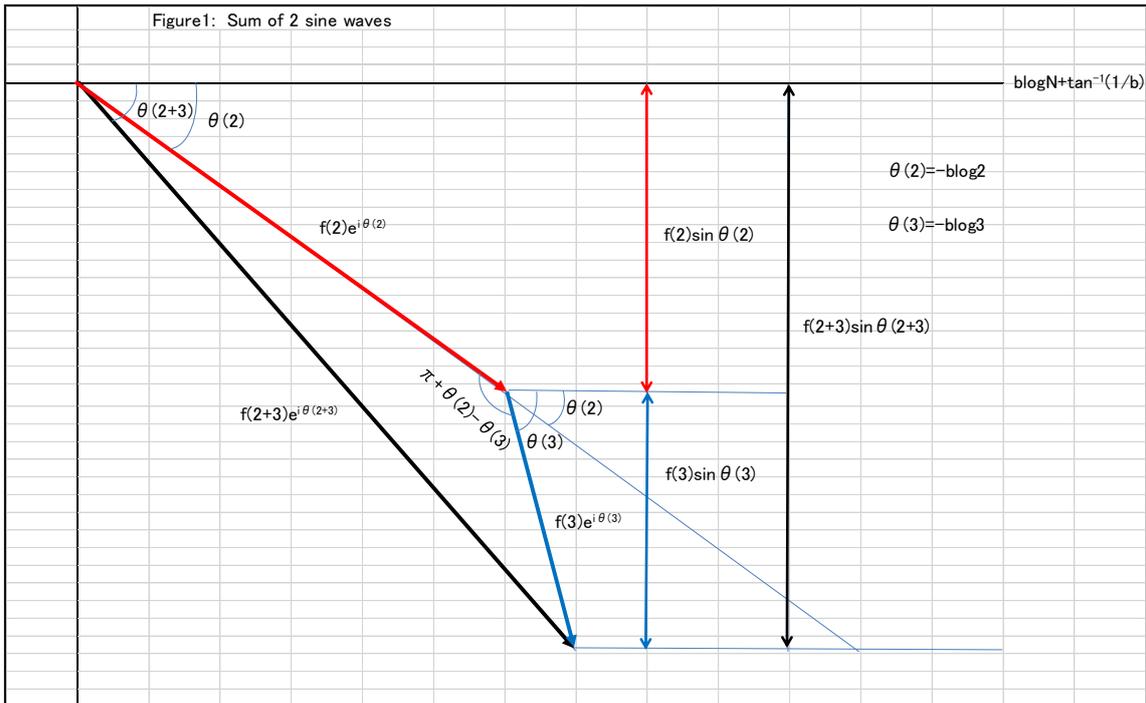
$= f(2\pm3)\sin\{b\log N+\theta(2\pm3)+\tan^{-1}(1/b)\}$ \hfill (71)

If we calculate sum or difference of 2 sine waves which have the same period and obtain the new sine wave like the right side of the above (71), the amplitude $f(2\pm3)$ and the phase $\theta(2\pm3)$ of the new sine wave become as follows. Sum (difference) of 2 sine waves takes $+(-)$ sign in (71) and the following equations.

$$f(2\pm3)=\sqrt{f(2)^2+f(3)^2\pm2f(2)f(3)\cos\{\theta(2)-\theta(3)\}}$$

$$\theta(2\pm3)=\tan^{-1}\frac{f(2)\sin\theta(2)\pm f(3)\sin\theta(3)}{f(2)\cos\theta(2)\pm f(3)\cos\theta(3)}$$

Therefore sum of infinite series of sine waves which have the same period becomes one sine wave which has the same period as shown in the above (18). In (18) $A(a,b)$ and $B(a,b)$ are constant which depends on a and b. If $A(a,b)=0$ is true, we have the following (18-1) which is identity regarding N.

$f(2)\sin\{b\log N-b\log2+\tan^{-1}(1/b)\}-f(3)\sin\{b\log N-b\log3+\tan^{-1}(1/b)\}$

$+f(4)\sin\{b\log N-b\log4+\tan^{-1}(1/b)\}-f(5)\sin\{b\log N-b\log5+\tan^{-1}(1/b)\}+---$

$= A(a,b)\sin\{b\log N+B(a,b)+\tan^{-1}(1/b)\}\equiv 0$ \hfill (18-1)

For the value of the leftmost side of (18-1) to be zero at any value of N the value of $f(n)$ must be zero at any value of n like the following (8-1). In other word $a=0$ must hold.

$$f(n) = \frac{1}{n^{1/2-a}}-\frac{1}{n^{1/2+a}} \equiv 0 \qquad (n=2,3,4,5,6,------)$$ \hfill (8-1)

Now we can say if $A(a,b)=0$ is true, $a=0$ holds true and if $A(a,b)\neq0$ is true, $0<a<1/2$ holds true.

Reference

[1]: Yukio Kusunoki, Introduction to infinite series, Asakura syoten, 1972, page
    22, (written in Japanese).