The Field Transformation Method

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Abstract

Currently, mathematicians are developing new methods for solving some nonlinear partial differential equations, or NPDEs. This paper introduces the field transformation method, or FTM, which utilizes transformations from, then back to a field to assist one in generating solutions to various NPDEs. First, it shows how the technique is implemented in the solving of a very notable NPDE, the Bateman-Burgers' equation. Then the study utilizes the method on three more examples, the Good Boussinesq, Cahn-Hilliard, and Rayleigh-like equations. Finally, the paper concludes with a very terse description of FTM attributes.

1. Introduction

Over the past two decades, there were several techniques developed to solve NPDEs. For instance, the exponential method used a truncated dividing series of exponentials functions to establish a general solution to the NPDEs [1]. On the other hand, the simplest equation method utilized different powers of the solution to the Riccati equation to serve as the general solution to the same NPDEs [2]. Often, they both assumed that solution take the form of a traveling-wave or soliton.

FTM utilizes an abstract transformative process involving the moving away from, then returning to a field via function [3]. The methods described above do not use integral or derivative transformative processes to solve NPDEs [4]. Transformations allow for an individual to transfer the differential equation into another spacetime, where it is easier to solve. Then (s)he uses inverse transforms to pull the solution back to our spacetime. In FTM, an inverse norm transform with functional differentiation gives one the ability to move away from the field through which the solution exists over. On the other hand, the norm transform with functional integration in FTM allows one to return to field through which the solution exists over.

This paper is ordered in the following fashion. The next section will show how the technique of FTM is implemented on a well-known NPDE called the Bateman-Burgers' equation. Also, it will provide simple rules that must be considered when using field transformations. Next, section 3 will display how
FTM is implemented on two more examples, the Good Boussinesq, Cahn-Hilliard, and Rayleigh-like equations. Finally, the study will end with a quick discussion of the power of FTM.

2. **FTM and its application on the Bateman-Burgers’ equation.**

First, an individual must limit the differential equation $F$ to its leading terms. Consider a differential equation $F$:

$$ F(x, t, u, u_x, u_t, ...) = 0, $$

where $u$ is the solution. For instance, one considers the Burgers’ equation as one of the most basic nonlinear differential equations:

$$ F(x, t, u, u_x, u_t, ...) = u^{(0,1)}(x, t) + u(x, t) u^{(1,0)}(x, t) - u^{(2,0)}(x, t) = 0. $$

The leading terms of the differential equation $F$ is the highest order linear term and highest degree $m$ nonlinear term, generally. The two leading terms to the Burgers’ equation, the individual establishes the following expression:

$$ u(x, t) u^{(1,0)}(x, t) - u^{(2,0)}(x, t) = 0. $$

Note: $m$ is equal to 2. Optionally, one could include more linear terms as leading terms if (s)he does not add terms whose order is equivalent to the leading nonlinear term. Also, the individual should apply coefficients to all linear terms if (s)he intends to use more than one leading linear term.

Next, one performs field transform, which entails transitioning from a $p$-norm function to a part of a sum, on the remaining basic equation established by the leading terms, thus:

$$ \mathcal{J} \left[ \frac{1}{(\Sigma f^p)} \mathcal{F} \right] \rightarrow f, $$

where $J$ is a regular and functional integration operator [5,6,7]. Assuming, the ansatz transformed general solution $u$, which is a traveling-wave or soliton, is a polynomial ring to completion over a field $K$, hence the name field transformation, that is defined as the following:

$$ u(\xi) = c_0 \left( \sum_i f(i) \int g(i, \xi)^p d\xi \right)^{\frac{1}{p}} + c_1, $$

where $c_0$ and $c_1$ are arbitrary constants, function $g(i, \xi)$ is equal to $e^{i\xi}$, the $i$-th integer is any natural number $N_0$, and the transform variable $\xi$ is equal to $\alpha t + \beta x$[8]. There are some rules associated with [inverse] field transformations. (More explicit details of these rules are discussed in the appendix section A.1.) For instance, the field transform of the ansatz transformed solution $u$ involve a few steps:
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\[ R_1(u) = R_1 \left[ e_0 \left( \sum \limits_i f(i)^p \int g(i, \xi)^p \, d\xi \right)^{\frac{1}{p}} + e_1 \right] \leq \left( \sum \limits_i f(i) \right) \sum \limits_i g(i, \xi) \, d\xi \]

\[ R_2 \left( \left( \sum \limits_i f(i) \right) \sum \limits_i g(i, \xi) \, d\xi \right) = \frac{\delta (\sum \limits_i f(i)) \sum \limits_i g(i, \xi) \, d\xi}{\delta \sum \limits_i g(i, \xi) \, d\xi} \]

\[ R_2 \left( \left( \sum \limits_i f(i) \right) \sum \limits_i g(i, \xi) \, d\xi \right) = \sum \limits_i f(i) \]

\[ R_3 \left( \sum \limits_i f(i) \right) = f(i) \]

where \( R_1, R_2, \) and \( R_3 \) are the individual steps. \( R_1 \) transforms a \( p \)-norm function into a Cauchy-Schwartz inequality [9]. Then \( R_2 \) utilizes functional differentiation of the Cauchy-Schwartz inequality which results in just the summation of function \( f \) [10]. Thus, with respect to the segment that incorporates the function \( g \), one establishes a derivative of the Cauchy-Schwartz inequality. Finally, \( R_3 \) breaks down the summation of function \( f \) into just a part/term \( f \).

Field transformations can be applied to the derivatives, such as leading terms in NPDEs. Knowing function \( g(i, \xi) = e^{i\xi} \), field transforms can be applied to the \( n \)-order derivatives of the ansatz transformed solution \( u \):
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\[ R(u^n(\xi)) = R_3 \circ R_2 \circ R_1 (u^n(\xi)) \]

\[ R(u^n(\xi)) = R_3 \circ R_2 \circ R_1 \left( c_0 \left( \sum_{i=1}^{n} f(i) \int g(i, \xi)^p \, d\xi \right)^{\frac{1}{p}} + \varepsilon_1 \right) \]

\[ R(u^n(\xi)) = R_3 \left( \sum_{i=1}^{n} f(i) \int g(i, \xi)^p \, d\xi \right) \]

\[ R(u^n(\xi)) = R_3 \left( \sum_{i=1}^{n} f(i) \right) \]

Note: the numerical value \(2^a\) can be dropped since it will be absorbed by the arbitrary constant \(c_0\). If one applies FTM on the leading terms of the Bateman-Burgers’ equation after stripping the ansatz transformed leading terms of all arbitrary constants \(\alpha\) and \(\beta\), (s)he would obtain:

\[ R(u(\xi)) \, u'(\xi) - u''(\xi) = 0 \]

or

\[ \int f^2 - i^2 f = 0. \]

All arbitrary constants \(\alpha\) and \(\beta\) of the ansatz transformed leading terms are removed since they will be eventually absorbed by the arbitrary constant \(c_0\).

After applying the field transform, the individual must solve for function \(U\) or \(f\), then implement the inverse field transformation. For the Burgers’ equation, \(f\) is equal to \(i\). Now, one must generate the \(p\)-norm of the functional integration/antiderivative of the sum of function \(f\) to [re-]establish the ansatz general solution \(u\):

\[ R^{-1} \left( f \right) \rightarrow (\Sigma J f^p)^{\frac{1}{p}}, \]

and

\[ u = R^{-1}(U) = c_1 + c_0 \| J(U) \|_p = c_0 \left( \sum_{i=0}^{\infty} \int d\xi \, f(i)^p \, \varepsilon^j \xi^p \right)^{\frac{1}{p}} + \varepsilon_1, \]

where \(J\) is a regular and functional integral operator, and the value \(p\) is equal to \(m - 1\). There are some conditions which the individual must consider: 1.) if \(i\) appears in the numerator, then \(p\) is equal to \(+ (m-\)
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1) if \( i \) appears in the denominator, then \( p \) is equal to \( -(m-1) \), thus a negative value; and 3.) again, if \( p \) does not equal one or \( m \) does not equal two, then \( c_1 \) is general equal to zero. Therefore, the general solution for the Bateman-Burgers' equation takes the following form:

\[
\begin{align*}
    u(\xi) = R^{-1}(U) &= \epsilon_1 + \epsilon_0 [\mathcal{J}(U)]_1 = \epsilon_0 \sum_{j=0}^\infty \int d\xi \ f(ji) \ e^{ij} + \epsilon_1 = c_0 - \frac{c_0}{e^\xi - 1}
\end{align*}
\]

since the function \( f \) is equal to \( i \).

Next, one must solve for the arbitrary constants to acquire the exact solution of the Bateman-Burgers' equation. To do so, (s)he must obtain the total ansatz transformed the Bateman-Burgers' equation and plug in the transformed general solution \( u \), or:

\[
F = \alpha u'(\xi) - \beta u''(\xi) + \beta uu'(\xi) = 0.
\]

By multiplying the differential equation \( F \) by its denominator, one forms a new exponential polynomial ring to completion that serves a non-trivial ideal \([11,12]\). If the individual establishes algebraic equations about the generators of the non-trivial ideal \( I \), or \( e^{\xi/2} \), then solve for the constants (i.e. \( c_0, \alpha, \beta \), etc.) assuming the algebraic equations are equal to zero, (s)he will produce:

\[
\begin{align*}
    \alpha &= -\beta^2 - \beta c_1 \\
    c_0 &= 2 \beta
\end{align*}
\]

(Sometimes, one must divide the non-trivial ideal \( I \) by its general solution to assist in the elucidation of the algebraic equations.) By plugging in the constants to the general solution, one would finally yield the exact solution:

\[
u(x, t) = c_1 - \frac{2 \beta}{e^{\beta t - \beta x}} - 1.
\]

3. More examples

3.1 the Good Boussinesq[like] equation:

Consider the following differential equation \( F \):

\[
u_{tt} + u_{xx} + u_{xxxx} + (u_t^2/2)_{xx} = 0.
\]

The leading terms of the above generates the following equation:

\[
u_{xxxx} + (u_t^2/2)_{xx} = 0.
\]

Applying FTM to the previous expression after using ansatz to transform the leading terms, then stripping them of all arbitrary constants yields:

\[
i^4 f + i^4 f^2 = 0,
\]
where the function $U$ or $f$ is equal to $-i^2$ (and $m$ is equal 1). The transformed general solution to the differential equation is:

$$u(\xi) = R^{-1}(U) = e_1 + c_0 \sum_{i=0}^{\infty} d_{i} f(i) e^{i\xi} + e_1 = \frac{c_0 \, e^\xi}{(e^\xi - 1)^2} + e_1.$$

The next step is ascertaining the total ansatz transformed differential equation $F$ assuming the solution is a traveling-wave or soliton:

$$\alpha^2 u''(\xi) + \beta^2 u''(\xi) + \beta^4 u''(\xi) u(\xi) = 0$$

and deriving the exact solution to the differential equation $F$. Substituting the solution into the transformed differential equation $F$, multiplying the differential equation by its common denominator, and ascertaining the algebraic equations about $e^{\alpha^2}$, an individual would obtain the following expressions:

$$-6 \alpha^2 c_0 + 6 \beta^4 c_0 + 6 \beta^2 c_0 - 6 \beta^2 c_0 - 6 \beta^2 c_0 c_1 = 0,$$

$$a^2 c_0 + \beta^4 c_0 + \beta^2 c_0 + \beta^2 c_0 c_1 = 0,$$

and

$$2 \alpha^2 c_0 + 2 \beta^4 c_0 + 2 \beta^2 c_0 + 2 \beta^2 c_0 + 2 \beta^2 c_0 c_1 = 0.$$

Finally, if one uses the above algebraic equations to derive the value of the constants $\alpha$, $\beta$, and $c_0$, (s)he will produce the exact solution to the differential equation $F$ in the following form:

$$u(x, t) = \frac{a^2}{\beta^2} + \beta^2 \left( \frac{6}{\cosh(\alpha t + \beta x) - 1} - 1 \right).$$

### 3.2 the Cahn-Hilliard[-like] equation

Consider the following differential equation $F$:

$$u_t + u_x + u_{xx} + u_{xxxx} + (u^3/3)_{xx} = 0.$$

The leading terms of the above generates the following equation:

$$u_{xxxx} + (u^3/3)_{xx} = 0.$$

The field transform of the previous expression yields after stripping the ansatz transformed leading terms of all arbitrary constants:

$$i^4 f + i^2 f^3 = 0,$$

where the function $U$ or $f$ is equal to $i$ (and $m$ is equal 3). ($^i$ is the imaginary unit $(-1)^{1/2}$). The transformed general solution to the differential equation is:
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\[ u(\xi) = R^{-1}(U) = c_0 \|J(U)\|_2 = c_0 \sum_{-\infty}^{\infty} \int d\xi \ f(t) \ e^{2 \imath \xi} = \frac{c_0 \sqrt{\xi^2}}{\sqrt{2} (\xi^2 - 1)^2}. \]

The next step is ascertaining the transformed differential equation \( F \) assuming the solution is a traveling wave:

\[ \alpha u'(\xi) + \beta u'(\xi) + \beta^2 u''(\xi) + \beta^2 (u''(\xi) u(\xi)^2 + 2 u'(\xi)^2 u(\xi)) = 0 \]

and deriving the exact solution to the differential equation \( F \). Substituting the solution into the ansatz transformed differential equation \( F \), multiplying the differential equation by its common denominator, and ascertaining the algebraic equations about \( e^{\imath \xi} \), an individual would obtain the following expressions:

\[-2 \sqrt{2} \alpha c_0 + 2 \sqrt{2} \beta^4 c_0 + 2 \sqrt{2} \beta^2 c_0 - 2 \sqrt{2} \beta c_0 = 0,\]
\[2 \sqrt{2} \alpha c_0 + 2 \sqrt{2} \beta^4 c_0 + 2 \sqrt{2} \beta^2 c_0 + 2 \sqrt{2} \beta c_0 = 0,\]
\[-4 \sqrt{2} \alpha c_0 + 152 \sqrt{2} \beta^4 c_0 + 3 \sqrt{2} \beta^2 c_0 + 8 \sqrt{2} \beta^2 c_0 - 4 \sqrt{2} \beta c_0 = 0,\]
\[4 \sqrt{2} \alpha c_0 + 152 \sqrt{2} \beta^4 c_0 + 3 \sqrt{2} \beta^2 c_0 + 8 \sqrt{2} \beta^2 c_0 + 4 \sqrt{2} \beta c_0 = 0,\]

and
\[460 \sqrt{2} \beta^4 c_0 + 10 \sqrt{2} \beta^2 c_0 - 20 \sqrt{2} \beta^2 c_0 = 0.\]

Finally, if one uses the above algebraic equations to derive the value of the constants \( \alpha, \beta, \) and \( c_0 \) he will produce the exact solution to the differential equation \( F \) in the following form:

\[ u(x, t) = i \sqrt{6} \ \csc(t - x). \]

3.3 the Rayleigh-like equation

Consider the following differential equation \( F \):

\[ u_{tt} + u_{xx} + u_t + u_t^3 = 0. \]

The leading terms of the above equation gives rise to:

\[ u_{xx} + u_t^3 = 0. \]

Using FTM directly will not solve the differential equation, thus one can use a substitution function \( v \) for the general solution \( u \):

\[ V(\xi) = U'(\xi). \]

After stripping the ansatz transformed leading terms of all arbitrary constants and substituting in function \( v \) into the expression of leading terms, one obtains:
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\[ i f + f' = 0, \]

where the function \( V \) or \( f \) is equal to \( f' = \sqrt{-i} \) (and \( m \) is equal 3). The transformed general solution to the differential equation is:

\[ v(\xi) = \mathcal{R}^{-1}(V) = e_0 \|J(V)\|_2 = e_0 \sqrt{\sum_{-\infty}^{\infty} f(i\xi) e^{2i\xi}}, \]

where,

\[ u(\xi) = c_1 + \int v(\xi) d\xi. \]

The general solution \( u \) in terms of the transformed variable \( \xi \) is as follows:

\[ u(\xi) = c_1 - c_0 \tanh^{-1}\left(\sqrt{1 - e^{2\xi}}\right). \]

The next step is ascertaining the total ansatz transformed differential equation \( F \) assuming the solution is a traveling-wave or soliton:

\[ \alpha^2 u''(\xi) + \beta^2 u''(\xi) + \alpha u'(\xi) + \alpha^3 u'(\xi)^3 = 0 \]

and deriving the exact solution to the differential equation \( F \). Substituting the general solution \( u \) into the transformed differential equation \( F \), multiplying the differential equation by its common denominator, and ascertaining the algebraic equations about \( e^{\frac{\xi}{4}} \), an individual would obtain the following expressions:

\[ \alpha^2 (-c_0) + \alpha c_0 - \beta^2 c_0 = 0, \]

and

\[ -\alpha^3 c_0^3 - \alpha c_0 = 0. \]

Finally, if one uses the above algebraic equations to derive the value of the constants \( \alpha, \beta, \) and \( c_0 \) he will produce the exact solution \( u \) to the differential equation \( F \) in the following form:

\[ u(x, t) = c_1 - \frac{\sqrt{2} \beta^2 + \sqrt{1 - 4 \beta^2} - 1 \tanh^{-1}\left(\sqrt{1 - e^{\frac{\xi}{2} \sqrt{1 + 4 \beta^2 \tan^2 \beta x}}}\right)}{\sqrt{2} \beta^2}. \]

4. Conclusion

FTM appears to be quick and easy method for solving a subset of nonlinear partial differential equations yet possesses some limitations. Even though conceptually this new technique seems more elegant than the simplest equation method, it is an easier to apply in terms of practice. On the other hand, FTM may produce a more limited number solutions in comparison to other techniques. To help provide a greater number of solutions to NPDEs and solve linear partial differential equations from FTM check the appendix A.2.
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References


Appendix

A.1 Rules regarding field transforms of an ansatz transformed general solution \( u \) and its derivatives

Consider the field transform of the first order derivative of the ansatz transformed general solution \( u \):
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\[ \mathcal{R}(u'(\xi)) = \mathcal{R}
\left( \frac{\sum \int d\xi f(i)^p e^{i\xi^p}}{i^p \int d\xi f(i)^p e^{i\xi^p}} \right) \]

\[ \mathcal{R}(u''(\xi)) = \mathcal{R}
\left( \frac{\sum \int d\xi f(i)^p e^{i\xi^p}}{i^p \int d\xi f(i)^p e^{i\xi^p}} \right) \]

Then consider the field transform of the second order derivative of the ansarz transformed general solution \( u' \):

\[ \mathcal{R}(u''(\xi)) = \mathcal{R}
\left( \frac{\sum \int d\xi f(i)^p e^{i\xi^p}}{i^p \int d\xi f(i)^p e^{i\xi^p}} \right) \]

\[ \mathcal{R}(u''(\xi)) = \frac{f(i)^{1-p}}{i^p} \left( \frac{f(i)^{1-p}}{i^p} \right)^p \]

\[ \mathcal{R}(u''(\xi)) = i^p \frac{f(i)^{1-p}}{i^p} \left( \frac{f(i)^{1-p}}{i^p} \right)^p \]

\[ \mathcal{R}(u''(\xi)) = i f(i) \]

Then consider the field transform of the second order derivative of the ansarz transformed general solution \( u' \):
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\[ \mathcal{R}(u''(\xi)) = \mathcal{R}\left( \frac{\sum \int d\xi \sum f(\xi)^p \, e^{i\xi P} \sum i \int d\xi \, f(\xi)^p \, e^{i\xi P} \left( i \int d\xi \sum f(\xi)^p \, e^{i\xi P} \right)^2}{p^2} \right) \]

\[ \mathcal{R}(u''(\xi)) = \frac{1}{p^2} \left[ \left( \int d\xi \, \sum f(\xi)^p \, e^{i\xi P} \right)^{1 + \frac{1}{2} p} \right] \frac{1}{p^2} \left[ \left( \sum i \int d\xi \, f(\xi)^p \, e^{i\xi P} \right)^{1 + \frac{1}{2} p} \right] + (1 - p) \mathcal{R}(\sum i \int d\xi \sum f(\xi)^p \, e^{i\xi P}) \]

\[ \mathcal{R}(u''(\xi)) = \frac{f(t)^{1 + \frac{1}{2} p} \left( f(t) \left( i \int d\xi \sum f(\xi)^p \, e^{i\xi P} \right)^{1 + \frac{1}{2} p} \right) + (1 - p) \left( f(t) \left( i \int d\xi \sum f(\xi)^p \, e^{i\xi P} \right)^{1 + \frac{1}{2} p} \right)}{p^2} \]

\[ \mathcal{R}(u''(\xi)) = \frac{1}{p^2} f(t) \]

Thus, one ultimately obtains the following:

\[ \mathcal{R}(u(\xi)) = f(t) \]

\[ \mathcal{R}(u'(\xi)) = i \, f(t) \]

\[ \mathcal{R}(u''(\xi)) = i^2 \, f(t) \]

\[ \mathcal{R}(u'''(\xi)) = f(t) \, p \]

The rules dictating the field transform of the product of two functions \( u_1 \) and \( u_2 \) is similar to other transforms:
A.2 More complex FTM and using FTM to solve linear partial differential equations.

To increase the number of solutions produced by FTM, an individual should incorporate Chebyshev $U$ and/or square root Fibonacci combinatorial numbers about zero, or $\cos(i\pi/2)$ and/or $\sin(i\pi/2)$, respectively, within the formulation of the general solution:

$$u(\xi) = R^{-1}(U) = e_2 + e_0 \|JC_1(U)\|_p + e_1 \|JC_2(U)\|_p = e_1 \left( \sum_{i=0}^{m} \int_{0}^{\pi/2} \cos \left( \frac{\pi i}{2} \right) f(i\xi) \xi^2 \right)^{1/2} + e_0 \left( \sum_{i=0}^{m} \int_{0}^{\pi/2} \sin \left( \frac{\pi i}{2} \right) f(i\xi) \xi^2 \right)^{1/2} + e_2$$

or

$$u(\xi) = R^{-1}(U) = e_2 + e_0 \|JC_1(U)\|_p + e_1 \|JC_2(U)\|_p = e_1 \left( \sum_{i=0}^{m} \int_{0}^{\pi/2} \sin \left( \frac{\pi i}{2} \right) f(i\xi) \xi^2 \right)^{1/2} + e_0 \left( \sum_{i=0}^{m} \int_{0}^{\pi/2} \cos \left( \frac{\pi i}{2} \right) f(i\xi) \xi^2 \right)^{1/2} + e_2,$$

where $c_2$ is equal to zero if $p$ does not equal 1 or $m$ is not equal to 2. Finally, to solve linear partial differential equations, one just needs to consider the following general solution:

$$u(\xi) = R^{-1}(U) = e_2 + e_0 \|JC_1(U)\|_1 + e_1 \|JC_2(U)\|_1 = -2 e_1 \sinh(\xi) + 2 e_0 \cosh(\xi) + e_2$$

or

$$u(\xi) = R^{-1}(U) = e_2 + e_0 \|JC_1(U)\|_1 + e_1 \|JC_2(U)\|_1 = e_0 \left( e^2 \xi + 1 \right) + \frac{2 e_1}{1 - \coth(\xi)} + e_2,$$

where function $f$ is equal to $1/\xi$. 