Proof of Riemann hypothesis

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Abstract

This paper is a trial to prove Riemann hypothesis which says "All non-trivial zero points of Riemann zeta function $\zeta(s)$ exist on the line of $\mathrm{Re}(s)=1/2$." according to the following process.

1 We create the infinite number of infinite series from the following (1) that gives $\zeta(s)$ analytic continuation to $\mathrm{Re}(s)>0$ and the following (2) and (3) that show non-trivial zero point of $\zeta(s)$.

$$1-2^{-s}+3^{-s}-4^{-s}+5^{-s}-6^{-s}+ \text{-----} = (1-2^{1-s})\ \zeta(s) \quad (1)$$

$$S_0 = 1/2+a+bi \quad (2)$$

$$S_1 = 1-S_0 = 1/2-a-bi \quad (3)$$

2 We find that the value of the following F(a) must be zero from the above infinite number of infinite series.

$$F(a) = f(2)-f(3)+f(4)-f(5)+f(6)- \text{-----} \quad (15)$$

$$f(n) = \frac{1}{n^{1/2-a}}-\frac{1}{n^{1/2+a}} \geqq 0 \qquad (n=2,3,4,5,6,\ \text{------}) \quad (8)$$

3 We find that F(a)=0 has only one solution of a=0. Therefore zero point of $\zeta(s)$ must be $1/2\pm bi$ and other zero point does not exist.

1 Introduction

The following (1) gives Riemann zeta function $\zeta(s)$ analytic continuation to $\mathrm{Re}(s)>0$. "+ -----" means a series with infinite terms in all equations in this paper.

$$1-2^{-s}+3^{-s}-4^{-s}+5^{-s}-6^{-s}+ \text{-----} = (1-2^{1-s})\zeta(s) \quad (1)$$

The following (2) shows non-trivial zero point of $\zeta(s)$. S_0 is the zero points of the left side of (1) and also zero points of $\zeta(s)$.

$$S_0 = 1/2+a+bi \quad (2)$$

The range of a is $0\leqq a<1/2$ by the critical strip of $\zeta(s)$. The range of b is $b>0$ due to the following reasons. And i is $\sqrt{-1}$.

1.1 There is no zero point on the real axis of the critical strip.

1.2 [Conjugate complex number of S_0] = 1/2+a−bi is also zero point of $\zeta(s)$. Therefore $b>0$ is necessary and sufficient range for investigation.

The following (3) shows also zero points of $\zeta(s)$ by the functional equation of $\zeta(s)$.

$$S_1 = 1-S_0 = 1/2-a-bi \tag{3}$$

We have the following (4) and (5) by substituting S_0 for s in the left side of (1) and putting both the real part and the imaginary part of the left side of (1) at zero respectively.

$$1 = \frac{\cos(b\log 2)}{2^{1/2+a}} - \frac{\cos(b\log 3)}{3^{1/2+a}} + \frac{\cos(b\log 4)}{4^{1/2+a}} - \frac{\cos(b\log 5)}{5^{1/2+a}} + \frac{\cos(b\log 6)}{6^{1/2+a}} - \text{-----} \tag{4}$$

$$0 = \frac{\sin(b\log 2)}{2^{1/2+a}} - \frac{\sin(b\log 3)}{3^{1/2+a}} + \frac{\sin(b\log 4)}{4^{1/2+a}} - \frac{\sin(b\log 5)}{5^{1/2+a}} + \frac{\sin(b\log 6)}{6^{1/2+a}} - \text{-----} \tag{5}$$

We also have the following (6) and (7) by substituting S_1 for s in the left side of (1) and putting both the real part and the imaginary part of the left side of (1) at zero respectively.

$$1 = \frac{\cos(b\log 2)}{2^{1/2-a}} - \frac{\cos(b\log 3)}{3^{1/2-a}} + \frac{\cos(b\log 4)}{4^{1/2-a}} - \frac{\cos(b\log 5)}{5^{1/2-a}} + \frac{\cos(b\log 6)}{6^{1/2-a}} - \text{-----} \tag{6}$$

$$0 = \frac{\sin(b\log 2)}{2^{1/2-a}} - \frac{\sin(b\log 3)}{3^{1/2-a}} + \frac{\sin(b\log 4)}{4^{1/2-a}} - \frac{\sin(b\log 5)}{5^{1/2-a}} + \frac{\sin(b\log 6)}{6^{1/2-a}} - \text{-----} \tag{7}$$

2 Infinite number of infinite series

We define f(n) as follows.

$$f(n) = \frac{1}{n^{1/2-a}} - \frac{1}{n^{1/2+a}} \geqq 0 \qquad (n=2,3,4,5,6,\ \text{------}) \tag{8}$$

We have the following (9) from (4) and (6) with the method shown in item 1 of [Appendix 1: Equation construction].

$$0 = f(2)\cos(b\log 2) - f(3)\cos(b\log 3) + f(4)\cos(b\log 4) - f(5)\cos(b\log 5) + \text{-----} \tag{9}$$

We have also the following (10) from (5) and (7) with the method shown in item 2 of [Appendix 1: Equation construction].

$$0 = f(2)\sin(b\log 2) - f(3)\sin(b\log 3) + f(4)\sin(b\log 4) - f(5)\sin(b\log 5) + \text{-----} \tag{10}$$

We can have the following (11) (which is the function of real number x) from the above (9) and (10) with the method shown in item 3 of [Appendix 1: Equation construction]. And the value of (11) is always zero at any value of x.

$$\begin{aligned} 0 &\equiv \cos x\{\text{right side of (9)}\} + \sin x\{\text{right side of (10)}\} \\ &= \cos x\{f(2)\cos(b\log 2) - f(3)\cos(b\log 3) + f(4)\cos(b\log 4) - f(5)\cos(b\log 5) + \text{-----}\} \\ &\quad + \sin x\{f(2)\sin(b\log 2) - f(3)\sin(b\log 3) + f(4)\sin(b\log 4) - f(5)\sin(b\log 5) + \text{-----}\} \\ &= f(2)\cos(b\log 2 - x) - f(3)\cos(b\log 3 - x) + f(4)\cos(b\log 4 - x) - f(5)\cos(b\log 5 - x) + \text{----} \end{aligned} \tag{11}$$

We have (12-1) by substituting blog1 for x in (11).

$$0 = f(2)\cos(b\log2-b\log1)-f(3)\cos(b\log3-b\log1)+f(4)\cos(b\log4-b\log1)$$
$$-f(5)\cos(b\log5-b\log1)+f(6)\cos(b\log6-b\log1)+ ----- \quad (12\text{-}1)$$

We have (12-2) by substituting blog2 for x in (11).

$$0 = f(2)\cos(b\log2-b\log2)-f(3)\cos(b\log3-b\log2)+f(4)\cos(b\log4-b\log2)$$
$$-f(5)\cos(b\log5-b\log2)+f(6)\cos(b\log6-b\log2)+ ----- \quad (12\text{-}2)$$

We have (12-3) by substituting blog3 for x in (11).

$$0 = f(2)\cos(b\log2-b\log3)-f(3)\cos(b\log3-b\log3)+f(4)\cos(b\log4-b\log3)$$
$$-f(5)\cos(b\log5-b\log3)+f(6)\cos(b\log6-b\log3)+ ----- \quad (12\text{-}3)$$

In the same way as above we can have (12-n) by substituting blogn for x in (11). (n = 4,5,6,7,8,-----)

$$0 = f(2)\cos(b\log2-b\log n)-f(3)\cos(b\log3-b\log n)+ f(4)\cos(b\log4-b\log n)$$
$$-f(5)\cos(b\log5-b\log n)+ ----- \quad (12\text{-}n)$$

3 Verification of F(a)=0

We define g(k) as follows. (k = 2,3,4,5,6 -----)

$$g(k) = \cos(b\log k-b\log1)+\cos(b\log k-b\log2)+\cos(b\log k-b\log3)+\cos(b\log k-b\log4)+ ----$$
$$= \cos(b\log1-b\log k)+\cos(b\log2-b\log k)+\cos(b\log3-b\log k)+\cos(b\log4-b\log k)+ ----$$
$$= \cos(b\log1/k)+\cos(b\log2/k)+\cos(b\log3/k)+\cos(b\log4/k)+\cos(b\log5/k)+ ---- \quad (13)$$

We can have the following (14) from infinite equations of (12-1),(12-2),(12-3), ----------,(12-n),------------ with the method shown in item 4 of [Appendix 1: Equation construction].

$$0 = f(2)\{\cos(b\log2-b\log1)+\cos(b\log2-b\log2)+\cos(b\log2-b\log3)+\cos(b\log2-b\log4)+ ---\}$$
$$-f(3)\{\cos(b\log3-b\log1)+\cos(b\log3-b\log2)+\cos(b\log3-b\log3)+\cos(b\log3-b\log4)+ ---\}$$
$$+f(4)\{\cos(b\log4-b\log1)+\cos(b\log4-b\log2)+\cos(b\log4-b\log3)+\cos(b\log4-b\log4)+ ---\}$$
$$-f(5)\{\cos(b\log5-b\log1)+\cos(b\log5-b\log2)+\cos(b\log5-b\log3)+\cos(b\log5-b\log4)+ ---\}$$
$$+f(6)\{\cos(b\log6-b\log1)+\cos(b\log6-b\log2)+\cos(b\log6-b\log3)+\cos(b\log6-b\log4)+ ---\}$$
$$- ----------$$
$$= f(2)g(2)-f(3)g(3)+f(4)g(4)-f(5)g(5)+f(6)g(6)-f(7)g(7)+ ------ \quad (14)$$

$g(2)\neq0$ and $g(k)/g(2)=1$ (k=3,4,5,6,7 ----) are true as shown in [Appendix 2: Proof of $g(2)\neq0$] and [Appendix 3: Proof of $g(k)/g(2)=1$].

Here we define F(a) as follows.

$$F(a) = f(2)-f(3)+f(4)-f(5)+f(6)- ----- \quad (15)$$

From (14), $g(2)\neq0$, $g(k)/g(2)=1$ (k=3,4,5,6,7 ----) and (15) we have the following (16).

$$0 = g(2)\{f(2)-\frac{f(3)g(3)}{g(2)}+\frac{f(4)g(4)}{g(2)}-\frac{f(5)g(5)}{g(2)}+\frac{f(6)g(6)}{g(2)}-\frac{f(7)g(7)}{g(2)}+ \text{------}\}$$
$$= g(2)\{f(2)-f(3)+f(4)-f(5)+f(6)- \text{------}\}$$
$$= g(2)F(a) \tag{16}$$

In (16) $F(a)=0$ must be true because of $g(2)\neq 0$.

4 Riemann hypothesis shown from F(a)=0

F(a)=0 has the only one solution of a=0 as shown in [Appendix 4: Solution for F(a)=0 (1)] or [Appendix 5: Solution for F(a)=0 (2)]. a has the range of $0\leqq a<1/2$ by the critical strip of $\zeta(s)$. But a cannot have any value but zero because a is the solution for F(a)=0.

$$S_0 = 1/2+a+bi \tag{2}$$
$$S_1 = 1-S_0 = 1/2-a-bi \tag{3}$$

Due to a=0 non-trivial zero point of Riemann zeta function $\zeta(s)$ shown by the above 2 equations must be $1/2\pm bi$ and other zero point does not exist. Therefore Riemann hypothesis which says "All non-trivial zero points of Riemann zeta function $\zeta(s)$ exist on the line of Re(s)=1/2." is true.

In (16) F(a)=0 must be true and F(a) is a monotonically increasing function as shown in [Appendix 5: Solution for F(a)=0 (2)]. So F(a)=0 has the only one solution. If the solution were not a=0, there would not be any zero points on the line of Re(s)=1/2. This assumption is contrary to the following (Fact 1) or (Fact 2). Therefore the only one solution for F(a)=0 must be a=0 and Riemann hypothesis must be true.

Fact 1: In 1914 G. H. Hardy proved that there are infinite zero points on the line of Re(s)=1/2.

Fact 2: All zero points found until now exist on the line of Re(s)=1/2.

Appendix 1: Equation construction

We can construct (9), (10), (11) and (14) by applying the following existing theorem 1(*).

Theorem 1: On condition that the following (Series 1) and (Series 2) converge, the following (Series 3) and (Series 4) are true.

(Series 1) = $a_1+a_2+a_3+a_4+a_5+$ ----- = A

(Series 2) = $b_1+b_2+b_3+b_4+b_5+$ ----- = B

(Series 3) = $(a_1+b_1)+(a_2+b_2)+(a_3+b_3)+(a_4+b_4)+(a_5+b_5)+$ ----- = A+B

(Series 4) = $(a_1-b_1)+(a_2-b_2)+(a_3-b_3)+(a_4-b_4)+(a_5-b_5)+$ ----- = A−B

1 Construction of (9)

We can have the following (9) as (Series 4) by regarding (6) and (4) as (Series 1) and (Series 2) respectively.

$$\text{(Series 1)} = \frac{\cos(b\log 2)}{2^{1/2-a}} - \frac{\cos(b\log 3)}{3^{1/2-a}} + \frac{\cos(b\log 4)}{4^{1/2-a}} - \frac{\cos(b\log 5)}{5^{1/2-a}} + \frac{\cos(b\log 6)}{6^{1/2-a}} - \text{-----} = 1 \qquad (6)$$

$$\text{(Series 2)} = \frac{\cos(b\log 2)}{2^{1/2+a}} - \frac{\cos(b\log 3)}{3^{1/2+a}} + \frac{\cos(b\log 4)}{4^{1/2+a}} - \frac{\cos(b\log 5)}{5^{1/2+a}} + \frac{\cos(b\log 6)}{6^{1/2+a}} - \text{-----} = 1 \qquad (4)$$

$$\text{(Series 4)} = f(2)\cos(b\log 2) - f(3)\cos(b\log 3) + f(4)\cos(b\log 4) - f(5)\cos(b\log 5) + \text{-----} = 1-1 = 0 \qquad (9)$$

Here
$$f(n) = \frac{1}{n^{1/2-a}} - \frac{1}{n^{1/2+a}} \geqq 0 \qquad (n=2,3,4,5,6, \text{------}) \qquad (8)$$

2 Construction of (10)

We can have the following (10) as (Series 4) by regarding (7) and (5) as (Series 1) and (Series 2) respectively.

$$\text{(Series 1)} = \frac{\sin(b\log 2)}{2^{1/2-a}} - \frac{\sin(b\log 3)}{3^{1/2-a}} + \frac{\sin(b\log 4)}{4^{1/2-a}} - \frac{\sin(b\log 5)}{5^{1/2-a}} + \frac{\sin(b\log 6)}{6^{1/2-a}} - \text{-----} = 0 \qquad (7)$$

$$\text{(Series 2)} = \frac{\sin(b\log 2)}{2^{1/2+a}} - \frac{\sin(b\log 3)}{3^{1/2+a}} + \frac{\sin(b\log 4)}{4^{1/2+a}} - \frac{\sin(b\log 5)}{5^{1/2+a}} + \frac{\sin(b\log 6)}{6^{1/2+a}} - \text{-----} = 0 \qquad (5)$$

$$\text{(Series 4)} = f(2)\sin(b\log 2) - f(3)\sin(b\log 3) + f(4)\sin(b\log 4) - f(5)\sin(b\log 5) + \text{-----} = 0-0 = 0 \qquad (10)$$

(*): Please refer to page 22 in "Introduction to infinite series" by Yukio Kusunoki, published in 2004, (written in Japanese)

3 Construction of (11)

We can have the following (11) as (Series 3) by regarding the following equations as (Series 1) and (Series 2).

(Series 1) = cosx{right side of (9)}

$$= \cos x\{f(2)\cos(b\log 2)-f(3)\cos(b\log 3)+f(4)\cos(b\log 4)-f(5)\cos(b\log 5)+ \text{-----}\} = 0$$

(Series 2) = sinx{right side of (10)}

$$= \sin x\{f(2)\sin(b\log 2)-f(3)\sin(b\log 3)+f(4)\sin(b\log 4)-f(5)\sin(b\log 5)+ \text{-----}\} = 0$$

$$(\text{Series 3}) = f(2)\cos(b\log 2-x)-f(3)\cos(b\log 3-x)+f(4)\cos(b\log 4-x)-f(5)\cos(b\log 5-x)+ \text{----} = 0+0 \qquad (11)$$

4 Construction of (14)

4.1 We can have the following (12-1*2) as (Series 3) by regarding (12-1) and (12-2) as (Series 1) and (Series 2) respectively.

$$(\text{Series 1}) = f(2)\cos(b\log 2-b\log 1)-f(3)\cos(b\log 3-b\log 1)+f(4)\cos(b\log 4-b\log 1)$$
$$-f(5)\cos(b\log 5-b\log 1)+f(6)\cos(b\log 6-b\log 1)+ \text{-----} = 0 \qquad (12\text{-}1)$$

$$(\text{Series 2}) = f(2)\cos(b\log 2-b\log 2)-f(3)\cos(b\log 3-b\log 2)+f(4)\cos(b\log 4-b\log 2)$$
$$-f(5)\cos(b\log 5-b\log 2)+f(6)\cos(b\log 6-b\log 2)+ \text{-----} = 0 \qquad (12\text{-}2)$$

$$(\text{Series 3}) = f(2)\{\cos(b\log 2-b\log 1)+\cos(b\log 2-b\log 2)\}$$
$$-f(3)\{\cos(b\log 3-b\log 1)+\cos(b\log 3-b\log 2)\}$$
$$+f(4)\{\cos(b\log 4-b\log 1)+\cos(b\log 4-b\log 2)\}$$
$$-f(5)\{\cos(b\log 5-b\log 1)+\cos(b\log 5-b\log 2)\}$$
$$+f(6)\{\cos(b\log 6-b\log 1)+\cos(b\log 6-b\log 2)\}- \text{-----} = 0+0 \qquad (12\text{-}1*2)$$

4.2 We can have the following (12-1*3) as (Series 3) by regarding (12-1*2) and (12-3) as (Series 1) and (Series 2) respectively.

$$(\text{Series 2}) = f(2)\cos(b\log 2-b\log 3)-f(3)\cos(b\log 3-b\log 3)+f(4)\cos(b\log 4-b\log 3)$$
$$-f(5)\cos(b\log 5-b\log 3)+f(6)\cos(b\log 6-b\log 3)+ \text{-----} = 0 \qquad (12\text{-}3)$$

$$(\text{Series 3}) = f(2)\{\cos(b\log 2-b\log 1)+\cos(b\log 2-b\log 2)+\cos(b\log 2-b\log 3)\}$$
$$-f(3)\{\cos(b\log 3-b\log 1)+\cos(b\log 3-b\log 2)+\cos(b\log 3-b\log 3)\}$$
$$+f(4)\{\cos(b\log 4-b\log 1)+\cos(b\log 4-b\log 2)+\cos(b\log 4-b\log 3)\}$$
$$-f(5)\{\cos(b\log 5-b\log 1)+\cos(b\log 5-b\log 2)+\cos(b\log 5-b\log 3)\}$$
$$+f(6)\{\cos(b\log 6-b\log 1)+\cos(b\log 6-b\log 2)+\cos(b\log 6-b\log 3)\}$$
$$- \text{------} = 0+0 \qquad (12\text{-}1*3)$$

4.3 We can have the following (12-1*4) as (Series 3) by regarding (12-1*3) and (12-4) as (Series 1) and (Series 2) respectively.

$$(\text{Series 2}) = f(2)\cos(b\log 2-b\log 4)-f(3)\cos(b\log 3-b\log 4)+f(4)\cos(b\log 4-b\log 4)$$
$$-f(5)\cos(b\log 5-b\log 4)+f(6)\cos(b\log 6-b\log 4)+ \text{-----} = 0 \qquad (12\text{-}4)$$

$$
\begin{aligned}
\text{(Series 3)} = {} & f(2)\{\cos(b\log 2-b\log 1)+\cos(b\log 2-b\log 2)+\cos(b\log 2-b\log 3)+\cos(b\log 2-b\log 4)\} \\
& -f(3)\{\cos(b\log 3-b\log 1)+\cos(b\log 3-b\log 2)+\cos(b\log 3-b\log 3)+\cos(b\log 3-b\log 4)\} \\
& +f(4)\{\cos(b\log 4-b\log 1)+\cos(b\log 4-b\log 2)+\cos(b\log 4-b\log 3)+\cos(b\log 4-b\log 4)\} \\
& -f(5)\{\cos(b\log 5-b\log 1)+\cos(b\log 5-b\log 2)+\cos(b\log 5-b\log 3)+\cos(b\log 5-b\log 4)\} \\
& +f(6)\{\cos(b\log 6-b\log 1)+\cos(b\log 6-b\log 2)+\cos(b\log 6-b\log 3)+\cos(b\log 6-b\log 4)\} \\
& - \text{------} = 0+0 \qquad (12\text{-}1{*}4)
\end{aligned}
$$

4.4 In the same way as above we can have (12-1*n) as (Series 3) by regarding (12-1*n-1) and (12-n) as (Series 1) and (Series 2) respectively. If we repeat this operation infinitely i.e. we do $n\to\infty$, we can have (12-1*∞)=(14).

Appendix 2: Proof of $g(2)\neq 0$

1 Proof (1)

1.1 Investigation of g(2)

We define g(2,N) as the partial sum from the first term of g(2) to the N-th term of g(2). (N=1,2,3,4,5,-----) From (15) g(2,N) is as follows. $\lim_{N\to\infty} g(2,N)$ means g(2).

$$g(2,N) = \cos(b\log 1/2)+\cos(b\log 2/2)+\cos(b\log 3/2)+\cos(b\log 4/2)+\cos(b\log 5/2) + ---- +\cos(b\log N/2)$$

$$= N\left(\frac{1}{N}\right)\left[\cos\left\{b\log\left(\frac{1}{N}\right)\left(\frac{N}{2}\right)\right\}+\cos\left\{b\log\left(\frac{2}{N}\right)\left(\frac{N}{2}\right)\right\}+\cos\left\{b\log\left(\frac{3}{N}\right)\left(\frac{N}{2}\right)\right\}+\cos\left\{b\log\left(\frac{4}{N}\right)\left(\frac{N}{2}\right)\right\}\right.$$

$$\left.+\cos\left\{b\log\left(\frac{5}{N}\right)\left(\frac{N}{2}\right)\right\}+ ----- +\cos\left\{b\log\left(\frac{N}{N}\right)\left(\frac{N}{2}\right)\right\}\right]$$

$$= N(1/N)\{\cos(b\log 1/N+b\log N/2)+\cos(b\log 2/N+b\log N/2)+\cos(b\log 3/N+b\log N/2)$$
$$+\cos(b\log 4/N+b\log N/2)+\cos(b\log 5/N+b\log N/2)+ ---- +\cos(b\log N/N+b\log N/2)\}$$

$$= N(1/N)\{\cos(b\log N/2)\}\{\cos(b\log 1/N)+\cos(b\log 2/N)+\cos(b\log 3/N)+ ---- +\cos(b\log N/N)\}$$
$$- N(1/N)\{\sin(b\log N/2)\}\{\sin(b\log 1/N)+\sin(b\log 2/N)+\sin(b\log 3/N)+ ---- +\sin(b\log N/N)\}$$

Here we do $N\to\infty$ as follows.

$$\lim_{N\to\infty} g(2,N)= g(2)$$

$$= \lim_{N\to\infty}\{N\cos(b\log N/2)\}\lim_{N\to\infty}(1/N)\{\cos(b\log 1/N)+\cos(b\log 2/N)+\cos(b\log 3/N)+ ---- +\cos(b\log N/N)\}$$
$$- \lim_{N\to\infty}\{N\sin(b\log N/2)\}\lim_{N\to\infty}(1/N)\{\sin(b\log 1/N)+\sin(b\log 2/N)+\sin(b\log 3/N)+ ---- +\sin(b\log N/N)\}$$

$$=\lim_{N\to\infty}\{N\cos(b\log N/2)\}\int_0^1\cos(b\log x)\,dx - \lim_{N\to\infty}\{N\sin(b\log N/2)\}\int_0^1\sin(b\log x)\,dx \qquad (21)$$

We define A and B as follows.

$$A = \int_0^1\cos(b\log x)\,dx \qquad B = \int_0^1\sin(b\log x)\,dx$$

We calculate A and B.

$$A = [x\cos(b\log x)]_0^1 + bB = 1 + bB$$

$$B = [x\sin(b\log x)]_0^1 - bA = -bA$$

Then we can have the values of A and B from the above equations as follows.

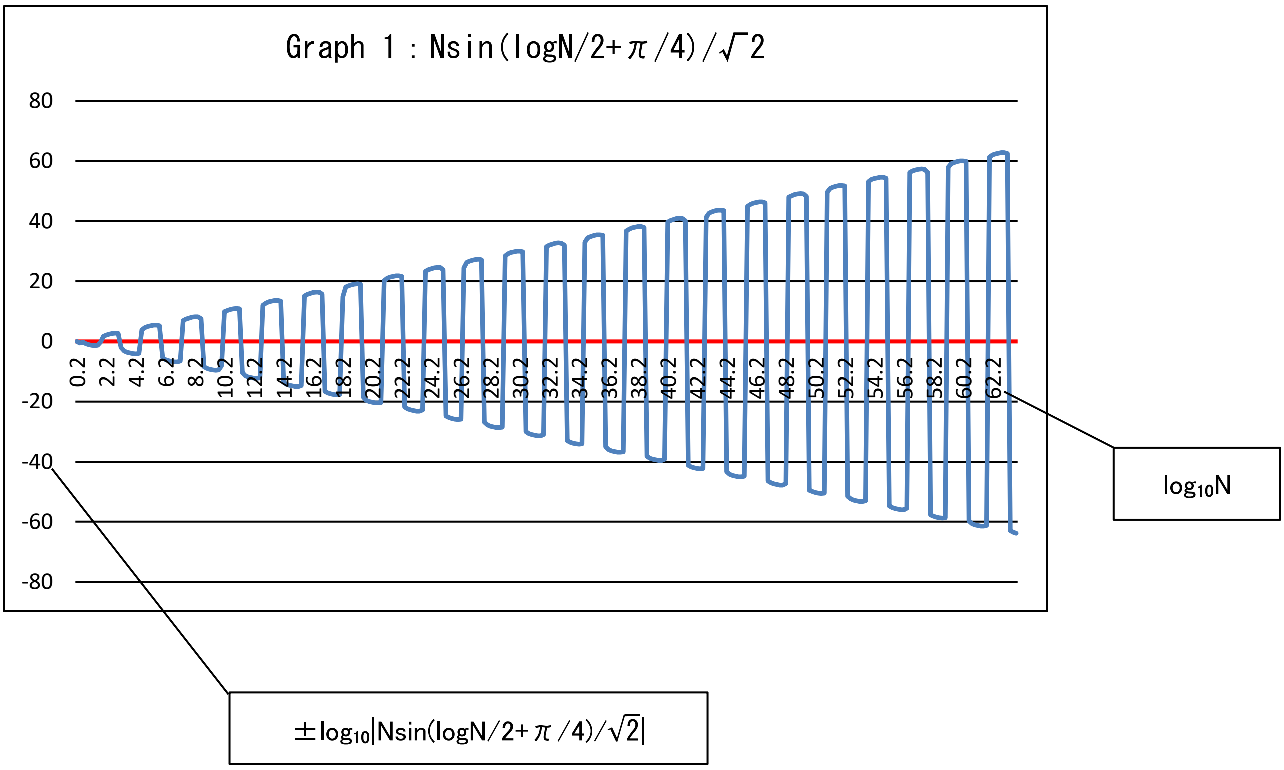
$$A = 1/(1+b^2) \qquad B = -b/(1+b^2)$$

We have the following (22) by substituting the above values of A and B for $\int_0^1\cos(b\log x)dx$ and $\int_0^1\sin(b\log x)dx$ in (21).

$$g(2) = \lim_{N\to\infty} \{N\cos(b\log N/2)\}\{1/(1+b^2)\} - \lim_{N\to\infty} \{N\sin(b\log N/2)\}\{-b/(1+b^2)\}$$

$$= \frac{\lim_{N\to\infty} N\{\cos(b\log N/2)+b\sin(b\log N/2)\}}{1+b^2} = \frac{\lim_{N\to\infty} N\sin\{b\log N/2+\tan^{-1}(1/b)\}}{\sqrt{1+b^2}} \qquad (22)$$

(Graph 1) shows the value of $[N\sin\{b\log N/2+\tan^{-1}(1/b)\}/\sqrt{1+b^2}$ at $b=1]$. The scale of horizontal axis is $\log_{10}N$ and the scale of vertical axis is $\pm\log_{10}|N\sin(\log N/2+\pi/4)/\sqrt{2}|$. $\pm$ is subject to the sign of $\sin(\log N/2+\pi/4)$.



1.2 Verification of $\sin\{b\log N/2+\tan^{-1}(1/b)\}\neq 0$

If we assume $\sin\{b\log N/2+\tan^{-1}(1/b)\}=0$ (N=3, 4, 5, 6, 7, ------), the following (23) is supposed to be true.

$$b\log N/2+\tan^{-1}(1/b) = k\pi \qquad (k=1, 2, 3, 4, ------) \qquad (23)$$

In (23) k is natural number because of $0<$ {left side of (23)} that is due to $0<b$, $0<\log N/2$ and $0<\tan^{-1}(1/b)<\pi/2$ as shown in item 1.2.1.

1.2.1 $\tan^{-1}(1/b)$ has the value of $L\pi$ as shown in (Table 1) and the range of L is $0<L<1/2$.

Table 1 : Value of $\tan^{-1}(1/b)$

b	0	$1/\sqrt{3}$	1	$\sqrt{3}$	∞
$\tan^{-1}(1/b)$	$\pi/2$	$\pi/3$	$\pi/4$	$\pi/6$	0

1.2.2 From (23)

$$b\log N/2 + L\pi = k\pi$$

$$\log N/2 = \frac{(k-L)\pi}{b} = M\pi$$

$k-L>1/2$ due to $1\leqq k$ and $0<L<1/2$. $(k-L)\pi/b=M>0$ due to $0<b$ and $k-L>1/2$.

$$N/2 = e^{M\pi}$$

$$N = 2e^{M\pi} \qquad (24)$$

1.2.3 N is natural number. (24) has impossible formation like (natural number) = (irrational number). Therefore (24) is false and (23) (which is the original formula of (24)) is also false. Now we can have the following (25).

$$\sin\{b\log N/2+\tan^{-1}(1/b)\} \neq 0 \quad (N=3,4,5,6,7,\text{-----}) \qquad (25)$$

1.3 Verification of $g(2)\neq 0$

$$g(2) = \frac{\lim_{N\to\infty} N\sin\{b\log N/2+\tan^{-1}(1/b)\}}{\sqrt{1+b^2}} \neq 0$$

The above inequality is true due to the following reasons.

1.3.1 $\lim_{N\to\infty}\sin\{b\log N/2+\tan^{-1}(1/b)\}$ fluctuates between -1 and 1 during $N\to\infty$.

So $\lim_{N\to\infty} N\sin\{b\log N/2+\tan^{-1}(1/b)\}$ diverges to $\pm\infty$ as shown in (Graph 1) in the previous page. Therefore g(2) does not converge to zero.

1.3.2 g(2) cannot be zero during $N\to\infty$ due to the above (25) as verified in item 1.2.

2 Proof (2)

If we assume g(2)=0,the following (26) is supposed to be true from (22).

$$g(2) = \lim_{N\to\infty}\{N\cos(b\log N/2)\}\{1/(1+b^2)\} - \lim_{N\to\infty}\{N\sin(b\log N/2)\}\{-b/(1+b^2)\} = 0 \qquad (26)$$

The following (27) and (28) are true because of the following reasons.

2.1 $\lim_{N\to\infty}\{N\cos(b\log N/2)\}$ and $\lim_{N\to\infty}\{N\sin(b\log N/2)\}$ diverge to $\pm\infty$ and does not converge to zero.

2.2 In (N=3,4,5,6,7,-----) we can confirm $\sin(b\log N/2)\neq 0$ by putting L=0 in item 1.2. Hence $\lim_{N\to\infty}\{N\sin(b\log N/2)\}$ cannot be zero during $N\to\infty$.

In (N=3,4,5,6,7,-----) we can confirm

$$\cos(b\log N/2) = \sin(b\log N/2+\pi/2)\neq 0$$

by putting L=1/2 in item 1.2. Hence $\lim_{N\to\infty}\{N\cos(b\log N/2)\}$ cannot be zero during $N\to\infty$.

(N=3,4,5,6,7,-----)
$$\lim_{N\to\infty}\{N\cos(b\log N/2)\}\{1/(1+b^2)\}\neq 0 \tag{27}$$

$$\lim_{N\to\infty}\{N\sin(b\log N/2)\}\{-b/(1+b^2)\}\neq 0 \tag{28}$$

From (26),(27) and (28) we have the following (29).

$$\frac{\lim_{N\to\infty}\{N\sin(b\log N/2)\}\{-b/(1+b^2)\}}{\lim_{N\to\infty}\{N\cos(b\log N/2)\}\{1/(1+b^2)\}} = 1 \tag{29}$$

From (29) we have the following (30).

$$\frac{\lim_{N\to\infty}\{N\sin(b\log N/2)\}}{\lim_{N\to\infty}\{N\cos(b\log N/2)\}} = \frac{\lim_{N\to\infty}\{\sin(b\log N/2)\}}{\lim_{N\to\infty}\{\cos(b\log N/2)\}} = \lim_{N\to\infty}\tan(b\log N/2) = \frac{-1}{b} \tag{30}$$

But tangent function fluctuates between $-\infty$ and $+\infty$ during $N\to\infty$ and does not converge to the fixed value. So (30) is false and (26) (which is the original formula of (30)) is also false. Therefore we can confirm $g(2)\neq 0$.

Appendix 3: Proof of g(k)/g(2)=1

1. Introduction

We can have the following equation for g(k) by calculating in the same way as that for g(2) in item 1.1 of Appendix 2.

$$g(k) = \frac{\lim_{N\to\infty} N\sin\{b\log N/k+\tan^{-1}(1/b)\}}{\sqrt{1+b^2}} \qquad (k=3,4,5,6,7 \text{ ----}) \tag{31}$$

We define h(2, N) and h(k, N) as follows.

$$h(2,N) = b\log N/2 + \tan^{-1}(1/b)$$

$$h(k,N) = b\log N/k + \tan^{-1}(1/b)$$

We have the following 2 equations from the above definition.

$$\lim_{N\to\infty}\frac{h(2,N)}{h(k,N)} = \lim_{N\to\infty}\frac{b\log N/2 + \tan^{-1}(1/b)}{b\log N/k + \tan^{-1}(1/b)} = \lim_{N\to\infty}\frac{1-\log 2/\log N+\tan^{-1}(1/b)/b\log N}{1-\log k/\log N+\tan^{-1}(1/b)/b\log N} = 1$$

$$\lim_{N\to\infty}\lim_{n\to\infty}\left\{\frac{h(2,N)}{h(k,N)}\right\}^{2n-1} = \lim_{N\to\infty}\left\{\frac{h(2,N)}{h(k,N)}\right\}^{\infty} = 1^{\infty} = 1$$

We have the following (32) from the above equation.

$$\lim_{N\to\infty}\lim_{n\to\infty}\left\{\frac{h(2,N)}{h(k,N)}\right\}^{2n-1} = \lim_{N\to\infty}\lim_{n\to\infty}\left\{\frac{1/h(k,N)}{1/h(2,N)}\right\}^{2n-1} = \frac{\lim_{N\to\infty}\lim_{n\to\infty}\{1/h(k,N)^{2n-1}\}}{\lim_{N\to\infty}\lim_{n\to\infty}\{1/h(2,N)^{2n-1}\}} = 1 \tag{32}$$

From (22), (31) and (32) g(k)/g(2) is calculated as follows.

$$\frac{g(k)}{g(2)} = \frac{\lim_{N\to\infty} N\sin\{b\log N/k+\tan^{-1}(1/b)\}}{\lim_{N\to\infty} N\sin\{b\log N/2+\tan^{-1}(1/b)\}} = \frac{\lim_{N\to\infty}\sin\{b\log N/k+\tan^{-1}(1/b)\}}{\lim_{N\to\infty}\sin\{b\log N/2+\tan^{-1}(1/b)\}}$$

$$= \frac{\lim_{N\to\infty}\sin\{h(k,N)\}}{\lim_{N\to\infty}\sin\{h(2,N)\}} = \frac{\lim_{N\to\infty}\lim_{n\to\infty}\{1/h(k,N)^{2n-1}\}\lim_{N\to\infty}\sin\{h(k,N)\}}{\lim_{N\to\infty}\lim_{n\to\infty}\{1/h(2,N)^{2n-1}\}\lim_{N\to\infty}\sin\{h(2,N)\}}$$

$$= \frac{\lim_{N\to\infty}[\sin\{h(k,N)\}/\lim_{n\to\infty}h(k,N)^{2n-1}]}{\lim_{N\to\infty}[\sin\{h(2,N)\}/\lim_{n\to\infty}h(2,N)^{2n-1}]} \tag{33}$$

2 verification of $\lim_{N\to\infty}\frac{\sin\{h(2,N)\}}{\lim_{n\to\infty}h(2,N)^{2n-1}} = \lim_{n\to\infty}\frac{(-1)^{n-1}}{(2n-1)!}$ (1)

The denominator of (33) is calculated by performing Mclaughlin expansion for sin{h(2, N)} as follows.

$$\lim_{N\to\infty}\frac{\sin\{h(2,N)\}}{\lim\limits_{n\to\infty}h(2,N)^{2n-1}}$$

$$=\lim_{N\to\infty}\frac{\lim\limits_{n\to\infty}\{h(2,N)-\frac{h(2,N)^3}{3!}+\frac{h(2,N)^5}{5!}-\frac{h(2,N)^7}{7!}+\cdots+\frac{(-1)^{n-2}h(2,N)^{2n-3}}{(2n-3)!}+\frac{(-1)^{n-1}h(2,N)^{2n-1}}{(2n-1)!}\}}{\lim\limits_{n\to\infty}h(2,N)^{2n-1}}$$

$$=\lim_{N\to\infty}\lim_{n\to\infty}\frac{h(2,N)-\frac{h(2,N)^3}{3!}+\frac{h(2,N)^5}{5!}-\frac{h(2,N)^7}{7!}+\cdots+\frac{(-1)^{n-2}h(2,N)^{2n-3}}{(2n-3)!}+\frac{(-1)^{n-1}h(2,N)^{2n-1}}{(2n-1)!}}{h(2,N)^{2n-1}}$$

$$=\lim_{N\to\infty}\lim_{n\to\infty}\{h(2,N)^{2-2n}-\frac{h(2,N)^{4-2n}}{3!}+\frac{h(2,N)^{6-2n}}{5!}-\frac{h(2,N)^{8-2n}}{7!}+\cdots+\frac{(-1)^{n-2}h(2,N)^{-2}}{(2n-3)!}+\frac{(-1)^{n-1}}{(2n-1)!}\}\ (*)$$

$$=\lim_{N\to\infty}\lim_{n\to\infty}\{h(2,N)^{2-2n}-\frac{h(2,N)^{4-2n}}{3!}+\frac{h(2,N)^{6-2n}}{5!}-\frac{h(2,N)^{8-2n}}{7!}+\cdots+\frac{(-1)^{n-2}h(2,N)^{-2}}{(2n-3)!}\}$$

$$+\lim_{N\to\infty}\lim_{n\to\infty}\frac{(-1)^{n-1}}{(2n-1)!}$$

$$=\lim_{N\to\infty}\{h(2,N)^{-\infty}-\frac{h(2,N)^{-\infty}}{3!}+\frac{h(2,N)^{-\infty}}{5!}-\frac{h(2,N)^{-\infty}}{7!}+\cdots\}+\lim_{N\to\infty}\lim_{n\to\infty}\frac{(-1)^{n-1}}{(2n-1)!}$$

$$=\lim_{N\to\infty}\lim_{n\to\infty}\frac{(-1)^{n-1}}{(2n-1)!}$$

$$=\lim_{n\to\infty}\frac{(-1)^{n-1}}{(2n-1)!} \tag{34}$$

The 6th equal sine (=) of (34) is true due to $\lim\limits_{N\to\infty}h(2,N)=\infty$.

3 verification of $\lim\limits_{N\to\infty}\frac{\sin\{h(2,N)\}}{\lim\limits_{n\to\infty}h(2,N)^{2n-1}}=\lim\limits_{n\to\infty}\frac{(-1)^{n-1}}{(2n-1)!}$ (2)

From the 3rd formula(*) of (34) we have the following (35).

$$\lim_{N\to\infty}\lim_{n\to\infty}\{h(2,N)^{2-2n}-\frac{h(2,N)^{4-2n}}{3!}+\frac{h(2,N)^{6-2n}}{5!}-\frac{h(2,N)^{8-2n}}{7!}-\cdots+\frac{(-1)^{n-2}h(2,N)^{-2}}{(2n-3)!}+\frac{(-1)^{n-1}}{(2n-1)!}\}\ (*)$$

$$=\lim_{N\to\infty}\{h(2,N)^{-\infty}-\frac{h(2,N)^{-\infty}}{3!}+\frac{h(2,N)^{-\infty}}{5!}-\frac{h(2,N)^{-\infty}}{7!}+\cdots\}$$

$$=0 \tag{35}$$

Here we exchange $\lim\limits_{N\to\infty}$ with $\lim\limits_{n\to\infty}$ each other in the 3rd formula(*) of (34) as follows.

$$\lim_{n\to\infty}\lim_{N\to\infty}\{h(2,N)^{2-2n}-\frac{h(2,N)^{4-2n}}{3!}+\frac{h(2,N)^{6-2n}}{5!}-\frac{h(2,N)^{8-2n}}{7!}+\cdots+\frac{(-1)^{n-2}h(2,N)^{-2}}{(2n-3)!}+\frac{(-1)^{n-1}}{(2n-1)!}\}$$

$$=\lim_{n\to\infty}\frac{(-1)^{n-1}}{(2n-1)!}$$

$$= 0 \tag{36}$$

The 1st equal sign (=) of (36) is true due to $\lim_{N\to\infty} h(2,N) = \infty$.

We can have the following (37) from (35) and (36) as follows.

$$\lim_{N\to\infty}\lim_{n\to\infty}\{h(2,N)^{2-2n}-\frac{h(2,N)^{4-2n}}{3!}+\frac{h(2,N)^{6-2n}}{5!}-\frac{h(2,N)^{8-2n}}{7!}------+\frac{(-1)^{n-2}h(2,N)^{-2}}{(2n-3)!}+\frac{(-1)^{n-1}}{(2n-1)!}\}\ (*)$$

$$=\lim_{n\to\infty}\lim_{N\to\infty}\{h(2,N)^{2-2n}-\frac{h(2,N)^{4-2n}}{3!}+\frac{h(2,N)^{6-2n}}{5!}-\frac{h(2,N)^{8-2n}}{7!}+--------+\frac{(-1)^{n-2}h(2,N)^{-2}}{(2n-3)!}+\frac{(-1)^{n-1}}{(2n-1)!}\}$$

$$= 0 \tag{37}$$

We can have the following (38) from (34), (36) and (37).

$$\lim_{N\to\infty}\frac{\sin\{h(2,N)\}}{\lim_{n\to\infty}h(2,N)^{2n-1}}$$

$$=\lim_{N\to\infty}\lim_{n\to\infty}\{h(2,N)^{2-2n}-\frac{h(2,N)^{4-2n}}{3!}+\frac{h(2,N)^{6-2n}}{5!}--------+\frac{(-1)^{n-2}h(2,N)^{-2}}{(2n-3)!}+\frac{(-1)^{n-1}}{(2n-1)!}\}\ (*)$$

$$=\lim_{n\to\infty}\lim_{N\to\infty}\{h(2,N)^{2-2n}-\frac{h(2,N)^{4-2n}}{3!}+\frac{h(2,N)^{6-2n}}{5!}----------+\frac{(-1)^{n-2}h(2,N)^{-2}}{(2n-3)!}+\frac{(-1)^{n-1}}{(2n-1)!}\}$$

$$=\lim_{n\to\infty}\frac{(-1)^{n-1}}{(2n-1)!} \tag{38}$$

3 Conclusion

From (34) or (38) we can have the following (39).

$$\lim_{N\to\infty}\frac{\sin\{h(2,N)\}}{\lim_{n\to\infty}h(2,N)^{2n-1}} = \lim_{n\to\infty}\frac{(-1)^{n-1}}{(2n-1)!} \tag{39}$$

The numerator of (33) is calculated in the same way as that for the denominator of (33). The result is the following (40).

$$\lim_{N\to\infty}\frac{\sin\{h(k,N)\}}{\lim_{n\to\infty}h(k,N)^{2n-1}} = \lim_{n\to\infty}\frac{(-1)^{n-1}}{(2n-1)!} \tag{40}$$

From (33), (39) and (40) we can have $g(k)/g(2)=1$ as follows.

$$\frac{g(k)}{g(2)} = \frac{\lim_{N\to\infty}[\sin\{h(k,N)\}/\lim_{n\to\infty}h(k,N)^{2n-1}]}{\lim_{N\to\infty}[\sin\{h(2,N)\}/\lim_{n\to\infty}h(2,N)^{2n-1}]} = \frac{\lim_{n\to\infty}\frac{(-1)^{n-1}}{(2n-1)!}}{\lim_{n\to\infty}\frac{(-1)^{n-1}}{(2n-1)!}} = \lim_{n\to\infty}\frac{\frac{(-1)^{n-1}}{(2n-1)!}}{\frac{(-1)^{n-1}}{(2n-1)!}} = \lim_{n\to\infty}1 = 1$$

Appendix 4 : Solution for F(a)=0 (1)

1 Preparation for verification of F(a)＞0

1.1 Investigation of f(n)

$$f(n) = \frac{1}{n^{1/2-a}} - \frac{1}{n^{1/2+a}} \geqq 0 \qquad (n=2,3,4,5,\text{--------}) \tag{8}$$

$$F(a) = f(2)-f(3)+f(4)-f(5)+f(6)- \text{-----} \tag{15}$$

a=0 is the solution for F(a)=0 due to f(n)≡0 at a=0. Hereafter we define the range of a as 0＜a＜1/2 to verify F(a)＞0. The alternating series F(a) converges due to $\lim_{n\to\infty} f(n)=0$.

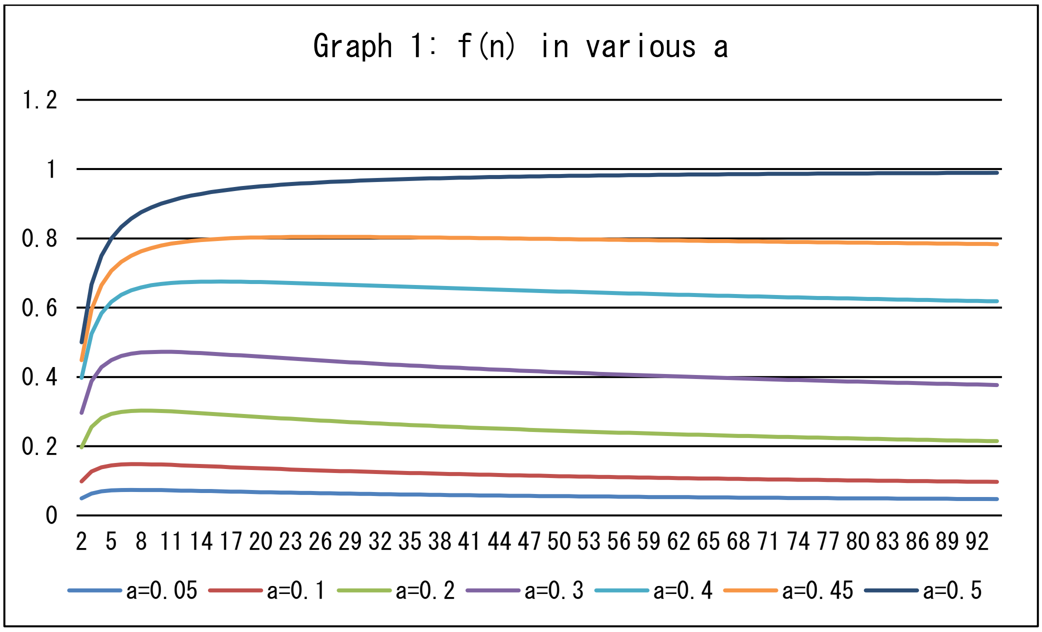
We have the following equation by differentiating f(n) regarding n.

$$\frac{df(n)}{dn} = \frac{1/2+a}{n^{a+3/2}} - \frac{1/2-a}{n^{3/2-a}} = \frac{1/2+a}{n^{a+3/2}}\left\{1 - \left(\frac{1/2-a}{1/2+a}\right)n^{2a}\right\}$$

The value of f(n) increases with the increase of n and reaches the maximum value $f(n_{max})$ at $n=n_{max}$. Afterward f(n) decreases to zero through n→∞.

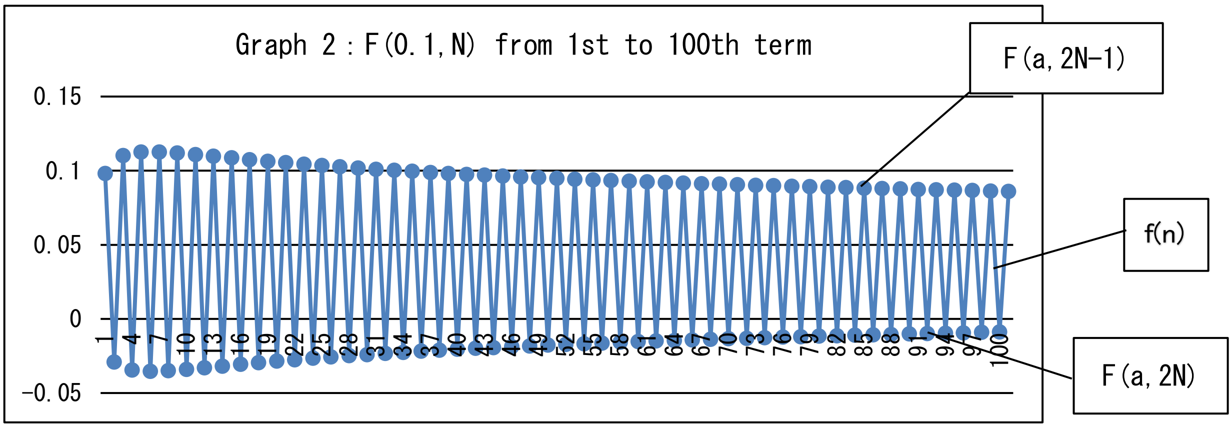
n_{max} is the nearest natural number to $\left(\frac{1/2+a}{1/2-a}\right)^{1/2a}$.

(Graph 1) shows f(n) in various value of a. At a=1/2 f(n) does not have $f(n_{max})$ and increases to 1 through n→∞ due to n_{max} =∞.



1.2 Verification method for $F(a)>0$

We define $F(a,N)$ as the partial sum from the first term of $F(a)$ to the N-th term of $F(a)$. ($N=1,2,3,4,5,$-----) $F(a,N)$ repeats increase and decrease by $f(n)$ with increase of N as shown in (Graph 2), because $F(a)$ is the alternating series. In (Graph 2) upper points mean $F(a,2N-1)$ and lower points mean $F(a,2N)$. $F(a,2N-1)$ decreases and converges to $F(a)$. $F(a,2N)$ increases and also converges to $F(a)$ due to $\lim_{n\to\infty} f(n)=0$.



$F1(a,2N)$ which is the partial sum from the first term of the following $F1(a)$ to the 2N-th term of $F1(a)$ is equal to $F(a,2N)$.

$$F1(a) = \{f(2)-f(3)\}+\{f(4)-f(5)\}+\{f(6)-f(7)\}+\{f(8)-f(9)\}+ \text{-----}$$

Therefore $\lim_{N\to\infty} F1(a,2N)$ also converges to $F(a)$. That means $F(a)=F1(a)$. We use $F1(a)$ instead of $F(a)$ for verifying $F(a)>0$.

On the condition of $n_{max}=k$ or $n_{max}=k+1$ (k:odd number) ,after enclosing 2 terms of $F(a)$ each from the first term with { } as follows, the inside sum of { } from $f(2)$ to $f(k)$ is negative value and the inside sum of { } after $f(k+1)$ is positive value.

$$F(a) = f(2)-f(3)+f(4)-f(5)+f(6)-f(7)+ \text{------}$$
$$=\{f(2)-f(3)\}+\{f(4)-f(5)\}+ \text{---} +\{f(k-1)-f(k)\}+\{f(k+1)-f(k+2)\} + \text{----}$$

(inside sum of { })<0 ← | → (inside sum of { })>0

(total sum of { }) = -B ← | → (total sum of { }) = A

We define as follows.

[the partial sum from $f(2)$ to $f(k)$] = $-B<0$

[the partial sum from $f(k+1)$ to $f(\infty)$] = $A>0$

$F(a) = A-B$

So we can verify $F(a)>0$ by verifying $A>B$.

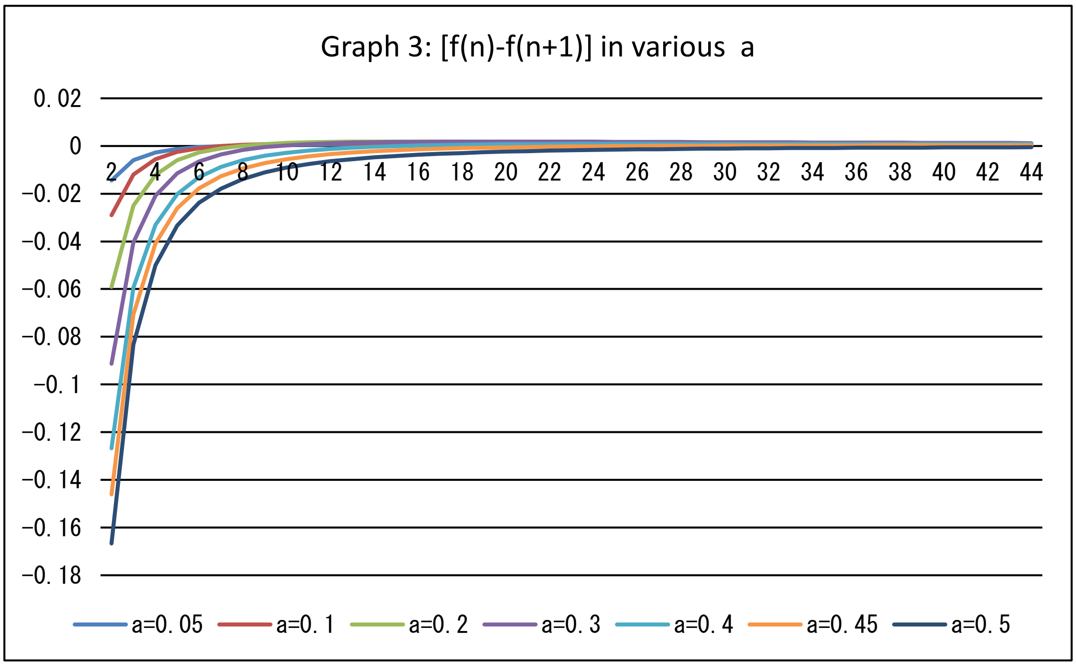
1.3 Investigation of f(n)-f(n+1)

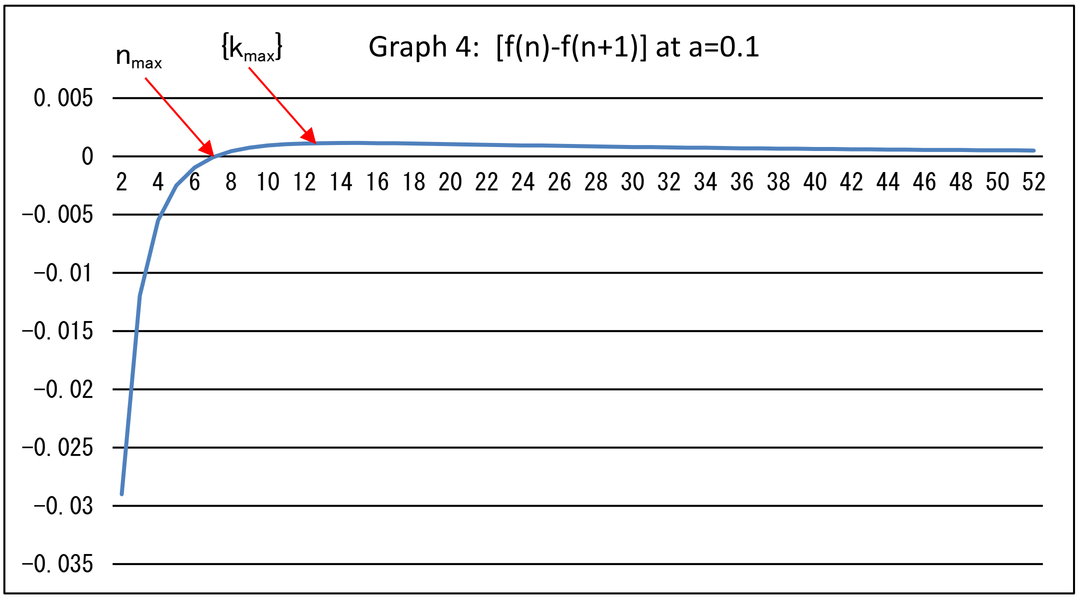
We have the following equation by differentiating [f(n)-f(n+1)] regarding n.

$$\frac{df(n)}{dn} - \frac{df(n+1)}{dn} = \frac{1/2+a}{n^{3/2+a}}\left\{1 - \left(\frac{n}{n+1}\right)^{3/2+a}\right\} - \frac{1/2-a}{n^{3/2-a}}\left\{1 - \left(\frac{n}{n+1}\right)^{3/2-a}\right\}$$

$$= C(n) - D(n)$$

"Convergence velocity to zero" of $n^{-a-3/2}$ is larger than that of $n^{a-3/2}$. When n is small number the value of [f(n)-f(n+1)] increases due to [C(n)>D(n)]. As n increases the value reaches the maximum value $\{k_{max}\}$ at C(n)≒D(n). (n is natural number. The situation cannot be C(n)=D(n).) After that the situation changes to C(n)<D(n) and the value decreases to zero through n→∞. (Graph 3) shows the value of [f(n)-f(n+1)] in various value of a. (Graph 4) shows the value of [f(n)-f(n+1)] at a=0.1.





We can find the following from (Graph 3) and (Graph 4).

1.3.1 The maximum value of $|f(n)-f(n+1)|$ is $f(3)-f(2)$ at same value of a.

1.3.2 In increasing of n the sign of $[f(n)-f(n+1)]$ changes minus to plus at $n=n_{max}$ ($n=n_{max}+1$) when n_{max} is even(odd) number.

1.3.3 After that the value reaches the maximum value $\{k_{max}\}$ and the value decreases to zero through $n\to\infty$.

2 Verification of A>B ($f(n_{max})$ is even-numbered term.)

Hereafter a is fixed within $0<a<1/2$ to find the condition of A>B. $f(n_{max})$ is even-numbered term as follows.

$$F(a) = f(2)-f(3)+f(4)-f(5)+f(6)- \text{------}$$
$$= \{f(2)-f(3)\}+\{f(4)-f(5)\}+ \text{---} +\{f(n_{max}-3)-f(n_{max}-2)\}+\{f(n_{max}-1)-f(n_{max})\}$$
$$+\{f(n_{max}+1)-f(n_{max}+2)\}+\{f(n_{max}+3)-f(n_{max}+4)\}+\{f(n_{max}+5)-f(n_{max}+6)\}+ \text{-----}$$

We can have A and B as follows.

$$B = \{f(3)-f(2)\}+\{f(5)-f(4)\}+\{f(7)-f(6)\}+ \text{---} +\{f(n_{max}-2)-f(n_{max}-3)\}+\{f(n_{max})-f(n_{max}-1)\}$$
$$A = \{f(n_{max}+1)-f(n_{max}+2)\}+\{f(n_{max}+3)-f(n_{max}+4)\}+\{f(n_{max}+5)-f(n_{max}+6)\}+ \text{-----}$$

2.1 Condition of B

We define as follows.

{ } is included within B.

{ } is not included within B.

We have the following equation.

$f(n_{max})-f(2) = \{f(n_{max})-f(n_{max}-1)\}+\{f(n_{max}-1)-f(n_{max}-2)\}+\{f(n_{max}-2)-f(n_{max}-3)\}+ ---$
$+\{f(7)-f(6)\}+\{f(6)-f(5)\}+\{f(5)-f(4)\}+\{f(4)-f(3)\}+\{f(3)-f(2)\}$

And we have the following inequalities from (graph 3) and (graph 4).

$\{f(3)-f(2)\} > \{f(4)-f(3)\} > \{f(5)-f(4)\} > \{f(6)-f(5)\} > \{f(7)-f(6)\} > -----$
$> \{f(n_{max}-2)-f(n_{max}-3)\} > \{f(n_{max}-1)-f(n_{max}-2)\} > \{f(n_{max})-f(n_{max}-1)\} > 0$

Then

$f(n_{max})-f(2)+\{f(3)-f(2)\}$

$= \{f(3)-f(2)\}+\{f(5)-f(4)\}+\{f(7)-f(6)\}+ --- +\{f(n_{max}-2)-f(n_{max}-3)\}+\{f(n_{max})-f(n_{max}-1)\}$

$\|$ $\wedge$ $\wedge$ $\wedge$ $\wedge$ ←Value comparison

$+ \{f(3)-f(2)\}+\{f(4)-f(3)\}+\{f(6)-f(5)\}+ --- +\{f(n_{max}-3)-f(n_{max}-4)\}+\{f(n_{max}-1)-f(n_{max}-2)\}$

$> 2B$ (41)

Due to [Total sum of upper row of (41) = B < Total sum of lower row of (41)], we have the following inequality.

$f(n_{max})-f(2)+\{f(3)-f(2)\} > 2B$ (42)

2.2 Condition of A ($\{k_{max}\}$ is included within A.)

We abbreviate $\{f(n_{max}+k)-f(n_{max}+k+1)\}$ to $\{k\}$ for easy description. (k=0, 1, 2, 3-----) All $\{k\}$ is positive as shown in item 1.2.

We define as follows.

{ } is included within A.

{ } is not included within A.

$\{k_{max}\}$ is the maximum value in all $\{k\}$.

$\{k_{max}\}$ is included within A. Then value comparison of $\{k\}$ is as follows.

$\{1\}<\{2\}<\{3\}<-----<\{k_{max}-3\}<\{k_{max}-2\}<\{k_{max}-1\}<\{k_{max}\}>\{k_{max}+1\}>\{k_{max}+2\}>\{k_{max}+3\}>----$

We have the following equation.

$f(n_{max}+1) = \{f(n_{max}+1)-f(n_{max}+2)\}+\{f(n_{max}+2)-f(n_{max}+3)\}+\{f(n_{max}+3)-f(n_{max}+4)\}$
$+\{f(n_{max}+4)-f(n_{max}+5)\}+ -----$

$= \{1\}+\{2\}+\{3\}+\{4\}+ --- +\{k_{max}-3\}+\{k_{max}-2\}+\{k_{max}-1\}+\{k_{max}\}+\{k_{max}+1\}+\{k_{max}+2\}+\{k_{max}+3\}+ -----$

From the above equation

$f(n_{max}+1)-\{k_{max}-1\}$

$= \{1\}+\{2\}+\{3\}+\{4\}+ --- +\{k_{max}-3\}+\{k_{max}-2\}+\{k_{max}\}+\{k_{max}+1\}+\{k_{max}+2\}+\{k_{max}+3\}+ -----$

←------------- Range 1 ------------→ | ←-------------- Range 2 --------------

(Range 1) and (Range 2) are determined as above.

In (Range 1) value comparison is as follows.

$\{1\}<\{2\}<\{3\}<\{4\}-----\{k_{max}-4\}<\{k_{max}-3\}<\{k_{max}-2\}$

And

Total sum of { } = {1} + {3} + {5} + {7}+ --------- +{k_{max}−4}+{k_{max}−2}

∨ ∨ ∨ ∨ ∨ ←value comparison

Total sum of { } = {2} + {4} + {6} + -------- +{k_{max}−5}+{k_{max}−3}

Therefore Total sum of { } > Total sum of { }

In (Range 2) value comparison is as follows.

{k_{max}} > {k_{max}+1} > {k_{max}+2} > {k_{max}+3} > {k_{max}+4} > {k_{max}+5} -----

And

Total sum of { } = {k_{max}} + {k_{max}+2} + {k_{max}+4} + {k_{max}+6} + ---------

∨ ∨ ∨ ∨ ←value comparison

Total sum of { } = {k_{max}+1} + {k_{max}+3} + {k_{max}+5} + {k_{max}+7} + --------

Therefore Total sum of { } > Total sum of { }

In (Range 1)+(Range 2) we have [A=Total sum of { } >Total sum of { }].

We have the following inequality.

$$f(n_{max}+1) - \{k_{max}-1\} < 2A \qquad (43)$$

2.3 Condition of A ({k_{max}} is not included within A.)

We have the following equations. {k_{max}} is not included within A.

f(n_{max}+1) = {f(n_{max}+1)−f(n_{max}+2)}+{f(n_{max}+2)−f(n_{max}+3)}+{f(n_{max}+3)−f(n_{max}+4)}

+{f(n_{max}+4)−f(n_{max}+5)} + -----

= {1}+{2}+{3}+{4}+ --- +{k_{max}−3}+{k_{max}−2}+{k_{max}−1}+{k_{max}}+{k_{max}+1}+{k_{max}+2}+{k_{max}+3}+ -----

f(n_{max}+1)− {k_{max}}

= {1}+{2}+{3}+{4}+ --- +{k_{max}−3}+{k_{max}−2}+{k_{max}−1}+{k_{max}+1}+{k_{max}+2}+{k_{max}+3}+{k_{max}+4}+ -----

←----------------- Range 1 ----------------→ | ←------------- Range 2 -------------

(Range 1) and (Range 2) are determined as above.

In (Range 1) value comparison is as follows.

{1} < {2} < {3} < {4} <-----< {k_{max}−3} < {k_{max}−2} < {k_{max}−1}

And

Total sum of { } = {1} + {3} + {5} + {7}+ --------- +{k_{max}−3}+{k_{max}−1}

∨ ∨ ∨ ∨ ∨ ←value comparison

Total sum of { } = {2} + {4} + {6} + -------- +{k_{max}−4}+{k_{max}−2}

Therefore Total sum of { } > Total sum of { }

In (Range 2) value comparison is as follows.

$\{k_{max}+1\} > \{k_{max}+2\} > \{k_{max}+3\} > \{k_{max}+4\} > \{k_{max}+5\} > \{k_{max}+6\}$ -----

Total sum of { } = $\{k_{max}+1\} + \{k_{max}+3\} + \{k_{max}+5\} + \{k_{max}+7\} +$ ---------

$\qquad\qquad\qquad\qquad \vee \qquad\quad \vee \qquad\quad \vee \qquad\quad \vee$ ←value comparison

Total sum of { } = $\{k_{max}+2\} + \{k_{max}+4\} + \{k_{max}+6\} + \{k_{max}+8\} +$ --------

Therefore Total sum of { } > Total sum of { }

In (Range 1)+(Range 2) we have [A=total sum of { } >Total sum of { }].

We have the following inequality.

$$f(n_{max}+1) - \{k_{max}\} < 2A \tag{44}$$

2.4 Condition of A>B

From (43) and (44) we have the following inequality.

$$f(n_{max}+1) - [\{k_{max}\} \text{ or } \{k_{max}-1\}] < 2A$$

As shown in item 1.3.1 {f(3)−f(2)} is the maximum in all { }. Then

$\{f(3)-f(2)\} > [\{k_{max}\} \text{ or } \{k_{max}-1\}]$

$\{f(3)-f(2)\} > f(n_{max}) - f(n_{max}+1)$

We have the following inequality from the above conditions.

$$2A > f(n_{max}+1) - [\{k_{max}\} \text{ or } \{k_{max}-1\}] > f(n_{max}+1) - \{f(3)-f(2)\}$$
$$> f(n_{max}) - \{f(3)-f(2)\} - \{f(3)-f(2)\} = f(n_{max}) - 2\{f(3)-f(2)\} \tag{45}$$

We have the following condition for A>B from (42) and (45).

$$2A > f(n_{max}) - 2\{f(3)-f(2)\} > f(n_{max}) - f(2) + \{f(3)-f(2)\} > 2B \tag{46}$$

From (46) we can have the final condition as follows.

$$(4/3)f(2) > f(3) \tag{47}$$

(Graph 6) shows (4/3)f(2)−f(3) = $(4/3)(2^{a-1/2}-2^{-a-1/2})-(3^{a-1/2}-3^{-a-1/2})$.

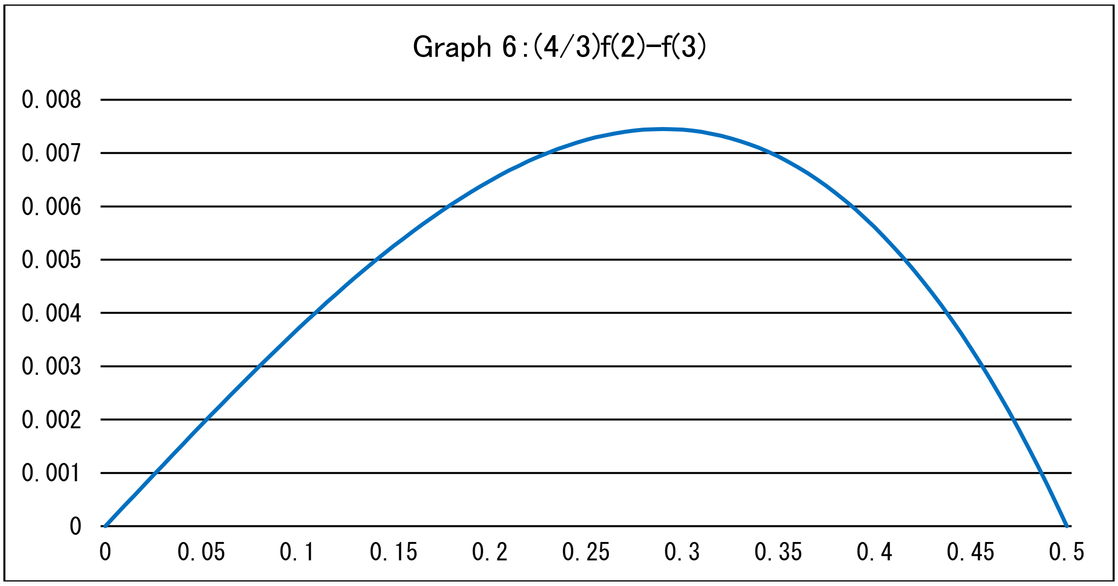


Table 1 : The values of (4/3)f(2)−f(3)

a=	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
(4/3)f(2)−f(3)	0	0.001903	0.003694	0.005257	0.00648	0.007246	0.007437	0.006933	0.005611	0.003343	0

(Graph 7) shows [differentiated (4/3)f(2)−f(3) regarding a] i.e.

(4/3)f\`(2)−f\`(3) = $(4/3)\{\log 2(2^{a-1/2}+2^{-a-1/2})\}-\{\log 3(3^{a-1/2}+3^{-a-1/2})\}$.

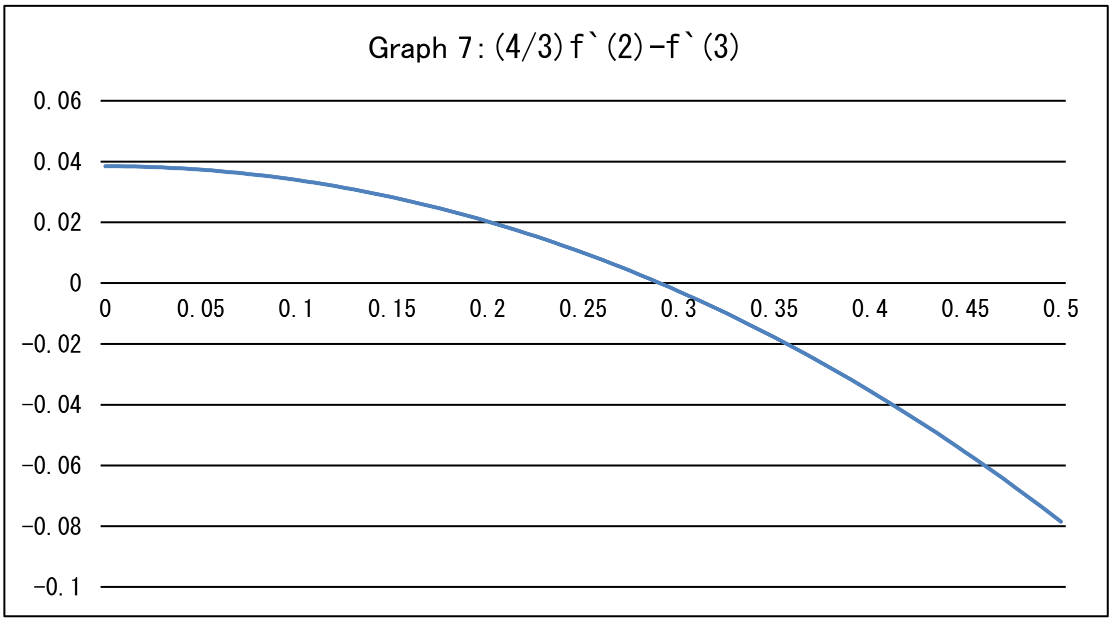


Table 2 : The values of (4/3)f\`(2)−f\`(3)

a=	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
(4/3)f'(2)−f'(3)	0.038443	0.037313	0.033921	0.02825	0.020277	0.009967	−0.00272	−0.01785	−0.03547	−0.05567	−0.07852

From (Graph 6) and (Graph 7) we can find [(4/3)f(2)−f(3)>0 in 0<a<1/2] that means A>B i.e. F(a)>0 in 0<a<1/2.

3 Verification of A>B ($f(n_{max})$ is odd-numbered term.)

$f(n_{max})$ is odd-numbered term as follows.

$F(a) = f(2)-f(3)+f(4)-f(5)+f(6)-$ ------

$= \{f(2)-f(3)\}+\{f(4)-f(5)\}+$ --- $+\{f(n_{max}-4)-f(n_{max}-3)\}+\{f(n_{max}-2)-f(n_{max}-1)\}$

$+\{f(n_{max})-f(n_{max}+1)\}+\{f(n_{max}+2)-f(n_{max}+3)\}$ + -----

And

$B = \{f(3)-f(2)\}+\{f(5)-f(4)\}+$ --- $+\{f(n_{max}-3)-f(n_{max}-4)\}+\{f(n_{max}-1)-f(n_{max}-2)\}$

$A = \{f(n_{max})-f(n_{max}+1)\}+\{f(n_{max}+2)-f(n_{max}+3)\}+\{f(n_{max}+4)-f(n_{max}+5)\}+$ -----

$f(n_{max}) = \{f(n_{max})-f(n_{max}+1)\}+\{f(n_{max}+1)-f(n_{max}+2)\}+\{f(n_{max}+2)-f(n_{max}+3)\}+\{f(n_{max}+3)-f(n_{max}+4)\}$ + -----

$= \{0\}+\{1\}+\{2\}+\{3\}+$ --- $+\{k_{max}-3\}+\{k_{max}-2\}+\{k_{max}-1\}+\{k_{max}\}+\{k_{max}+1\}+\{k_{max}+2\}+\{k_{max}+3\}+$ -----

After the same process as in item 2 we can have the following condition.

$$f(n_{max}-1)-f(2)+\{f(3)-f(2)\} > 2B \tag{48}$$

As shown in item 1.3.1 $\{f(3)-f(2)\}$ is the maximum in all { }. Then

$\{f(3)-f(2)\} > [\{k_{max}\}$ or $\{k_{max}-1\}]$

$f(n_{max}) > f(n_{max}-1)$

We have the following inequality from the same process as in item 2 and the above conditions.

$$2A > f(n_{max}) - [\{k_{max}\} \text{ or } \{k_{max}-1\}] > f(n_{max}) - \{f(3)-f(2)\} > f(n_{max}-1) - \{f(3)-f(2)\} \tag{49}$$

We have the following condition for A>B from (48) and (49).

$$2A > f(n_{max}-1)-\{f(3)-f(2)\} > f(n_{max}-1)-f(2)+\{f(3)-f(2)\} > 2B \tag{50}$$

From (50) we can have the final condition as follows.

$$(3/2)f(2) > f(3) \tag{51}$$

In the inequality of $(3/2)f(2) > (4/3)f(2) > f(3) > 0$, $(3/2)f(2) > (4/3)f(2)$ is true self-evidently and in item 2.4 we already confirmed that the following (47) is true in $0<a<1/2$.

$$(4/3)f(2) > f(3) \tag{47}$$

Therefore (51) is true in $0<a<1/2$.

4 Conclusion

$F(a)=0$ has the only one solution of $a=0$ due to $[0 \leqq a < 1/2]$, $[F(0)=0]$ and $[F(a)>0$ in $0<a<1/2]$.

Appendix 5 : Solution for F(a)=0 (2)

1 Investigation of $F(a)_N$

$$f(n) = \frac{1}{n^{1/2-a}} - \frac{1}{n^{1/2+a}} \geqq 0 \qquad (n=2,3,4,5,\text{--------}) \qquad (8)$$

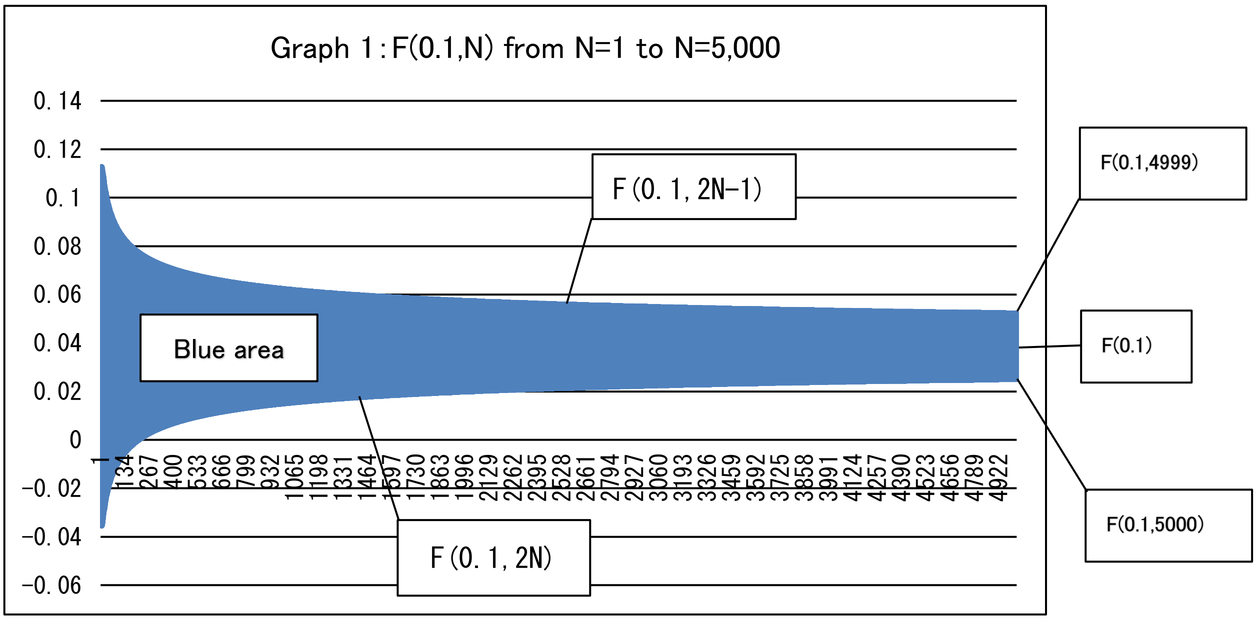
$$F(a) = f(2)-f(3)+f(4)-f(5)+f(6)- \text{-----} \qquad (15)$$

F(a, N) : the partial sum from the first term of F(a) to the N-th term of F(a)

a=0 is the solution for F(a)=0 because of $f(n)\equiv 0$ at a=0. F(a) is the alternating series. So F(a, N) repeats increase and decrease by f(n) with increase of N. $\lim_{N\to\infty} F(a,N)$ converges to F(a) due to $\lim_{n\to\infty} f(n)=0$.

(Graph 1) shows F(0.1, N) from N=1 to N=5,000. The upper edge of blue area shows F(0.1, 2N-1) and lower edge of blue area shows F(0.1, 2N).

((Graph 1) is line graph. Graph has so many data points that the area surrounded by data points becomes blue.)



Upper-right point of blue area, F(0.1, 4999) decreases to F(a) through N→∞ and lower-right point of blue area, F(0.1, 5000) increases to F(a) through N→∞. F(0.1) can be approximated with {F(0.1, 4999)+ F(0.1, 5000)}/2.

But {F(a, N-1)+F(a, N)}/2 is also the partial sum of alternating series. It repeats increase(decrease) of {f(n)-f(n-1)}/2 and decrease(increase) of {f(n+1)-f(n)}/2 when n is even(odd) number. So we approximate F(a) with the average of

{F(a, N−1)+F(a, N)}/2 i.e. $F(a)_N$ for better accuracy according to the following (61).

$$\frac{\frac{F(a,N)+F(a,N-1)}{2}+\frac{F(a,N+1)+F(a,N)}{2}}{2}=F(a)_N \qquad (61)$$

Left side of (61) converges to F(a) through N→∞. We can have the accurate $F(a)_N$ from F(a, N) of large N. (Graph 2) shows $F(a)_N$ calculated at 3 cases of N=500, 1000, 5000.

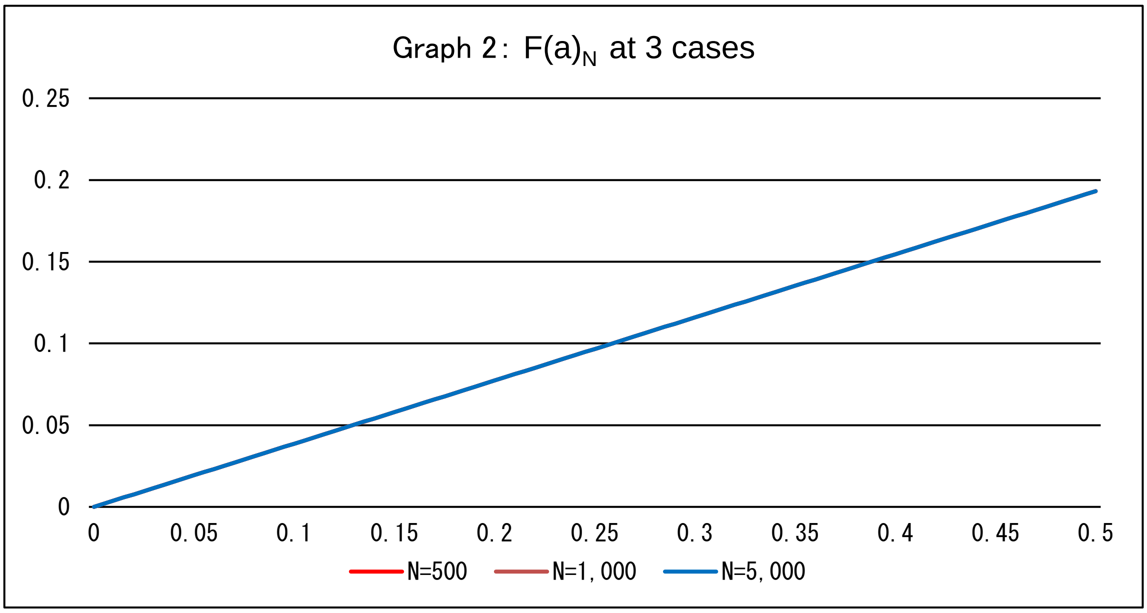


Table 1 : The values of $F(a)_N$

a	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
N=500	0	0.01932876	0.03865677	0.05798326	0.0773074	0.09662832	0.11594507	0.13525658	0.15456168	0.17385904	0.19314718
N=1,000	0	0.01932681	0.03865282	0.05797725	0.0772993	0.09661821	0.11593325	0.13524382	0.15454955	0.17385049	0.19314743
N=5,000	0	0.01932876	0.03865676	0.05798324	0.07730738	0.09662829	0.11594504	0.13525655	0.15456165	0.17385902	0.19314718

3 line graphs overlapped. Because $F(a)_N$ calculated at 3 cases of N=500, 1000, 5000 are equal to 4 digits after the decimal point.

The range of a is $0 \leqq a < 1/2$. a=1/2 is not included in the range. But we added $F(1/2)_N$ to calculation according to the following reason.

[f(n) at a=1/2] is (1−1/n) and $\lim_{n\to\infty}(1-1/n)$ does not converge to zero. Therefore F(1/2) fluctuates due to $\lim_{n\to\infty} f(n)=1$. But {F(a, N)+F(a, N−1)}/2 is partial sum of alternating series with the term of {f(n+1)−f(n)}/2 and it can converge to the fixed value on the condition of $\lim_{n\to\infty}\{f(n+1)-f(n)\}=0$. $\lim_{n\to\infty}\{f(n+1)-f(n)\}$ converges to zero due to $f(n+1)-f(n)=1/(n+n^2)$.

2 Investigation of $F`(a)_N$

We define as follows.

$$f`(n) = df(n)/da = n^{a-1/2}\log n + n^{-a-1/2}\log n = n^{a-1/2}\log n\ (1 + n^{-2a}) > 0$$

$$F`(a) = f`(2)-f`(3)+f`(4)-f`(5) + \text{-----}$$

$F`(a,N)$ : the partial sum from the first term of $F`(a)$ to the N-th term of $F`(a)$

$F`(a)$ converges due to $\lim_{n\to\infty} f`(n)=0$. $F`(a)$ is alternating series. We can calculate approximation of $F`(a)$ i.e. $F`(a)_N$ according to the following (62). $\lim_{N\to\infty} F`(a)_N$ converges to $F`(a)$.

$$\frac{\dfrac{F`(a,N)+F`(a,N-1)}{2} + \dfrac{F`(a,N+1)+F`(a,N)}{2}}{2} = F`(a)_N \tag{62}$$

(Graph 3) shows $F`(a)_N$ calculated by (62) at 5 cases of N=500, 1000, 2000, 5000, 10000. 5 line graphs overlapped. Because $F`(a)_N$ of 5 cases are equal to 6 digits after the decimal point.

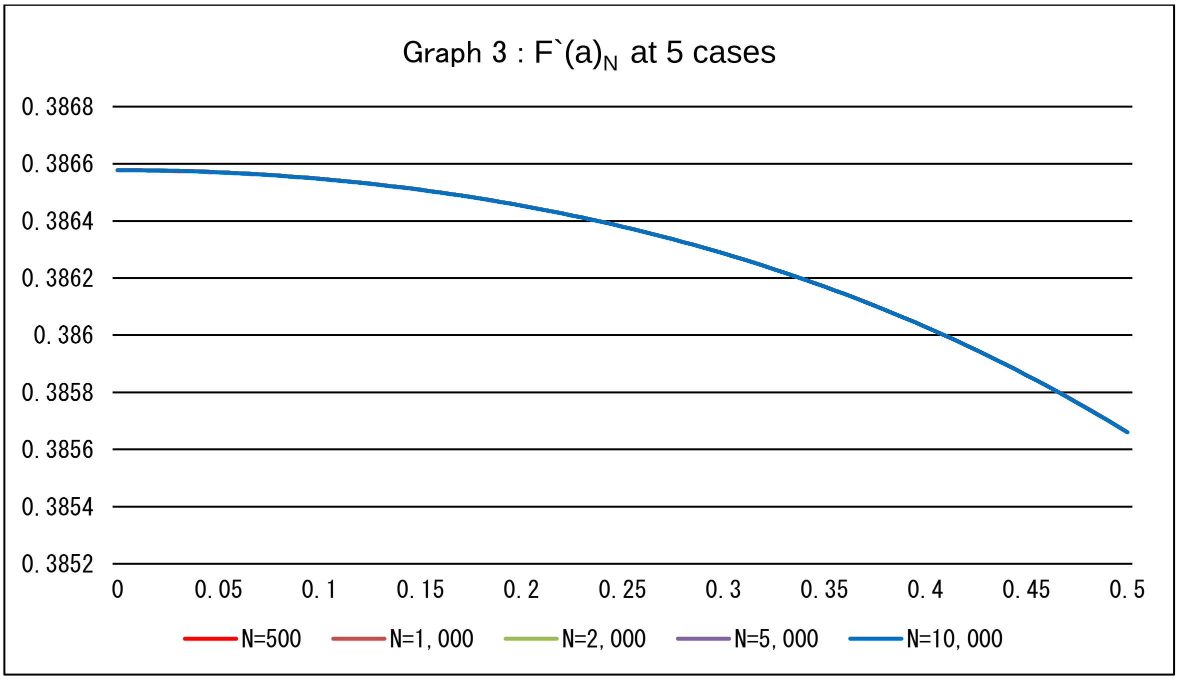


Table 2 : The values of $F`(a)_N$

a	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
N=500	0.38657754	0.38657004	0.38654734	0.38650882	0.38645348	0.3863799	0.38628625	0.38617032	0.3860295	0.38586078	0.38566075
N=1,000	0.38657764	0.38657014	0.38654743	0.38650891	0.38645355	0.38637995	0.38628627	0.3861703	0.3860294	0.38586057	0.38566038
N=2,000	0.38657766	0.38657016	0.38654745	0.38650893	0.38645357	0.38637996	0.38628628	0.3861703	0.38602938	0.38586052	0.38566029
N=5,000	0.38657766	0.38657016	0.38654745	0.38650893	0.38645358	0.38637997	0.38628628	0.3861703	0.38602938	0.38586051	0.38566026
N=10,000	0.38657766	0.38657016	0.38654745	0.38650893	0.38645358	0.38637997	0.38628629	0.3861703	0.38602938	0.3858605	0.38566026

The range of a is $0 \leqq a < 1/2$. $a=1/2$ is not included in the range. But we added $F`(1/2)_N$ to calculation according to the following reason.

[$f`(n)$ at $a=1/2$ ] is $(1+1/n)\log n$ and $\lim_{n\to\infty}(1+1/n)\log n$ does not converge to zero. $F(1/2)$ diverges to $\pm\infty$ due to $\lim_{n\to\infty} f`(n)=\infty$.

But $\{F`(a,N)+F`(a,N-1)\}/2$ is partial sum of alternating series with the term of $\{f`(n+1)-f`(n)\}/2$ and it can converge to the fixed value on the condition of $\lim_{n\to\infty}\{f`(n+1)-f`(n)\}=0$. $\lim_{n\to\infty}\{f`(n+1)-f`(n)\}=0$ is true as follows.

$f`(n)$ is the increasing function regarding n due to [ $\frac{df`(n)}{dn}=\frac{1+n-\log n}{n^2} > 0$].

It means $[0 < f`(n+1)-f`(n)]$.

$$0 < f`(n+1)-f`(n) = \{1+1/(n+1)\}\log(n+1) - (1+1/n)\log n$$
$$< (1+1/n)\log(n+1) - (1+1/n)\log n = (1+1/n)\log(1+1/n)$$

From the above inequality we can have $\lim_{n\to\infty}\{f`(n+1)-f`(n)\}=0$ due to $\lim_{n\to\infty}\{(1+1/n)\log(1+1/n)\}=0$.

3 Approximation of F`(a)

$F`(a)_N$ calculated by (62) converges to $F`(a)$ through $N\to\infty$. To confirm how large N we need to approximate $F`(a)$ accurately, we calculated $F`(a)_N$ with N from N=500 to N=100,000. (Graph 4) shows $F`(a)_N/F`(a)_{500}$ from N=500 to N=100,000 in various a.

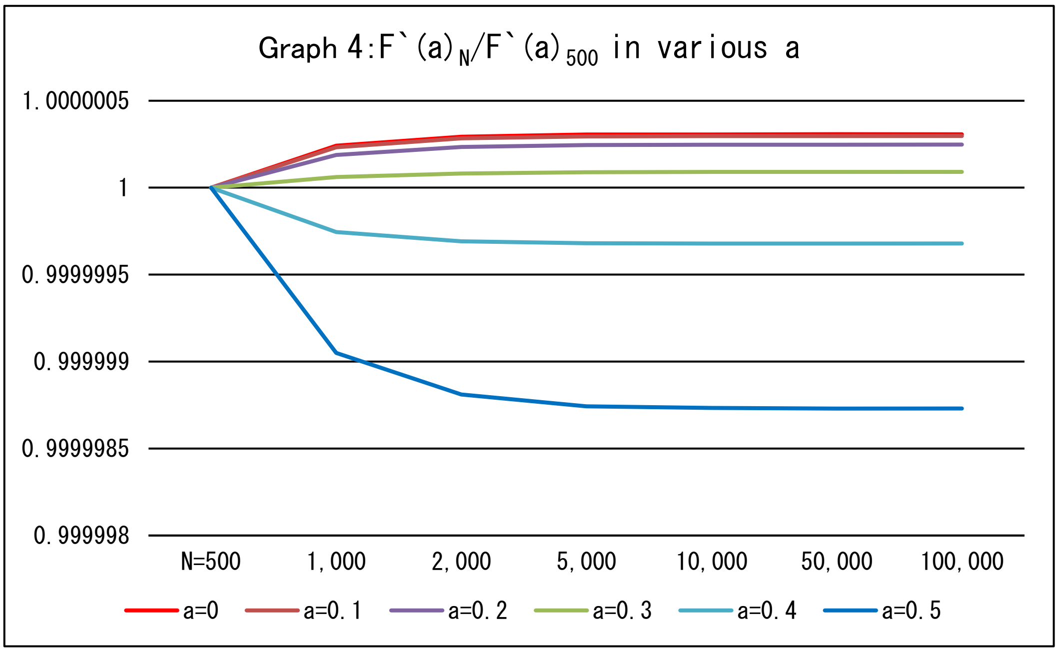


Table 3 : The values of $F`(a)_N/F`(a)_{500}$

a	0	0.1	0.2	0.3	0.4	0.5
N=500	1	1	1	1	1	1
1,000	1.000000242	1.000000232	1.000000189	1.000000061	0.999999745	0.999999051
2,000	1.000000294	1.000000284	1.000000234	1.000000082	0.999999692	0.999998811
5,000	1.000000306	1.000000296	1.000000246	1.000000089	0.999999681	0.999998743
10,000	1.000000307	1.000000297	1.000000248	1.000000091	0.999999679	0.999998734
50,000	1.000000307	1.000000297	1.000000248	1.000000091	0.999999679	0.999998731
100,000	1.000000307	1.000000297	1.000000248	1.000000091	0.999999679	0.999998731

We can find the following from (Graph 4) and (Table 3).

3.1 $F`(a)_{50,000}/F`(a)_{500}$ and $F`(a)_{100,000}/F`(a)_{500}$ have the same values. When N is larger than N=50,000 the values are as same as at N=50,000. So we can consider $F`(a)_{50,000} = F`(a)$.

3.2 The differences between $F`(a)_{500}$ and $F`(a)_{50,000}$ have the maximum value at a=1/2. The maximum difference is [1-0.999998731 = 0.00013%] as shown in (Table 3). Therefore $F`(a)_{500}$ is almost equal to $F`(a)_{50,000}$ i.e. $F`(a)$. N=500 is enough to obtain the accurate $F`(a)$.

From item 3.2 we can consider that (Graph 3) shows $F`(a)$ accurately. (Graph 3) illustrates [$0.3866 > F`(a) > 0.3856$ in $0 \leqq a < 1/2$]. Therefore F(a) is the monotonically increasing function in $0 \leqq a < 1/2$.

4 Conclusion

F(a)=0 has the only one solution of a=0 due to
[$0 \leqq a < 1/2$], [F(0)=0] and
[F(a) is the monotonically increasing function in $0 \leqq a < 1/2$.].