Theoretical study of a spherically symmetric exact solution without event horizon and its gravity loss

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Abstract

To provide solutions for the unresolved theoretical questions of black holes, such as the presence of an event horizon, we propose a new spherically symmetric exact solution (we call the Ryskmit(R) solution). The R solution can be obtained by applying Kruskal-Szekeres coordinates (referred to hereinafter as Kruskal coordinates) to the Schwarzchild solution. The R solution has no singularities other than the origin of coordinates and no "event horizon"; therefore, a black hole from which information could not be extracted from the outside need not be considered. Far from the origin, this solution is approximately equal to the Schwarzschild solution. Another characteristics of this solution is that the gravity reaches its maximum at the Schwarzschild radius, and at the half of this radius, it transits to Minkowski space, in which gravity does not exist. This means that the gravity gradually decreases with distance from the Schwarzschild radius. Based on the law of conservation of energy , we deduced a result that explains the production of sufficient kinetic energy for gamma-ray burst. Furthermore, the metric of this solution was remarkably similar to the Reissner–Nordström metric, and the presence and absence of an electrical charge lead to two different masses at the scale of Planck units where the two solutions match. This is an important relationship for answering questions about dark matter. As described above, this exact solution could be a useful basic equation that sheds light not only on astrophysics, but also on particle theory and the unified field theory.

Keywords: internal structure of black holes – spherically symmetric exact solution – Kruskal coordinate – event horizon – perihelion shift – Reissner–Nordström solution – dark matter – Planck mass

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1 Introduction

Theories of black holes is progressing rapidly these years, and their presence is close to being confirmed based on observations such as gravitational waves [1] and the images of accretion disks [2]. However, many unanswered questions remain, such as the internal structure of black holes and gamma-ray bursts. In addition, problems surrounding dark matter and dark energy, whose essence is elusive even in theory, affect the foundation of physics. Such issues present both exciting opportunities and challenges for physicists.

One reason for this theoretical difficulty is that many theoretical models of black holes rely on the Schwarzschild solution [3] and its extension, i.e., the Kerr solution [4]. Since their proposition, various types of spherically symmetrical solutions have been proposed [5][6] and various types of non-symmetrical solutions are also derived [7]. Although the Schwarzschild solution is, historically, the most well-known and simplest model, it is a highly unsatisfactory solution, since it does not address the many mysteries of black hole interiors, such as the event horizon and the essential singularity with infinite internal gravity.

In this study, we deduce a spherically symmetric exact solution similar to the Schwarzschild solution by applying a simple extension to the Kruskal coordinates [8] and explain how it works in various characteristics of the space-time concept. For example, this solution enables to reach continuously into the interior of a black hole without an event horizon. Here the gravity gradually decreases and completely vanishes at half of the Schwarzschild radius, at which point the system transits to Minkowski space where it is flat structure on which special relativity is formulated. Thus, the internal information of a black hole can be obtained from its external environment.

2 Method and Results

The Kruskal coordinates [8] can effectively explain the external solution of a black hole and are expressed as follows in spherical coordinates r, θ and φ. Here the signs of the metric are $(- + ++)$.

$$\begin{aligned} ds^2 &= f(r,t)^2(du^2 - dv^2) + r^2 d\Omega^2 \\ &= f(r,t)^2(g'(r)^2 dr^2 - g(r)^2 h'(ct)^2 c^2 dt^2) + r^2 d\Omega^2 \end{aligned} \tag{1}$$

where

$$\begin{aligned} & d\Omega^2 = d\theta^2 + \sin\theta^2 d\varphi^2 \\ & u(r,t) = g(r)\cosh h(ct), v(r,t) = g(r)\sinh h(ct) \end{aligned} \tag{2}$$

Assuming that, with an arbitrary function $A(r)$, the equation takes the form:

$$ds^2 = -A(r)c^2 dt^2 + \frac{1}{A(r)}dr^2 + d\Omega^2 \tag{3}$$

where

$$u = \left(\frac{r}{\alpha} - 1\right)^{\frac{1}{2}} \exp\frac{r}{2\alpha}\cosh\frac{ct}{2\alpha}, v = \left(\frac{r}{\alpha} - 1\right)^{\frac{1}{2}} \exp\frac{r}{2\alpha}\sinh\frac{ct}{2\alpha},$$
$$f(r)^2 = \frac{4\alpha^3}{r}\exp\left(-\frac{r}{\alpha}\right)$$

We can obtain the Schwarzschild solution

$$ds^2 = -\left(1 - \frac{\alpha}{r}\right)c^2dt^2 + \frac{1}{\left(1 - \frac{\alpha}{r}\right)}dr^2 + r^2d\Omega^2, \alpha = \frac{2GM}{c^2} \tag{4}$$

This is a unique solution with:

$$g(r) = (ar + b)^{\frac{1}{2}} \exp\frac{r}{2\alpha}, h(ct) = \frac{ct}{2\alpha}, \alpha = \frac{2GM}{c^2}, f(r)^2 = \frac{4\alpha^3}{r}\exp\left(-\frac{r}{\alpha}\right)$$
$$a \neq 0, a, b = const. \tag{5}$$

It is assumed that, in the denominator of the metric dr^2 calculated in $du^2 - dv^2$, all coefficients are 0 for the terms other than that having the highest order, r^2.

To further generalize eq. (5), the following equation can be used:

$$g(r) = (ar^2 + br + d)^{\frac{1}{2}} \exp\frac{r}{2\alpha}, h(ct) = \frac{ct}{2\alpha}, f(r)^2 = \frac{4\alpha^2}{ar^2}\exp\left(-\frac{r}{\alpha}\right)$$
$$a \neq 0, a, b, d = const. \tag{6}$$

Where it is assumed that, similar to eq. (5), in the denominator of the metric dr^2 calculated in $du^2 - dv^2$, all coefficients are 0 for the terms other than that having the highest order, r^4. Simple calculations based on (1) result in:

$$ds^2 = -\left(1 - \frac{2\alpha}{r} + \frac{2\alpha^2}{r^2}\right)c^2dt^2 + \frac{1}{1 - \frac{2\alpha}{r} + \frac{2\alpha^2}{r^2}}dr^2 + r^2d\Omega^2 \tag{7}$$

This is also a unique solution that satisfies the condition given by (6). We provisionally refer to this as the R solution (the Ryskmit solution).
Under a condition such that this solution agrees with the Schwarzschild solution far from the origin, the constant α in eq. (7) can be given as:

$$\alpha = \frac{GM}{c^2}.$$

Furthermore, by solving Einstein's equations using the metric (7), where the Einstein tensor is expressed as:

$$G^i_k = R^i_k - \frac{1}{2}g^i_k R$$

we obtain:

$$G^0_0 = G^1_1 = \frac{4\alpha^2}{r^4}, G^2_2 = G^3_3 = -\frac{4\alpha^2}{r^4} \tag{8}$$

3 Discussion

i) Versatility of the Kruskal coordinates

The Kruskal coordinates used in this paper are able to express many exact solutions of this form in a unified way.
For example, with

$$g(r) = \sqrt{\frac{1+\alpha r}{1-\alpha r}}, h(ct) = \alpha ct, \alpha = \sqrt{\frac{\Lambda}{3}}, f(r)^2 = \frac{(1-\alpha r)^2}{\alpha^2} \tag{9}$$

we can obtain the metric of the De Sitter universe [9]:

$$ds^2 = -\left(1 - \frac{\Lambda}{3}r^2\right) c^2 dt^2 + \frac{1}{1 - \frac{\Lambda}{3}r^2} dr^2 + r^2 d\Omega^2 \tag{10}$$

This also satisfies eq. (3).

In addition, with

$$\begin{aligned} &g(r) = \exp(\sin^{-1}\sqrt{K}r), h'(ct) = \frac{\partial h(ct)}{\partial ct}, \\ &\alpha(ct) = \frac{1}{h'(ct)}, f(r)^2 = \exp(-\sin^{-1}\sqrt{K}r) \end{aligned} \tag{11}$$

the following equation is obtained:

$$ds^2 = -h'(ct)c^2 dt^2 + \frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2$$

Furthermore, if this is rewritten as

$$\frac{ds^2}{h'(ct)^2} \Rightarrow ds^2,$$

we obtain the Robertson–Walker metric [10]:

$$ds^2 = -c^2 dt^2 + a(ct)^2 \left(\frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2\right) \tag{12}$$

This is an example where the Kruskal coordinates is applied without satisfying eq. (3).

Minkowski metric.can be expressed as a special case:

$$u = \exp\frac{r}{2\alpha}\cosh\frac{ct}{2\alpha}, v = \exp\frac{r}{2\alpha}\sinh\frac{ct}{2\alpha}, f(r)^2 = \exp\left(-\frac{r}{\alpha}\right) \tag{13}$$

In particular, when $f(r)^2 = 1$, the above equation can express the Rindler metric [11].

Moreover, in eq. (4), if it is assumed that h is a function of r and t, the following conversion can be made:

$$u(r,t) = g(r)\cosh h(ct,r), v(r,t) = g(r)\sinh h(ct,r) \tag{14}$$
$$h(ct,r) = \frac{ct}{2\alpha} - \frac{1}{2}\log\left(\frac{r}{\alpha} - 1\right) \Rightarrow \frac{ct'}{2\alpha}$$

to obtain the Eddington's form [12]. From this, we arrive at the Kerr solution [4] by generalizing the Schwarzschild solution and degenerating these metrics [13].

As mentioned above, utilizing the Kruskal coordinates is a simple and unified method that can derive various spherically symmetric exact solutions, such as the Schwarzschild solution, without using the Christoffel symbols. Here, we provisionally name the forms deduced using this method as "Kruskal forms."

The R solution derived in this study provides one of the simplest metrics to satisfy "Kruskal forms". It successfully explains certain previously observed physical phenomena, for example, the deflection of a light ray and the perihelion shift of star like the Schwarzschild solution., and is specialized by a "natural form" of extension including the Schwarzschild solution derived using Riemannian geometry. This solution has previously been explored by Eddington [14] and the metric used is similar to that of parametrized post-Newtonian formalism (PPN) [15] , which has recently featured in discussions concerning proposed modifications to the general theory of relativity [16]. For example:

$$ds^2 = -\left(1 - \frac{2\alpha}{r} + \beta\frac{2\alpha^2}{r^2} + \cdots\right)c^2dt^2 + \left(1 + \gamma\frac{2\alpha}{r} + \cdots\right)(dx^2 + dy^2 + dz^2)$$
$$\beta, \gamma = const.$$

According to the observation, the value of β of temporal metric is shown to be equal to 1 within the error 10^{-5} [17]. So the temporal metric is almost identical, comparison of the spatial components expects that they are approximately the same in nature using isotropic coordinates and have yet to be examined using the exact R solution. Therefore, transforming this R solution to isotropic coordinates may lead to new developments in PPN.

ii) Characteristics of the black hole space-time by the R solution

While the metrics of the R solution approximate to the Schwarzschild solution far from the origin, spatial structures inside the Schwarzschild radius are significantly different; the most prominent feature of the R solution is that the event horizon necessary for the formation of a black hole [14][18] does not exist. Various discussions are made about the existence and the effect of event horizon [19], but in the R solution, it is not necessary to assume the event horizon. This is because $A(r) = 1 - \frac{2\alpha}{r} + \frac{2\alpha^2}{r^2}$ is always positive and has no singularities other than that at $r = 0$. However, while the question of whether a structure without an event horizon can be called a black hole is under active discussion, we herein refer to space-time within the Schwarzschild radius is referred to as a black hole for purposes of simplicity. The absence of an event horizon indicates that information inside the Schwarzschild radius can be extracted from outside. As shown in Figure 1, the value of $A(r)$ in the R solution reaches its minimum $(\frac{1}{2})$ at Schwarzschild radius $r = 2\alpha$, and becomes 1 at $r = \alpha$. This indicates that the curvature of space due to gravity vanishes at half the Schwarzschild radius, at which point the condition is the same as in Minkowski space. In other words, in a black hole within the Schwarzschild radius, gravity gradually decreases and eventually vanishes. Such concept has not been proposed to the present. Furthermore, reduced gravity also leads to changes in gravitational mass, such that inside a black hole, there may be separation of inertial mass and gravitational mass. Various quasi-black hole models [20] have been proposed, such as the gravastar model [21], which combines the Schwarzschild solution (eq. (4) at outside of the black hole) and the de Sitter solution (eq. (10) at inside of the black hole), which does not require an event horizon, and the brane-world black hole model [22], which considers the second order term r^{-2} as $A(r) = 1 - \frac{\alpha}{r} + \frac{\beta}{r^2}$. β is the constant found by 5-dimentional Weyl tensor. These models are similar to the R solution in some aspects such as the possibilities of having no event horizon or of showing gravity decrease inside the black hole. In spite of these efforts, there have been no reports of a single, continuous, and exact solution wherein gravity decreases inside the Schwarzschild radius, and in which the r^{-2} term is defined. Therefore, new developments can be expected based on decreasing gravity using the R solution.

If the kinetic energy of substances inside the black hole subsequently overcomes the gravitational mass applied from the exterior part of the black hole, explosive destruction may occur from the inside of the black hole. It remains unclear whether gamma-ray bursts, the most explosive phenomenon known in the universe [23], are caused by an imbalance between two large stellar masses that formed when the universe was smaller than it is at present. If a star equivalent in size to the Sun converted its inertial mass to energy, E, it would yield:

$$E = mc^2 \sim 2 \times 10^{30} \times (3 \times 10^8)^2 \sim 2 \times 10^{47} J.$$

This represents a sufficient amount of energy for gamma-ray bursts and such explosive phenomenon can thus be explained using the R solution.

On close inspection, for the R solution, $A(r)$ diverges to infinity at the essential singularity $r = 0$, and the decrease of gravity inside the Schwarzschild radius implies that the internal spatial structure of a black hole can withstand the enormous gravity from outside, even beyond the Chandrasekhar limit [24]. It may be possible to construct a model that maintains equilibrium for $\alpha < r < 2\alpha$ according to this kinetic pressure from Minkowski space. Due to intervention from Minkowski space, in which gravity vanishes, it is questionable whether the space can shrink beyond $r = \alpha$. Consequently, the condition for $r < \alpha$ and the essential singularity at $r = 0$ may be mere mathematical artifacts, which would require significant modifications to the conventional concept of black holes, in which matter is simply condensed within.

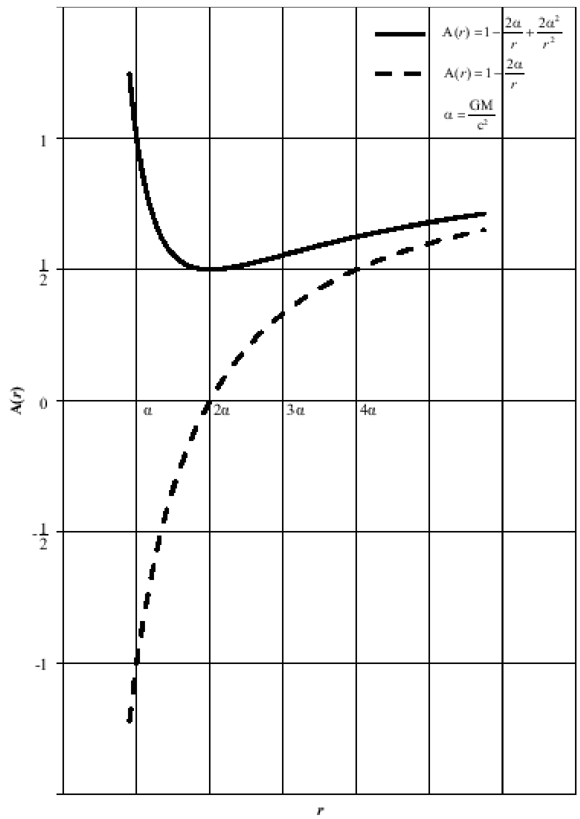

Figure 1: Plots of function $A(r)$ and distance r for the Schwarzschild solution and the R solution.

The R solution significantly influences the space within the Schwarzschild radius, causing the r^{-2} term to rapidly decrease with increasing distance r, and eventually approximate the Schwarzschild solution (Figure 1). This indicates that the effect of the R solution is mainly limited to within the Schwarzschild radius, and that the conventional Schwarzschild solution can address wide interstellar spaces.

In the following section, we report on the orbits of matter obtained with the R solution near the Schwarzschild radius by putting $\theta = \frac{1}{2}\pi$. Based on the motion of a test particle, m, by the Schwarzschild solution [13]

$$\left(\frac{dr}{d\varphi}\right)^2 = \frac{r^4}{L_\varphi}\left[\left(\frac{E}{c}\right)^2 - \left(1 - \frac{2\alpha}{r} + \frac{2\alpha^2}{r^2}\right)\left(\frac{L_\varphi^2}{r^2} + m^2c^2\right)\right] \tag{15}$$

$$\text{where, } \frac{E}{c} = m\left(1 - \frac{2\alpha}{r} + \frac{2\alpha^2}{r^2}\right)c\dot{t}, L_\varphi = -mr^2\dot{\varphi}, \alpha = \frac{GM}{c^2} \tag{16}$$

Using the transformation $u \equiv \frac{1}{r}$ and taking a derivative with respect to φ, with

$$a = \frac{L_\varphi}{m}, h = \frac{a^2}{GM},$$

we obtain the following:

$$\frac{d^2u}{d\varphi^2} + \left(1 + \frac{2\alpha}{h}\right) u = \frac{1}{h} + 3\alpha u^2 - 4\alpha^2 u^3 \tag{17}$$

For Mercury, where perihelion shift by the gravitation was solved for the first time by Einstein, the perihelion shift occurs in a substantially weaker gravitational field than in a black hole. In such cases, where $r \gg \alpha, h \gg \alpha$, we neglected $\frac{2\alpha}{h}$ on the left side of eq. (17) instead made an approximation using the terms of u^2 on the right side. This is equivalent to Einstein's equation [25]. However, in a strong gravitational field, the R solution indicates that other terms can also have an effect.

For instance, SgrA* [26], which is assumed to be the black hole of the Milky Way and a strong source of gravity, has many orbiting stellar and gas cloud [27]. In such cases, it is expected that eq. (17) explains the perihelion shift. Long-term observations will shed light on whether perihelion shifts of them orbiting SgrA* are different from those predicted by Einstein.

iii) Similarity between the R solution and the Reissner-Nordström solution [28][29]

The form of the R solution is similar to that of the Reissner-Nordström solution (18).

$$ds^2 = -\left(1 - \frac{2\alpha}{r} + \frac{Ge^2}{4\pi\varepsilon_0 c^4 r^2}\right) c^2dt^2 + \frac{1}{1 - \frac{2\alpha}{r} + \frac{Ge^2}{4\pi\varepsilon_0 c^4 r^2}} dr^2 + r^2 d\Omega^2 \tag{18}$$

where ε_0 is the permittivity of a vacuum, e is the electrical charge, and $\alpha = \frac{GM}{c^2}$. The Einstein tensor G^i_k in both solutions are the same except for the constants (see equation (8)).

Assuming that the two solutions with the same structure match exactly, as the only difference is the term of r^{-2},

This leads to $2\frac{G^2M^2}{c^4r^2} = \frac{Ge^2}{4\pi\varepsilon_0 c^4 r^2}$ and finally to $2\frac{GM^2}{r^2} = \frac{e^2}{4\pi\varepsilon_0 r^2}$, which is suggests gravity and Coulomb force.

With further transformation, we obtain an equation independent of r

$$M = \sqrt{\frac{e^2}{8\pi\varepsilon_0 G}}, (M > 0) \tag{19}$$

By further applying transformation, we obtain

$$M = \sqrt{\frac{1}{2}} \cdot \sqrt{\frac{e^2}{4\pi\varepsilon_0 \hbar c}} \cdot \sqrt{\frac{\hbar c}{G}} = \sqrt{\frac{1}{2}} \cdot \alpha \cdot M_p \tag{20}$$

Here, α is the fine structure constant and M_p is the Planck mass.

The Reissner-Nordström solution is an exact solution for a single charge. Therefore, there would be two masses in the quantum mechanical level i.e., both the mass of this solution and that of the R solution. It is not clear whether the two solutions can be identified; however, by calculating the constant part of eq.(20), we obtain

$$M \sim 0.06M_p$$

Thus, 6% of the Planck mass is the mass in the R solution. In other words, when a charge acts on the Planck mass, it is replaced by a small mass. This may contribute to our understanding of dark matter [30][31]. "Electrical charge" is a physical concept that is associated with a particle; however, at a particle level, charge plays a role in interacting with mass and changing mass. The Reissner-Nordström solution assumes the space-time of a stationary single electrical charge and does not consider interactions with magnetic and electromagnetic fields. A general solution may be obtained by including such interactions into account.

4 Conclusion

1) The present study proposes a simple extension of the Kruskal coordinates, which are used in general relativity. In addition, this study identified a new exact solution (termed the Ryskmit solution, it is the R solution for short), which is different from the Schwarzschild solution.

2) The most striking characteristic of this metric is that, while it can be approximated by the Schwarzschild solution far from the origin of coordinates, there is no event horizon at the Schwarzschild radius, allowing internal information to be extracted. This differs from a conventional black hole.

3) There are two values of r where the metric of the R solution matches the Minkowski metric: infinity and half the Schwarzschild radius. This implies that, within the Schwarzschild radius, the gravity gradually decreases and eventually vanishes.

4) It is implied that the motion of matter in the R solution differs from the Einstein's perihelion shift near the Schwarzschild radius.

5) The R solution is similar to the Reissner-Nordström solution, and their spacial structures are the same. When the two exactly match, the presence of an electrical charge results in two masses, which would shed a new light on the problem of dark matter.

6) As shown above, it is expected that further analysis of the R solution will make tremendous contributions not only to astrophysics but also to particle theory and the unified field theory.

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