A Note on the Barut Second-Order Equation

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The second-order equation in the (1/2, 0) ⊕ (0, 1/2) representation of the Lorentz group has been proposed by A. Barut in the 70s, ref. [1]. It permits to explain the mass splitting of leptons (e, µ, τ). The interest is growing in this model (see, for instance, the papers by S. Kruglov [2] and J. P. Vigier et al. [3, 4]). We noted some additional points of this model.

The Barut main equation is

$$[iγ^μ∂_μ + a_2γ^ρ∂_ρ - κ]Ψ = 0 ,$$

where $a_2$ and $κ$ are the constants later related to the anomalous magnetic moment and mass, respectively. The matrices $γ^μ$ are defined by the anticommutation relation:

$$γ^μγ^ν + γ^νγ^μ = 2δ^{μν} .$$

$δ^{μν}$ is the metrics of the Minkowski space, $μ, ν = 0, 1, 2, 3$. The equation represents a theory with the conserved current $E$ (e.g., the electromagnetic current), the correct relativistic relation $E = ± \sqrt{p^2 + m^2}$. In fact, it describes states of different masses (the second one is $m_2 = 1/a_2 - m_1 = m_e(1 + \frac{1}{2})$, $a$ is the fine structure constant), provided that the certain physical condition is imposed on $a_2 = (1/m_1)(2a_2/3)/(1 + 4a_2/3)$, the parameter (the anomalous magnetic moment should be equal to $4a_2/3$). One can also generalize the formalism to include the third state, the τ-lepton [1b]. Barut has indicated at the possibility of including $γ_s$ terms (e.g., $γ_5$).

The most general form of spinor relations in the (1/2, 0) ⊕ (0, 1/2) representation has been given by Dvoeglazov [5]. It was possible to derive the Barut equation from the first principles [6]. Let us reveal the connections with other models. For instance, in refs. [3, 7] the following equation has been studied:

$$[(i\hat{D} - e\hat{A})(i\hat{D} - e\hat{A}) - m^2]Ψ = [(i∂_μ - eA_μ)(i∂^μ - eA^μ) - 1/2eσ^{μν}F_{μν} - m^2]Ψ = 0$$

for the 4-component spinor $Ψ$. $A_μ = γ^μA_μ$; $A_μ$ is the 4-vector potential; $e$ is electric charge; $F_{μν}$ is the electromagnetic tensor. $σ^{μν} = \frac{i}{2}[γ^μ, γ^ν]$. This is the Feynman-Gell-Mann equation. In the free case we have the Lagrangian (see Eq. (9) of ref. [3c]):

$$L_0 = (\overline{Ψ}i\hat{D}Ψ)(i\hat{D}Ψ) - m^2\overline{Ψ}Ψ .$$

Let us re-write the equation (1) into the form:

$$[iγ^μ∂_μ + a_2γ^ρ∂_ρ + b]Ψ = 0 .$$

So, one should calculate $(p^2 = p_0^2 - p^2)$

$$Det \begin{pmatrix} b - ap^2 & p_0 + σ \cdot p \\ p_0 - σ \cdot p & b - ap^2 \end{pmatrix} = 0$$

in order to find energy-momentum-mass relations. Thus, $[(b - ap^2 - p^2)^2 = 0$ and if $a = 0$, $b = ±m$ we come to the well-known relation $p^2 = p_0^2 - p^2 = m^2$ with four Dirac solutions. However, in the general case $a ≠ 0$ we have

$$p^2 = (2ab + 1 \pm \sqrt{4ab + 1}/2a^2) > 0 ,$$

that signifies that we do not have tachyons. However, the above result implies that we cannot just put $a = 0$ in the solutions, while it was formally possible in the equation (5). When $a → 0$ then $p^2 → ∞$; when $a → ∞$ then $p^2 → 0$. It should be stressed that the limit in the equation does not always coincide with the limit in the solutions. So, the questions arise when we consider limits, such as Dirac → Weyl, and Proca → Maxwell. The similar method has also been presented by S. Kruglov for bosons [8]. Other fact should be mentioned: when $4ab = -1$ we have only the solutions with $p^2 = 4b^2$. For instance, $b = m/2$, $a = -1/2m$, $p^2 = m^2$. Next, I just want to mention one Barut omission. While we can write

$$\sqrt{4ab + 1}/a^2 = m_2 - m_1 \quad \text{and} \quad 2ab + 1/a^2 = m_2^2 + m_1^2 ,$$

but $m_2$ and $m_1$ not necessarily should be associated with $m_{ν,τ}$ (or $m_{ν,e}$). They may be associated with their superpositions, and applied to neutrino mixing, or quark mixing.

The lepton mass splitting has also been studied by Markov [9] on using the concept of both positive and negative masses in the Dirac equation. Next, obviously we can calculate anomalous magnetic moments in this scheme (on using, for instance, methods of [10, 11]).

We previously noted:

$^a$Of course, one could admit $p^5, p^6$ etc. in the Dirac equation too. The dispersion relations will be more complicated [6].

$^b$a has dimensionality [1/m], $b$ has dimensionality [m].
The Barut equation is a sum of the Dirac equation and the Feynman-Gell-Mann equation.

Recently, it was suggested to associate an analogue of Eq. (4) with the dark matter, provided that \( \Psi \) is composed of the self/anti-self charge conjugate spinors, and it has the dimension [energy] in the unit system \( c = \hbar = 1 \). The interaction Lagrangian is \( \mathcal{L}^H \sim g \bar{\Psi} \Psi \phi^2 \), \( \phi \) is a scalar field.

The term \( \bar{\Psi} \sigma^{\mu \nu} \Psi F_{\mu \nu} \) will affect the photon propagation, and non-local terms will appear in higher orders.

However, it was shown in [3b,c] that a) the Mott cross-section formula (which represents the Coulomb scattering up to the order \( \sim e^2 \)) is still valid; b) the hydrogen spectrum is not much disturbed; if the electromagnetic field is weak the corrections are small.

The solutions are the eigenstates of \( \gamma^5 \) operator.

In general, the current \( J_0 \) is not the positive-defined quantity, since the general solution \( \Psi = c_1 \Psi_+ + c_2 \Psi_- \), where \( [\gamma^\mu \partial_\mu \pm m] \Psi_\pm = 0 \), see also [9].

We obtained the Barut-like equations of the 2nd order and 3rd order in derivatives.

We obtained dynamical invariants for the free Barut field on the classical and quantum level.

We found relations with other models (such as the Feynman-Gell-Mann equation).

As a result of analysis of dynamical invariants, we can state that at the free level the term \( \sim \partial_\mu \bar{\Psi} \sigma_{\mu \nu} \partial_\nu \Psi \) in the Lagrangian does not contribute.

However, the interaction terms \( \sim \bar{\Psi} \sigma_{\mu \nu} \partial_\nu \Psi A_\mu \) will contribute when we construct the Feynman diagrams and the \( S \)-matrix. In the curved space (the 4-momentum Lobachevsky space) the influence of such terms has been investigated in the Skachkov works [10, 11]. Briefly, the contribution will be such as if the 4-potential were interact with some “renormalized” spin. Perhaps, this explains, why did Barut use the classical anomalous magnetic moment \( g \sim 4\alpha/3 \) instead of \( \pi \). 

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References
