An Attempt to approach mathematically the Concept of Time-Crystals.

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A. Abstract.

F. WILCZEK has published an article on the concept of time–crystals (please look into [1]). This article is to be understood as an attempt to drawn a picture mathematically, which approaches the contents of WILCZEK's article.

1. Introduction.

The concept of a crystal is mainly connected with two mayor properties:

- Symmetry and
- Spontaneous break of symmetry.

With regard to symmetry a certain class of transformations is essential:

- Transpositions are highlighted by individual increments of steps.

An idealized crystal linearly transposed by a multiple of the distance between its elements will preserve its layout beyond the transposition; but under this condition only. According to transpositions crystals are characterized as afflicted with reduced degrees of symmetry (in comparison with the symmetry of a continuum). The situation is similar to rotation–symmetry of square and cycle. In this respect the crystal's symmetry also is called broken in relation to that of continua. Because the degree of crystal–symmetry normally changes abruptly due to critical external influences (like temperature e.g.), the break is also called spontaneous.

- Spontaneous symmetry–breaks are decisive for crystals.

A symmetry–break occurs e.g. when a fluid or gas cools down and finally enters the crystal–state by a so–called phase–transition. Within this process the crystal will obtain a lower degree of symmetry as allowed by physical laws before this situation.

Crystals can be divided into two classes:

- Spatial crystals keep their symmetric properties in spatial transpositions and preserve them independently on elapsed time as long as this is compatible with the physical conditions.
- Time–crystals show their essential symmetric properties in space–time transpositions only.

In spatial crystals a spontaneous symmetry–break will occur, if from an energy point of view the new crystallization becomes more preferable. During a phase–transition energy will not be preserved. If the state of a lower energy–level breaks the symmetry of a crystal and a new crystallization has been settled, energy is again maintained and the captured state will exist as long as the actual situation does allow. This explains stability of a spatial crystal after a phase–transition. But this is still no longer valid for time–crystals. Here energy is preserved even in spontaneous symmetry–breaks and therefore an energy related measure to explain this kind of breaks is no longer suitable.
But there exists a more general conception appropriate to deal with spontaneous symmetry–breaks, which is also applicable for time–crystals.

- The reason why extended networks (connections of many parts) most often are tempted to resist a reorganisation and tries to keep its actual stability, based on the fact that most disordering influences act locally and long–range forces will them overrule.
- But material–states will not last forever, thus finally (sooner or later) a symmetry–break will occur and a new order will be established.

A network of parts may be identified as a time–crystal, if the following characteristics become apparent:

1^1. The network's symmetry will only be realized in space–time, regularities considered in space alone may change fluently by observations at different moments in time.

1^2. Most properties of the network are directly bounded to its regularities.

This is mainly extracted from the article of F. WILCZEK published in [1]. The following is to be understood as an attempt to approach the conceptual characteristics (1^1./2.) of time–crystals by a picture mathematically drawn.

2. A Fusion of SIERPINSKI-Gasket and PASCAL-Triangle.

The following conception mainly based on a fusion of an IFS–developed SIERPINSKI–gasket and patterns according to divisibility of numbers in a PASCAL–triangle relative to primes. Both basic objects (gasket and triangle–patterns) will be merged to form a geometrical model, which is to be understood as a mathematical picture comparable with WILCZEK conception.

2.1. SIERPINSKI-Gasket.

Unit–square (Q) in a (u,v)–plane maybe specified by:

2.1^1. $Q = \{(u,v) |\ [0 \le u \le 1] \wedge [0 \le v \le 1]\}.$

If contractions:

2.1^2. $w_{a_e b_e} = \langle\!\langle([u+a_e]/2) \wedge ([v+b_e]/2) \wedge (a_e, b_e \in [0,1]) \wedge (e \in [0,1])\rangle\!\rangle$

are iteratively applied on (Q), one will obtain the following congruent sub–squares of (Q):

- $\langle\!\langle[Q_{a_0b_0} = w_{a_0b_0}(Q)] \Rightarrow [Q = {}_{a_0b_0}\bigcup Q_{a_0b_0}]\rangle\!\rangle$
- $\langle\!\langle[Q_{a_1a_0b_1b_0} = w_{a_1b_1}(Q_{a_0b_0})] \Rightarrow [Q_{a_0b_0} = {}_{a_1b_1}\bigcup Q_{a_1a_0b_1b_0}] \Rightarrow [Q = {}_{a_0b_0}\bigcup({}_{a_1b_1}\bigcup Q_{a_1a_0b_1b_0})]\rangle\!\rangle$

This will lead to the following pictures:

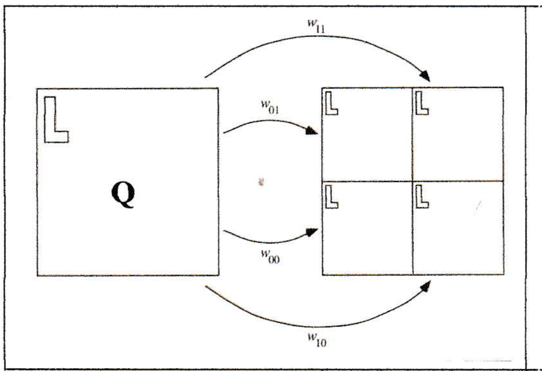


Figure 2.1^1.a.

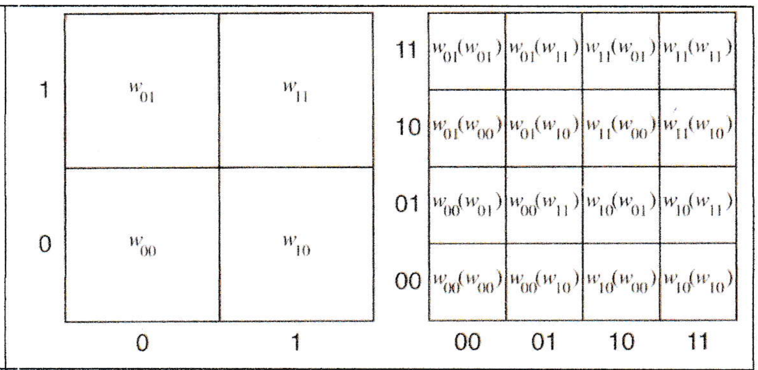


Figure 2.1^1.b.

If (u) and (v) from Equation (2.1^1.) are expressed in binary extension by:

2.1^3. $\langle u = {}_{(j=1)}\Sigma^{(j=\infty)}[a_j 2^{-j}]\rangle \wedge \langle v = {}_{(j=1)}\Sigma^{(j=\infty)}[b_j 2^{-j}]\rangle \wedge \langle a_j, b_j \in \{0,1\}\rangle$

and Equation (2.1^2.) becomes restricted in the following way:

- $w_{a_e b_e} = \langle\!\langle([u+a_e]/2) \wedge ([v+b_e]/2) \wedge (a_e, b_e \in \{0,1\}) \wedge (e \in \{0,1\}) \wedge (a_e+b_e \le 1\rangle\!\rangle,$

all sub–squares from set $\{Q_{a_1a_0b_1b_0}\}$ in Figures (2.1^1.[a/b]) are excluded, where the addition of (a_e) and (b_e) cause at least one carry (KUMMER's carry condition). One will get instead of Figures (2.1^1.[a/b]):

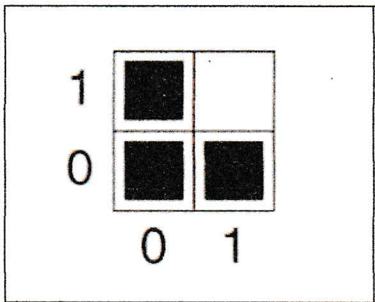


Figure 2.1^2.a.

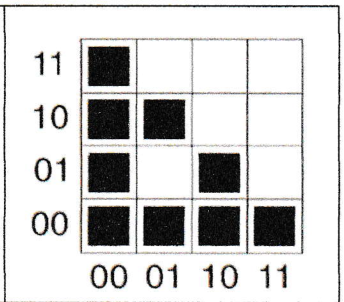


Figure 2.1^2.b.

These are the first (2) steps of an Iterated Function System (IFS) appropriate to create finally a 2–adic structure of SIERPINSKI–gasket. Subsequently the patterns will be considered from more general point of view.

With unit–square (Q) in (u,v)–plane in Equation (2.1^1.) and a binary expansion in Equation (2.1^3.) one can provide a number–theoretical description of the SIERPINSKI–gasket (S):

- $S = \{(u,v) \subset Q | \exists$ expansions $\langle[u = (0.a_0a_1...)_{p=2}] \wedge [v = (0.b_0b_1...)_{p=2}]\rangle \leftarrow \langle[a_e+b_e \leq 1] \leftarrow [e \in \{0,1,2....\}]\rangle\}$.

This can be expressed in IFS–form:

$$2.1^\wedge4. \quad S = \{\bigcup_e w_{a_e b_e}(S)] \leftarrow \langle[a_e,b_e \in \{0,1\}] \wedge [a_e+b_e \leq 1 \leftarrow e \in \{0,1\}]\}.$$

The binary representation allows one to pursue, how the iteration of Equation (2.1^4.), applied to an arbitrary point in the square (Q), yields a sequence of points that tends closer and closer to the SIERPINSKI–gasket. If the maps (w_{00}), (w_{01}) and (w_{10}) within IFS are applied again and again on $\langle(u.v) = 0.a_1a_2...,b_1b_2...\rangle$ with arbitrary $(a_{e\to\infty})$ and $(b_{e\to\infty})$, points are obtained with coordinates, whose leading binary decimals will more and more satisfy the condition $(a_e+b_e \leq 1)$. Starting from $(A_0 = Q)$ and running the IFS, one will generate the sequence:

- $A_x = w_{00}(A_{x-1}) \bigcup w_{01}(A_{x-1}) \bigcup w_{10}(A_{x-1})$,

where the coordinates $(\varkappa)$ of a point (A_x) satisfies $(a_x+b_x \leq 1)$ in the leading $(\varkappa)$ binary decimals. Furthermore the sequence will tend towards the SIERPINSKI–gasket $(S = A_\infty)$. The first steps are shown in the next figure:

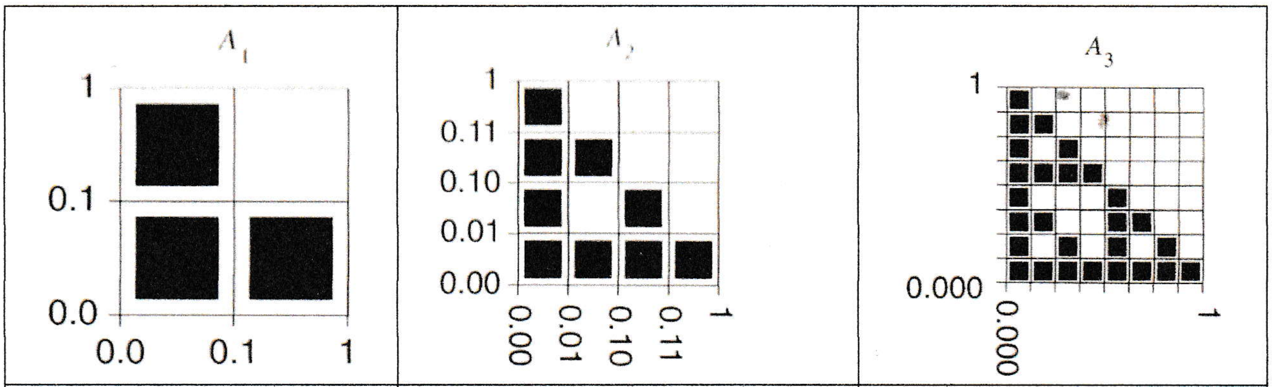


Figure 2.1^3.a.	Figure 2.1^3.b.	Figure 2.1^3.c.

One will observe, that this exactly matches with Figures (2.1^2.[a./b.]) with a step–3 Figure (2.1^2.c.) in addition:

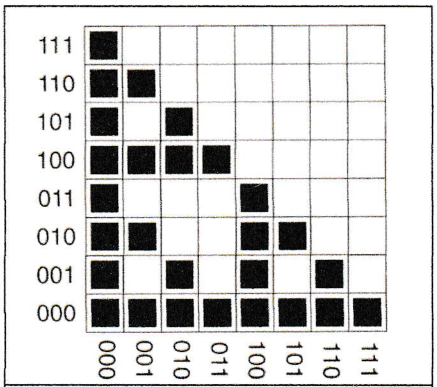


Figure 2.1^2.c.

if the coordinates used in Figures (2.1^2.[a./b./c.]) are preceded by a decimal point. In this case the patterns found on the (2 by 2)–, (4 by 4)– and (8 by 8)–grid would exactly match the steps (A_x) of the IFS. But introducing a decimal point in Figures (2.1^2.[a./b./c.]) means looking on a rescaled version of the PASCAL–triangle.

2.2. PASCAL-Triangle.

The PASCAL–triangle is an arithmetic triangle, an triangular array of numbers composed of the coefficients obtained by expansion of the polynomial $(1+z)^{\varkappa}$:

- $(1+z)^0 \;=\; 1$
  $(1+z)^1 \;=\; 1+z$
  $(1+z)^2 \;=\; 1+2z+z^2$
  $(1+z)^3 \;=\; 1+3z+3z^2+z^3$
  ……… .. …………………

The Figure (2.2^1.a.) contains the coefficients for the (8) expansion–steps organized in the following triangular–scheme:

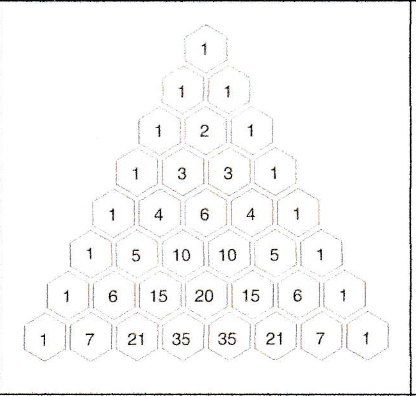


Figure 2.2^1.a.

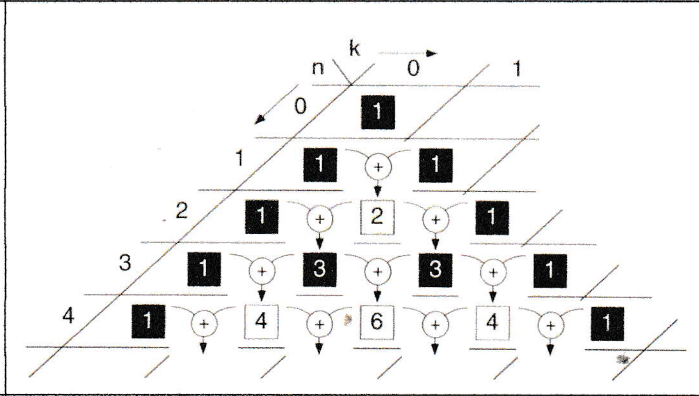


Figure 2.2^1.b.

The computation of the numbers in Figure (2.2^1.a) used the fact, that the entries in each row are determined by the entries of the previous row as demonstrated by Figure (2.2^1.b.).

- $(1+z)^{\varkappa} = a_0+a_1z+\ldots+a_{\varkappa}z^{\varkappa}$
  $(1+z)^{\varkappa+1} = b_0+b_1z+\ldots+b_{\varkappa+1}z^{\varkappa+1} = (1+z)^{\varkappa}(1+z) = a_0+a_1z+\ldots+ a_{\varkappa}z^{\varkappa}+a_0z +a_1z^2+\ldots+a_{\varkappa}\, z^{\varkappa+1}$
  $= a_0+(a_0+a_1)z+\ldots+(a_{\varkappa-1}+a_{\varkappa})z^{\varkappa}+a_{\varkappa}z^{\varkappa+1} \Rightarrow$
  $[b_0 = a_0] \wedge [b_1 = (a_0+a_1)] \wedge \ldots \wedge [b_{\varkappa} = (a_0+a_1)] \wedge [b_{\varkappa+1} = a_{\varkappa}]$.

The major question is now, how one can find out whether or not the coefficients are divisible by a prime (p) in a direct non–recursive computation. A solution for the problem was found by E. E. KUMMER in 1852. In order to follow KUMMER's idea, it will be more convenient to transpose the PASCAL–triangle into a new coordinate–system (n,k):

k \ n	0	1	2	3	4	5	6
6	1	7	28	84	240	462	924
5	1	6	21	56	126	252	462
4	1	5	15	35	70	126	210
3	1	4	10	20	35	56	84
2	1	3	6	10	15	21	28
1	1	2	3	4	5	6	7
0	1	1	1	1	1	1	1

Figure 2.2^2.

In the new coordinate–system at position (n,k) is now located a binomial coefficient with a value of:

- $(n+k)\hat{}k = (n+k)!/(n!\cdot k!)$.

In Figure (2.2^2.) entries of the triangle are coloured white or black depending on the fact whether or not the appropriate binomial coefficients are divisible by (2). In order to find a pattern–formation for a divisibility of the binomial coefficients with regard to any other prime, it is useful to start with the prime–factorization for an arbitrary integer (r):

2.2^1. $r = {}_{(e=1)}\Pi^{(e=s)}[p_e^{\tau_e}]$.

Herein primes (p_e) are different from each other and exponents (τ_e) are natural numbers. Subsequently one will take into consideration a set the following form:

- P(r) = {(n,k)|(n+k)^k is not divisible by r}.

In order to understand the pattern–formation according to a certain (r), it is sufficient to consider a sub–set of the appropriate prime–power from Equation (2.2^1.):

2.2^2. $P(p^\tau) = \{(n,k)|\ (n+k)\text{\textasciicircum}k$ is not divisible by $p^\tau\}$.

KUMMER realized that the solution for the set is encoded in the addition of (n) and (k) in their p–adic representation. A p–adic representation of an integer (q) looks like:

- $q = a_0+a_1p+a_2p^2+...+a_mp^m \quad \Rightarrow \quad q = (a_ma_{m-1}...\ a_1a_0)_p$.

KUMMER observed now that the numbers of carries $c_p(n,k)$ in the just mentioned addition of (n) and (k) is decisive for a solution of Equation (2.2^2.). He formulated the following statement:

- If $\tau = c_p(n,k)$ = number of carries in p–adic addition of (n) and (k),

then one will obtain:

- $P(p^\tau) = \{(n,k)_p \wedge (\tau = c_p(n,k))|(n+k)$^k is divisible by prime–power p^τ but not by $p^{\tau+1}\}$.

2.3. Divisibility of Binomial-Coefficients by Primes.

The global pattern–formation in:

- P(p) = {(n,k)|(n+k)^k is not divisible by p}

shall subsequently be formally described.

At first an appropriate IFS is to be constructed by considering the unit–square (Q) and subdividing it into (p^2) congruent sub–squares:

- $Q_{a,b}$ with $a,b \in \{0,1,2,...,p-1\}$,

which are obtained by introducing corresponding contractions:

- $\langle Q_{ab} = w_{ab}(Q)\rangle \leftarrow \langle w_{a,b}(u,v) = ([u+a]/p \wedge [v+b]/p)\ \rangle$.

This is to be considered as a generalization of what had already been specified for the case (p = 2) in Figures (2.1^1.[a/b.]). A set of admissible transformations will be defined next by imposing the restriction:

- $a+b \leq p-1$.

This yields a total number of (N = p(p+1) contractions, each with a contraction–factor (p^{-1}).
Additionally may be introduced:

- $W_p(A) = {}_{(a+b\leq p-1)}\bigcup w_{ab}(A)$

corresponding to the (N) contractions, where (A) is any sub–set of the plane. With the initial set (A_n = Q) one may start the iteration:

- $\langle A_x = W_p(A_{x-1})\rangle \leftarrow \langle \varkappa = 1,2,\ldots\rangle$

and Figure (2.3^1.) shows the first (2) steps for the choice $(p = 3)$:

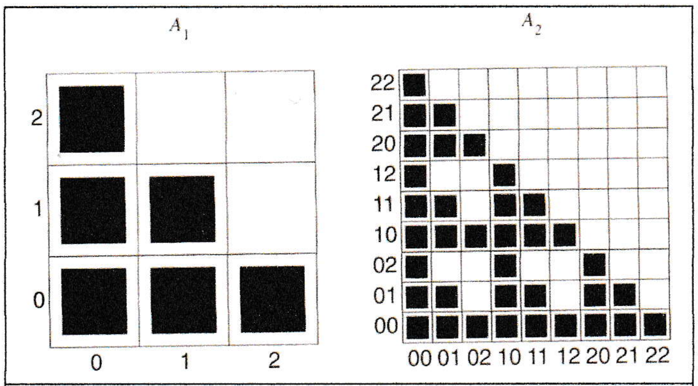


Figure (2.3^1.)

In order to keep track of the iteration, each of the (p^2) sub–squares of (Q) is subdivided into (p^2) even smaller ones, and so on repeatedly. Having indexed the first subdivision of (Q) by (Q_{ab}), one continues to label the sub–squares of the second subdivision by $(Q_{ab,cd})$ and so on. For the example of $(p = 3)$ from Figure (2.3^1.), the square $Q_{10,12}$ is identified in the following way:

- The pair (1,1), made from the leading digits in the index of $O_{10,12}$ determine the centre –square in the first subdivision and the pair (0,2) determines the upper left corner–square therein.

In a similar way the square $(Q_{a_{x-1}\ldots a_0,b_{x-1}\ldots b_0})$ is to be understood as a square of the $\varkappa$–th generation, where the double p–adic addresses are given by the pair $(a_{x-1}\ldots a_0,b_{x-1}\ldots b_0)$. This natural addressing–system helps to keep track of all iterations of W_p , e.g.:

- $\langle Q_{a_{x-1}\ldots a_0,b_{x-1}\ldots b_0} = w_{a_{x-1}\ldots a_0,b_{x-1}\ldots b_0}(Q)\rangle \leftarrow \langle a_e+b_e \leq p-1\rangle$.

In other words can be said, (A_x) is the collection of all those squares of the $\varkappa$–the subdivision of (Q) into $(p^{2\varkappa})$ sub–squares, whose addresses $(a_{x-1}\ldots a_0,b_{x-1}\ldots b_0)$ satisfy the condition $(a_e+b_e \leq p-1)$, i.e.:

2.3^1. $A_x = {}_{(a_e+b_e \leq p-1)}\bigcup Q_{a_{x-1}\ldots a_0,b_{x-1}\ldots b_0}$.

2.4. Rescaling the PASCAL-Triangle appropriately.

Now the sub–squares $(Q_{a_{x-1}\ldots a_0,b_{x-1}\ldots b_0})$ will be related to the entries of the PASCAL–triangle. In order to enable this, one has to generate first a geometric model of divisibility–pattern in the PASCAL–triangle. For this reason the first quadrant of the plane is equipped with a square–lattice in such a way, that each square of the lattice has side–length (1). Thus each square is indexed by the index–pair (n,k) and is called $(R_{n,k})$:

- $R_{n,k} = \{(u,v)|[n \leq u \leq n+1] \wedge [k \leq v \leq k+1]\}$.

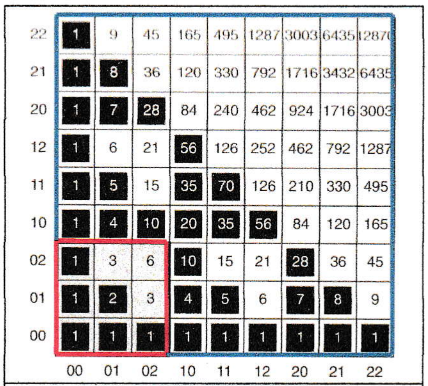
The geometrical model of [P(p)] will be obtained by selecting all squares $(R_{n,k})$ for which (p) does not divide [(n+k)^k]:

- $P(p) = \{R_{n,k}|(n+k)\text{^}k$ is not divisible p}.

This infinite pattern will be related to the evolutions of Sections (2.1./2./3.), i.e. to the sequence of the patterns (A_x), each with a length of $(p^{-\varkappa})$ and whose union will finally result in (Q). In order to recognize the relation between (A_x) and [P(p)], the latter will be considered though a sequence of filters $([0, p^\varkappa]\times[0, p^\varkappa])$ of length $(p^\varkappa)$. For $(e = 1,2,\ldots)$ that part from the geometrical model [P(p)] is picked–up which falls in the corresponding filter:

- $P_e(p) = P(p) \bigcap ([0, p^e] \times [0, p^e])$.

The next Figure (2.4^1.) display the filters $\langle P_1(p=3) \wedge P_2(p=3) \rangle$:



	00	01	02	10	11	12	20	21	22
22	1	9	45	165	495	1287	3003	6435	12870
21	1	8	36	120	330	792	1716	3432	6435
20	1	7	28	84	240	462	924	1716	3003
12	1	6	21	56	126	252	462	792	1287
11	1	5	15	35	70	126	210	330	495
10	1	4	10	20	35	56	84	120	165
02	1	3	6	10	15	21	28	36	45
01	1	2	3	4	5	6	7	8	9
00	1	1	1	1	1	1	1	1	1

Figure (2.4^1.)

If one compares $[P_1(p)]$ and $[P_2(p)]$ from Figure (2.4^1.) with the patterns (A_1) and (A_2) from Figure (2.3^1.) one will find them to be identical, although (A_1) and (A_2) are in the unit–square (Q) while $[P_1(p)]$fit into a square of side–length (p) and $[P_2(p)]$ into a square of side–length (p^2). In other words, during rescaling the pattern $[P_e(p)]$ by a factor (p^{-e}) one will obtain an object (S_e), which is identical with (A_e):

- $A_e \equiv S_e = p^{-e} \cdot P_e(p)$.

From IFS in Section (2.3.) it is known, that (A_e) is the collection of all those squares from e–th subdivision of (Q) into (p^{2e}) sub–squares, whose addresses $(a_{e-1} \ldots a_0, b_{e-1} \ldots b_0)$ satisfy the condition $(a_e + b_e \leq p-1)$. This collection for $(e \to \infty)$ will converge to the attractor of the IFS and in the rescaled geometric models (S_e) under the same condition $(e \to \infty)$ will do the same. Therefore it became obvious, that the rescaled geometric models have a limit–set, which represents the rescaled geometric model of PASCAL–triangle–pattern modulo (p), called P(p).

2.5. *Pattern-Formations and fractal Dimensions of the grometric Models P(p).*

In Figure (2.5^1.) the geometric models S(p) are shown, which result from running the IFS corresponding to $P(p \in \{2,3,5\})$:

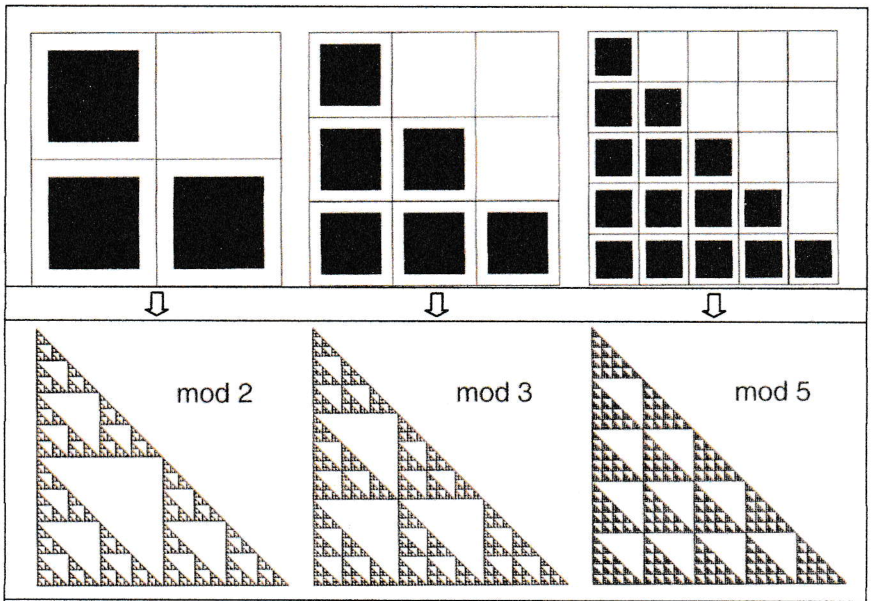


Figure (2.5^1.)

The geometrical model S(p) in Figure (2.5^1.) are self–similar fractals with self–similarity–dimensions of:

	Self–Similarity Dimension
S(2)	$\log\{3\}/\log\{2\} \approx 1.585$
S(3)	$\log\{6\}/\log\{3\} \approx 1.631$
S(5)	$\log\{15\}/\log\{5\} \approx 1.683.$

The black–pixel–patterns of S(p) are built according to the conditions:

	Black Pixels according to:
S(2)	$\{(n,k) \mid (n+k)^k$ is not divisible by $2\}$
S(3)	$\{(n,k) \mid (n+k)^k$ is not divisible by $3\}$
S(5)	$\{(n,k) \mid (n+k)^k$ is not divisible by $5\}$

.

3. Conclusion.

All patterns of the geometric model are in line with (1^1.). This becomes obvious in case of a specific example (can be modified by certain details in order to confirm the previous statement in general):

- Considering the symmetry among binomial coefficients [(n+k)^k] of (k_0z^0+ k_1z^1+...+$k_{n-1}z^{n-1}$+k_nz^n) ⟨from locations (n,k) in square–lattice (n,k)⟩ under condition P(p) = {(n,k)|(n+k)^k is not divisible by prime = p}, one will notice, that the pixel–regularity realized in pattern of step (n = j), most often differs in some confusing way from regularities of other steps ($n \neq j$). The reason for this is, the situation resembles glances into space only. Only if all these single patterns – for ($n \in \mathbb{N}$) – are put together into one common, all steps including pattern (similar to a look into space–time), the symmetry mentioned above will become obvious.

The following can be said about the geometric model in relation with (1^2.):

- The symmetries of the patterns are completely determined by the divisibility [(n+k)^k] at locations (n,k) in square–lattice (n,k) relative to a prime p.
- A symmetry–break occurs only if the divisibility–condition changes. Thus stability of a pattern as its fractal dimension as well is only guaranteed by a certain divisibility–condition.

Looking all over the contents of chapter (1.) one will notice further characteristics of the geometric model:

- Local similarities exist, in Figure (2,5^1.) e.g. between mod(3) and mod(5), but they are not capable to neither determine nor destroy the overall–symmetries of the patterns. They are overruled by the large–scale relations in form of overall divisibility–conditions. The latter are decisive and responsible alone for the stabilities of the patterns. The divisibility expressed in a pattern is too resistant for being disturbed by weaker local regularities.
- No preference exists among patterns of the geometric model. The patterns are not subjected to any further kind of overriding principles (like highest/lowest energy–level, highest/lowest order–level,...).

These characteristics of the geometric model may let it become suitable as an appropriate mathematical picture corresponding to the contents of chapter (1.).

4. References.

[1]	F. Wilczek:	Quantensysteme **Zeitkristalle**, Spektrum der Wissenschaft, 6.2020.
[2]	H. O. Pleitgen, H. Jürgens, D. Saupe:	Chaos and Fractals, New Frontiers of Science, Springer 1992.
[2]	U.E. Steinemann:	Phase–Transition means: Fractal invariant in Renormalizations mediates Transitions between Manifolds of different Topologies., http://vixra.org/2001.0569.