Solutions of the Navier-Stokes equations

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Abstract

We show how we can find the solutions of the Navier-Stokes equations.

1 The Navier-Stokes equations

For a vector $u = (u_i)$, the Navier-Stokes equations are the following ones:

$$\frac{\partial u_i}{\partial t} + \sum_j u_j \frac{\partial u_i}{\partial x_j} = \nu \Delta(u_i) - \frac{\partial p}{\partial x_i}$$

$$div(u) = \sum_i \frac{\partial u_i}{\partial x_i} = 0$$

with p the pression.
The problem is to find solutions of the Navier-Stokes equations with an initial function $u_0 = u(x, 0)$.

2 Geometrization

For a riemannian manifold (M, g), the Navier-Stokes equations can be written down as:

$$\dot{u} + \nabla_u u = \Delta(u) + df^*$$

$$d^* u = 0$$

with f a function depending on u.

3 The solutions of the NS equations

We recognize the equation of geodesics for $u = \dot{\gamma}$:

$$\nabla_{\dot{\gamma}} \dot{\gamma} = 0$$

So that we can define a Lagrangian over the curves γ:

$$L(\gamma) = \int_0^t (||\dot{\gamma}||^2 + \nu ||d\gamma||^2) dt$$

with constraint $d^* \dot{\gamma} = 0$.
If we minimize the Lagrangian over the curves γ, we find the solutions of the Navier-Stokes equations.

References

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