Young's Double-Slit and Wheeler's Delayed-Choice Experiments at a Single-Quantum Level: Wave-Particle Non-Duality

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A new 'wave-particle non-dualistic interpretation at a single-quantum level' is presented by showing the physical nature of Schrödinger's wave function as an 'instantaneous resonant spatial mode' to which a particle's motion is confined. The initial phase associated with a state vector is identified to be related to a particular position eigenstate of the particle and hence, the equality of quantum mechanical time to classical time is obtained; this equality automatically explains the emergence of classical world from the underlying quantum world. Derivation of the Born rule as a limiting case of *the relative frequency of detection* is provided for the first time, which automatically resolves the *measurement problem*. Also, the Born rule derivation is supplemented with a geometrical interpretation. 'What's really going on' in Young's double-slit and Wheeler's delayed-choice experiments is explained at a single-quantum level. Also, an interference experiment is proposed to verify the correctness of the non-dualistic interpretation.

I. INTRODUCTION

Prof. Feynmann remarked, "*We choose to examine a phenomenon which is impossible, absolutely impossible, to explain in any classical way, and which has in it the heart of quantum mechanics. In reality, it contains the only mystery*", where, the 'phenomenon' stands for wave-particle duality of a single quantum in Young's double-slit experiment [1]. Photons, electrons, neutrons, atoms, molecules, etc., are shown to exhibit the duality [2–7]. Providing an explanation to this mystery and also a causal explanation for Wheeler's delayed-choice experiment [8] by uniting the mutually exclusive wave and particle natures into a single non-dualistic entity, is the main purpose of the present article. This kind of uniting is possible only in quantum mechanics.

There are various interpretations of quantum mechanics [9–23]. The present 'wave-particle non-dualistic interpretation at a single-quantum level' provides a "derivation for Born's rule as a limiting case of the relative frequency of detection", which shows the absence of *measurement problem* in quantum mechanics. Importantly, non-duality never deviates from the quantum formalism and hence, it reproduces all aspects, like, expectation values of the observables, Heisenberg's uncertainty relation, etc., and successes of the same; because, it yields the Born rule when repeated measurements are made on a large number of identical states. It only brings out the picture of reality existing in the quantum world. In other words, it's just a 'quantum formalism as it is - interpretation'.

In section-II, the physical nature of Schrödinger's wave function is unraveled and the inner-product of a state vector with its dual is shown to be a kind of interaction arising in a measuring device. In sections-III & IV, the relation between the initial phase of a state vector and a particular outcome of an observable is worked out. Derivation of Born's rule and a solution to the 'measurement problem' are given in section-V. Equality of quantum mechanical time to classical time is shown in section-VI. "What's really happening?" in the Young's double-slit and Wheeler's delayed-choice experiments is explained in sections- VII and VIII, respectively. An interference experiment, to verify the physical nature of wave function is proposed in section-IX. Section-X contains the conclusions and discussions.

II. PHYSICAL NATURE OF SCHRÖDINGER'S WAVE FUNCTION

By analogy and contrast between the classical and quantum mechanical situations of a free particle in one-dimensional Euclidean space (1DES), the physical nature of the wave function is brought into light by a mathematical reasoning. The particle is emitted by a source and is absorbed at some later time. The same reasoning remains valid, as it can be straightforwardly verified, even for a non-free particle in 3DES.

The free-particle's classical and quantum mechanical Hamiltonians are given by,

$$H = \frac{p^2}{2m} = E \quad ; \text{and} \quad \hat{H}|\psi >= \frac{\hat{p}^2}{2m}|\psi >= E|\psi >, \tag{1}$$

respectively, where, m is the mass of the particle; p and E are momentum and total energy in the classical scenario and $\hat{p}$, $|\psi >$ and E are momentum operator, energy eigenstate and energy eigenvalue in the quantum mechanical case, respectively. The Hamiltonian equations of motion yield the following solutions,

$$x(t) = \frac{p(0)}{m}t + x(0) \quad ; \quad \text{and} \quad p(t) = p(0), \tag{2}$$

where, $x(0)$ and $p(0)$ are constants of integration corresponding to the initial position and initial momentum at time $t = 0$, whereas, Heisenberg's equations of motion results in,

$$\hat{x}(t) = \frac{\hat{p}(0)}{m}t + \hat{x}(0) \quad ; \quad \text{and} \quad \hat{p}(t) = \hat{p}(0); \tag{3}$$

here, $\hat{x}(t)$ & $\hat{p}(t)$ and $\hat{x}(0)$ & $\hat{p}(0)$ are time-dependent and time-independent position and momentum operators, respectively, such that one has the following commutation relations,

$$[\hat{x}(t) \ , \ \hat{p}(t)] = [\hat{x}(0) \ , \ \hat{p}(0)] = i\hbar. \tag{4}$$

$$[\hat{x}(0) \ , \ \hat{x}(t)] = \frac{i\hbar}{m}t \quad \text{and} \quad [\hat{p}(0) \ , \ \hat{p}(t)] = 0, \tag{5}$$

and also the eigenvalue equations,

$$\hat{x}(0)|\hat{x}(0) >= x(0)|x(0) > \quad ; \quad \hat{x}(t)|\hat{x}(t) >= x(t)|x(t) >, \tag{6}$$

where, $i = \sqrt{-1}$ and $\hbar$ is the reduced Plank's constant; $\{|x(0) > |x(0) \in \mathbf{R}\}$ & $\{x(0)|x(0) \in \mathbf{R}\}$ and $\{|x(t) > |x(t) \in \mathbf{R}\}$ & $\{x(t)|x(t) \in \mathbf{R}\}$ are the sets of eigenstates and eigenvalues of $\hat{x}(0)$ and $\hat{x}(t)$, respectively; $\mathbf{R}$ is the set of real numbers spanning the 1DES.

Using Eq. (4), the time-independent Schrödinger's wave equation can be written in the position bases as,

$$\text{either} \quad -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x(t))}{\partial x(t)^2} = E\psi(x(t)) \quad \text{or} \quad -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x(0))}{\partial x(0)^2} = E\psi(x(0)); \tag{7}$$

here, $\psi(x(t)) =< x(t)|\psi >$ and $\psi(x(0)) =< x(0)|\psi >$. Notice that, $\psi(x(t))$ does not explicitly depend on t. As long as the form of eigenvalue equations in Eq. (7) are considered, both $\psi(x(t))$and $\psi(x(0))$ describe the same physical situation.

When a particle appears at the source, in the classical scenario given in Eq. (2), $x(0)$ is a chosen unique value in $\mathbf{R}$. But, the same in the quantum mechanical case given in Eqs. (6) and (7), one has a set of all possible initial values, $\{x(0)|x(0) \in \mathbf{R}\}$, which are the eigenvalues of $\hat{x}(0)$ and hence, $\psi(x(0))$ is a function on $\mathbf{R}$, implying that, the moment the particle appears at the source, $\psi(x(0))$ appears instantaneously on the entire 1DES. Notice a symmetry that the reverse is also true, i.e., the moment the particle disappears at some later time t by absorption, then the wave function also disappears instantaneously, resembling the 'wave function collapse' advocated in the Copenhagen interpretation [9–11]. As well-known from experiments [2–7], the collapse occurs at some particular eigenvalue, say $x_p(t)$ (the subscript p stands for particle). Hence, by using the same symmetry, even the appearance of particle at the source can be inferred to occur at some definite eigenvalue, say $x_p(0)$. Note that, 'the appearance of ψ at the moment of particle's appearance and its disappearance at the moment of particle's disappearance' is analogous to a resonance process.

Using the position bases in Eq. (6), $|\psi >$ can be written as,

$$|\psi >= \int_{\mathbf{R}} dx(0)|x(0) >< x(0)|\psi >= \int_{\mathbf{R}} dx(t)|x(t) >< x(t)|\psi > . \tag{8}$$

Again following the same reasoning as above, the moment the particle appears at the source, $|\psi >$ $(= \{|x(0) >< x(0)|\psi > |x(0) \in \mathbf{R}\})$ appears instantaneously in a complex vector space (CVS) spanned by the continuous basis set $\{|x(0) > |x(0) \in \mathbf{R}\}$. After some time t, $\{|x(0) > |x(0) \in \mathbf{R}\}$ evolves to $\{|x(t) > |x(t) \in \mathbf{R}\}$ such that the initial position eigenvalue $x_p(0)$ of the particle changes to $x_p(t)$. $|x_p(0) >< x_p(0)|\psi > \in \{|x(0) >< x(0)|\psi > |x(0) \in \mathbf{R}\}$; here, $|x_p(0) >< x_p(0)|\psi >$ is the particular position eigenstate where the particle appeared initially. At t, the particle's position eigenstate is $|x_p(t) >< x_p(t)|\psi > \in \{|x(t) >< x(t)|\psi > |x(t) \in \mathbf{R}\}$. Hence, it's clear that the particle moves in the CVS but always confined to $|\psi >$. As it's known, $\psi(x(0))$ is in one-to-one correspondence with $\{|x(0) >< x(0)|\psi > |x(0) \in \mathbf{R}\}$

and $\psi(x(t))$, with $\{|x(t)><x(t)|\psi>|x(t)\in\mathbf{R}\}$. Also, the functional form of ψ is independent of its arguments, because, $\{x(0)|x(0)\in\mathbf{R}\}=\{x(t)|x(t)\in\mathbf{R}\}=\mathbf{R}$. Thus, it can be concluded that ψ is like a spatial mode in which the quantum particle moves akin to the case of a test particle moving in the curved space-time of the general theory of relativity [24]. Hence, it can be concluded that the the physical nature of Schrödinger's wave function (or equivalently the state vector) is '*Instantaneous Resonant Spatial Mode* (IRSM). Now onwards, whenever it's necessary, the IRSM is used synonymously to both Schrödinger's wave function and the state vector.

Like an eigenstate and its eigenvalue, the inseparable nature of IRSM and its particle is named as the wave-particle non-duality (WPND). Born's probabilistic interpretation [9], "*The wave function determines only the probability that a particle - which brings with itself energy and momentum - takes a path; but no energy and no momentum pertains to the wave*", resembles the IRSM, except for the notion of probability. In a recent experiment, the lower limit for the speed of collapse of a delocalized photon state is estimated to be 1550 times the speed of light [25]. But, according to the WPND, such a speed is infinity due to the instantaneous nature of the wave function.

During either absorption or scattering by collision, eigenvalues and their simultaneous eigenstate undergo a change. If a particle suddenly gets scattered at some position eigenvalue, say x_s, instead of absorption, then its IRSM disappears akin to the case of absorption and a new one with origin at x_s appears for the scattered particle.

A. Inner-Product as an Interaction

A classical-wave's intensity is proportional to the square of its amplitude. But, according to the WPND, Schrödinger's wave function can't be claimed to have such an intensity, because, it's an IRSM and is unlike a propagating classical wave.

Suppose that a particle ends up on a detector screen. A dual vector gets induced in the same screen and interacts according to the inner-product, which can be found within the quantum formalism. The scattering of $|\psi>$ into some other state, say $|\psi'>$, can be described by associating an operator, $\hat{O}=|\psi'><\psi|$, to the detector:

$$\hat{O}|\psi>=<\psi|\psi>|\psi'> . \tag{9}$$

Notice that, analogous to an image in a mirror, $<\psi|$ is confined only to the screen. Therefore, if the scattered state, $|\psi'>$, is discarded, then the particle must have interacted at some location in the region of the inner-product, $<\psi|\psi>$.

If a detector is associated with a projection operator $\hat{P}$, then the induced dual is $<\psi|\hat{P}^\dagger=<\psi|\hat{P}$ and Eq. (9) becomes,

$$\hat{O}|\psi>=\left(|\psi'><\psi|\hat{P}\right)|\psi>\,. \tag{10}$$

Therefore, the inner-product interaction is given by $<\psi|\hat{P}|\psi>$.

III. STATE VECTOR'S INITIAL PHASE AND A PARTICULAR OUTCOME OF AN OBSERVABLE

Prof. Dirac's statement [26], "*Questions about what decides whether the photon is to go through or not and how it changes its direction of polarization when it does go through can not be investigated by experiment and should be regarded as outside the domain of science*", is the actual inspiration behind the proposal of a relation between the initial/overall/global phase associated with the state vector and a particular eigenstate of an observable.

Consider the toss of a coin in the 3DES, in a CVS as shown in FIG. (1) and mapping of the coin in CVS to a charged spin-$\frac{1}{2}$ particle in the Stern-Gerlach (SG) apparatus [1, 27, 28] given in FIG. (2).

A. Toss of Coin in the 3D-Euclidean Space

Let $|n>$ be a normal vector to the head-surface, passing through coin's center-of-mass and α be an angle between $|n>$ and a vector, $|g>$, parallel to the gravitational field. Just before the landing of coin, consider its position at a height $h<r$ above the ground surface; here, r is the radius of the coin. If $-\pi/2<\alpha<\pi/2$, then head will be the outcome. Otherwise, tail occurs for $\pi/2<\alpha<3\pi/2$. Depending on the value of α, coin will jump into either head or tail state. Upon the outcome, $|n>$ will be pointing either parallel or anti-parallel to $|g>$. Note that, from the moment of toss to a point at h, $|n>$ itself will be varying from a given initial conditions, both in space and time, obeying Newton's equations of motion. The detailed dynamics of $|n>$ is immaterial for the probabilistic description, but only the value of α matters.

B. Toss of Coin in a Complex Vector Space

Since, the coin system is aimed to map onto spin-$\frac{1}{2}$ system in the SG apparatus, let's choose the eigenvalues $+\frac{1}{2}$ and $-\frac{1}{2}$ for the outcomes of head and tail, respectively. Also notice that, all the vectors considered in this subsection belongs to a CVS as shown in FIG. (1).

Let $|H>$ and $|T>$ be the eigenstates for the head and tail, respectively. Upon the outcome, $|n>$ will be pointing along either $|H>$ or $|T>$ which can also be regarded as anti-parallel vectors to $|g>$. Since, head and tail are mutually exclusive with respect to observation, one has $<T|H>=0$. The vector space above the ground can be taken as a direct sum of $|H>$ and $|T>$. Let α and β be the phase angles made by $|n>$ with $|H>$ and $|T>$, respectively, such that $|\alpha|+|\beta|=\pi$.

In any CVS of any dimensionality, one can always write $<a|b>=|<a|b>|.e^{i\theta}$ between any pair of vectors $|a>$ and $|b>$; where, $|<a|b>|$ is the absolute value of the complex number, $<a|b>$, and θ is the phase angle between the vectors:

$$<H|n>=|<H|n>|.e^{i\alpha}\ ;<T|n>=|<T|n>|.e^{i\beta}\ ;\ |\alpha|+|\beta|=\pi\ . \tag{11}$$

Let $\hat{C}$ be an observable of the coin:

$$\hat{C}=\frac{1}{2}(|H><H|-|T><T|)\ ;\ \hat{C}|H>=\frac{1}{2}|H>\ ;\ \hat{C}|T>=-\frac{1}{2}|T>, \tag{12}$$

where, $<H|H>=<T|T>=1$. Using the unit operator, $\hat{I}=|H><H|+|T><T|$ in the CVS above the ground-surface, $|n>$ can be expressed as,

$$\begin{aligned}|n>&=|H><H|n>+|T><T|n>\\&=|H>.|<H|n>|.e^{i\alpha}+|T>.|<T|n>|.e^{i\beta}.\end{aligned} \tag{13}$$

According to the criterion of the minimum phase given in subsection 3.1, if $|\alpha|<|\beta|$, then the coin enters into $|H>$ and if $|\alpha|>|\beta|$, then into $|T>$. Notice that, either α or β will be minimum at a time because $|\alpha|+|\beta|=\pi$ (the case of $|\alpha|=|\beta|$ is ruled out because, $h<r$). As an explicit example, consider $|\alpha|<|\beta|$; then, upon observation,

$$<n|n>\longrightarrow|<H|n>|^2\ ;\left(\text{occurrence of the eigenvalue }+\frac{1}{2}\right). \tag{14}$$

Consider another tossed coin represented by a state vector $|\tilde{n}>$ which is related to the previous coin as,

$$|\tilde{n}>=e^{i\phi}.|n>\ . \tag{15}$$

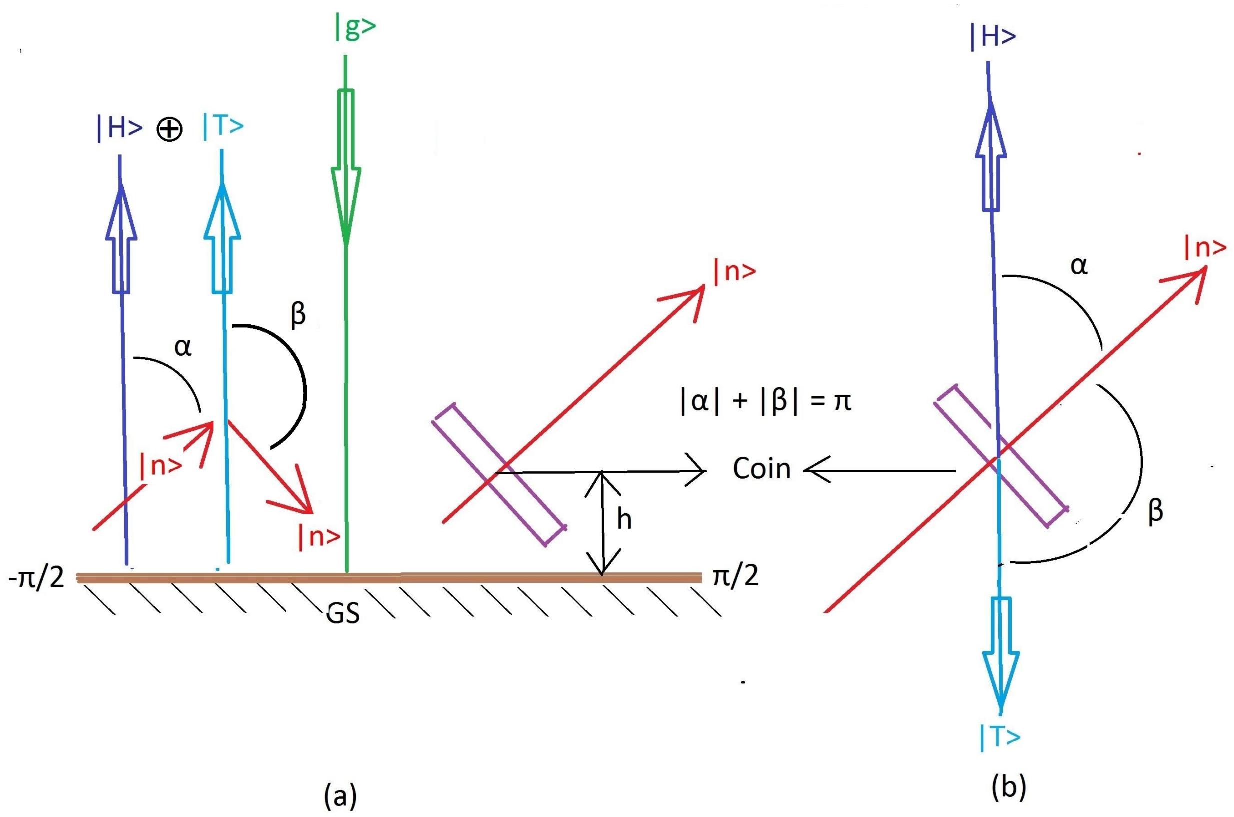

FIG. 1. **Schematic Diagram for the Toss of a Coin**: (a) h is the height of coin above the ground surface (GS) and is supposed to be less than the radius of coin. $|g>$ is a vector parallel to the gravitational field direction and perpendicular to the GS. $|n>$ is a vector normal to the head-surface passing through the center-of-mass of the coin. The outcomes, head and tail, are represented by the state vectors $|H>$ and $|T>$, respectively, which are taken to be anti-parallel to $|g>$. They are mutually exclusive with respect to the observation, i.e., $<T|H>=0$ (in the space above the GS). (b) α and β are the angles between $|H>\&|n>$ and $|T>\&|n>$, respectively; $|\alpha|+|\beta|=\pi$. If $|\alpha|<|\beta|$ ($|\beta|<|\alpha|$), then the coin enters into $|H>$ ($|T>$).

where, ϕ is the overall phase by which the second coin differs from the first one. Expressing $|\tilde{n}>$ akin to $|n>$ in Eq. (13):

$$|\tilde{n}>=|H>.|<H|n>|.e^{i(\phi+\alpha)}+|T>.|<T|n>|.e^{i(\phi+\beta)}. \tag{16}$$

Depending upon whether $|(\phi+\alpha)|<|(\phi+\beta)|$ or $|(\phi+\alpha)|>|(\phi+\beta)|$, the coin will enter into either $|H>$ or $|T>$, respectively.

Notice in Eq. (14) that, the absolute length of $|n>$ and hence the value of $|<H|n>|$

is immaterial for the case of single observation except for the eigenvalue. However, for an infinitely large number of tosses, the relative frequency of detection (RFD), $\frac{|<H|n>|^2}{<n|n>}$, must coincide with the probability of occurrence for heads, i.e., $\frac{1}{2}$, due to the constraint $|\alpha| + |\beta| = \pi$, which fixes $| < H|n > | = \frac{1}{\sqrt{2}}$.

C. Charged Spin-$\frac{1}{2}$ Particles in the SG Apparatus

All quantum phenomena actually happen in a CVS, while the eigenvalues live in 3DES. If quantum mechanics (QM) is taken to be more fundamental than the classical mechanics (CM), then obviously, any macroscopic object also lives in the CVS, because, it's a composite of 'quantum entities'. Hence, Nature herself dwells in the CVS, otherwise, the quantum mechanical commutation relations can't have any physical meaning. Hence, *the observables of a particle and the measuring device must commute with each other in order to detection to happen.* Therefore, the CVS of particle's observable can be used to represent the CVS of the measuring device and vice versa - which also explains the induced dual of a state vector as given in Eqs. (9) and (10. This is made use of hereafter.

Let SG_x, SG_y and SG_z be the SG apparatuses [1, 27], where the magnetic field directions are along X, Y and Z axises, respectively. By taking the gravitational and magnetic field directions along Z-axis, the states of the coin discussed in the subsection 3.2 can be mapped to that of a spin-$\frac{1}{2}$ particle in the SG_z as follows,

$$|H > \longrightarrow |S_z;\uparrow> \; ; |T > \longrightarrow |S_z;\downarrow>, \tag{17}$$

$$\hat{C} \longrightarrow \hat{S}_z = \frac{1}{2}(|S_z;\uparrow>< S_z;\uparrow| - |S_z;\downarrow>< S_z;\downarrow|), \tag{18}$$

$$\hat{I} \longrightarrow \hat{I}_z = |S_z;\uparrow>< S_z;\uparrow| + |S_z;\downarrow>< S_z;\downarrow|, \tag{19}$$

where, $\hat{S}_z$ is the Z-component of the total spin operator, $\hat{S}$, with eigenstates $|S_z;\uparrow>$ and $|S_z;\downarrow>$ corresponding to spin-up and spin-down states, respectively, and $\hat{I}_z$ is the unit operator in the CVS of $\hat{S}_z$. Consider an initial spin state 'up along Y', $|S_y;\uparrow>$, subjected to SG_z:

$$|n > \longrightarrow |S_y;\uparrow>= |S_z;\uparrow>< S_z;\uparrow|S_y;\uparrow> + |S_z;\downarrow>< S_z;\downarrow|S_y;\uparrow> . \tag{20}$$

Akin to the case of Eq. (13), the above equation becomes,

$$\begin{aligned} |S_y;\uparrow> &= |S_z;\uparrow> .|<S_z;\uparrow|S_y;\uparrow>|.e^{i\alpha} + |S_z;\downarrow> .|<S_z;\downarrow|S_y;\uparrow>|.e^{i\beta} \\ &= |S_z;\uparrow> .R.e^{i\alpha} + |S_z;\downarrow> .R.e^{i\beta}, \end{aligned} \tag{21}$$

where, $|<S_z;\uparrow|S_y;\uparrow>| = |<S_z;\downarrow|S_y;\uparrow>| = R$, $<S_z;\uparrow|S_y;\uparrow>= Re^{i\alpha}$ and $<S_z;\downarrow|S_y;\uparrow>= Re^{i\beta}$; here, R is a positive real number. Depending on whether $|\alpha| < |\beta|$ or $|\alpha| > |\beta|$, the particle enters into either $|S_z;\uparrow>$ or $|S_z;\downarrow>$, respectively. For example, let $|\alpha| < |\beta|$, then the particle will be in$|S_z;\uparrow>$ and $|S_z;\downarrow>$ will be remaining as an ontological empty state - see FIG. (2). Therefore, observation,

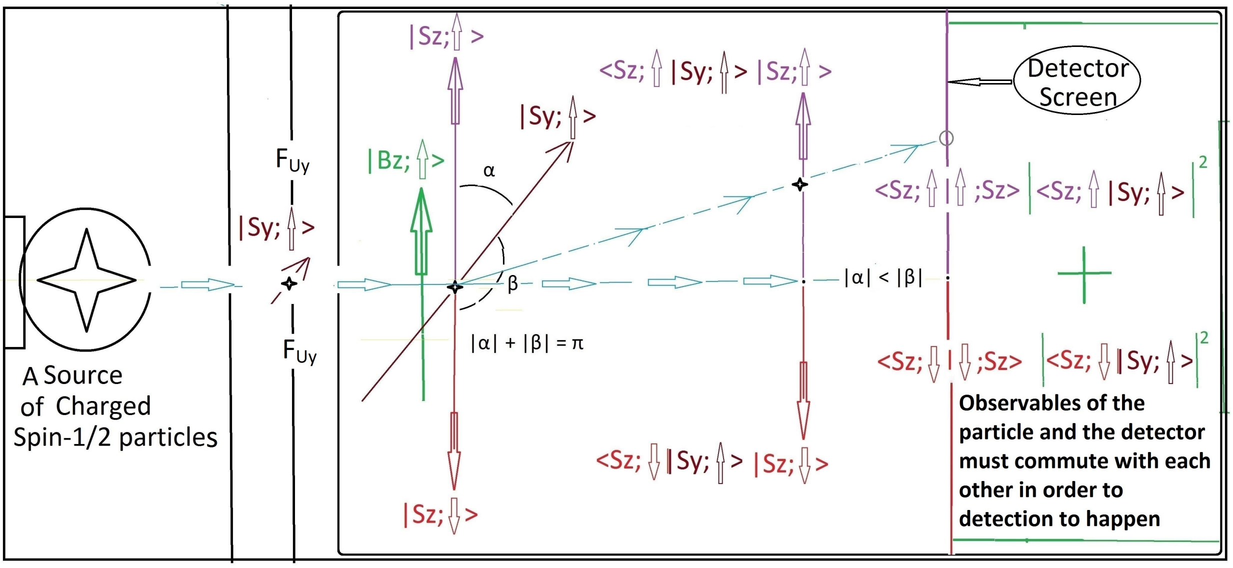

FIG. 2. **Schematic Diagram for the Stern-Gerlach Apparatus**: A source emits a charged spin-$\frac{1}{2}$ particle, whose initial state is filtered 'up along Y-axis', $|S_y;\uparrow>$, by a filter F_{Uy}. Then the particle is subjected to the Stern-Gerlach measurement along Z-axis. For the case of $|\alpha| < |\beta|$, the particle enters into $|S_z;\uparrow>$ and the state, $|S_z;\downarrow>$, remains without a particle. During the observation, the particle contributes a point to $|<S_z;\uparrow|S_y;\uparrow>|^2$, while the empty mode, $|S_z;\downarrow>$, contributes nothing.

$$\begin{aligned} <S_y;\uparrow|S_y;\uparrow> &= |<S_z;\uparrow|S_y;\uparrow>|^2 + |<S_z;\downarrow|S_y;\downarrow>|^2 \\ &\longrightarrow |<S_z;\uparrow|S_y;\uparrow>|^2, \end{aligned} \tag{22}$$

yields an eigenvalue $+\frac{1}{2}$; because, $|S_z;\downarrow>$ has no particle to contribute.

Consider another spin state prepared 'up along Y', $|\tilde{S}_y;\uparrow>$, which differs from the previous one only by an overall phase as,

$$|\tilde{S}_y;\uparrow>= e^{i\phi}.|S_y;\uparrow> . \tag{23}$$

The SG_z feels $|\tilde{S}_y;\uparrow>$ as,

$$|\tilde{S}_y;\uparrow>= |S_z;\uparrow> .R.e^{i(\alpha+\phi)} + |S_z;\downarrow> .R.e^{i(\beta+\phi)}. \tag{24}$$

Depending on whether $|(\alpha+\phi)| < |(\beta+\phi)|$ or $|(\alpha+\phi)| > |(\beta+\phi)|$, the particle enters into either $|S_z;\uparrow>$ or $|S_z;\downarrow>$, respectively. Therefore, it's sufficient to notice in Eq. (21) that, the values of α and β will be different for different 'up along Y' spin states. Similar to Eq. (21), let's write,

$$|S_y;\downarrow> = |S_z;\uparrow> .R.e^{i\alpha'} + |S_z;\downarrow> .R.e^{i\beta'}, \tag{25}$$

$$|S_x;\uparrow> = |S_z;\uparrow> .R.e^{i\gamma} + |S_z;\downarrow> .R.e^{i\delta}, \tag{26}$$

$$\text{and} \qquad |S_x;\downarrow> = |S_z;\uparrow> .R.e^{i\gamma'} + |S_z;\downarrow> .R.e^{i\delta'}. \tag{27}$$

Block the $|S_z;\downarrow>$ in Eq. (21) and subject $|S_z;\uparrow>$ to SG_x having the unit operator $\hat{I}_x = |S_x;\uparrow>< S_x;\uparrow| + |S_x;\downarrow>< S_x;\downarrow|$:

$$\begin{aligned} R.e^{i\alpha}.|S_z;\uparrow> &= R.e^{i\alpha}.|S_x;\uparrow>< S_x;\uparrow|S_z;\uparrow> + R.e^{i\alpha}.|S_x;\downarrow>< S_x;\downarrow|S_z;\uparrow> \\ &= R^2.e^{i(\alpha-\gamma)}.|S_x;\uparrow> + R^2.e^{i(\alpha-\delta)}.|S_x;\downarrow> . \end{aligned} \tag{28}$$

Again, depending on whether $|(\alpha-\gamma)|$ or $|(\alpha-\delta)|$ is minimum, which in turn depends on α, the particle will enter into either $|S_x;\uparrow>$ or $|S_x;\downarrow>$, respectively. All initial states prepared 'up along Y' will, in general, differ from each other by initial phases occurring randomly. Those random phases never contribute to the inner-product but responsible for the outcomes of different eigenvalues of an observable. As it can be easily seen, akin to the case described in Eq. (14), the Born rule emerges out here as a limiting case of RFD.

According to the requirement of WPND to describe a single-quantum behavior, a generalized representation for the $SU(2)$ algebra respecting the Eqs. (21), (25), (26) and (27) is explicitly worked in the appendix A.

IV. "PHASE-TUBE" STRUCTURE OF QUANTUM STATE VECTOR AND THE BORN RULE

In this section, any quantum state is shown to fall into a phase-hole, P_H, which sweeps a phase-tube, P_T, along the direction of particle's motion. If the quantum state becomes a superposition of, say, two orthogonal eigenstates of some observable, then the phase-tube branches into two smaller tubes as shown in FIG. 3. This kind of geometrical structures exist only in the complex vector space, where, the actual quantum phenomena happen.

A. Phase-Hole Representation of Quantum State Vector

As considered in Eq. (21), various spin states, filtered through F_{Uy}, can be written as given below:

$$|S_y(\alpha);\uparrow>= |S_z;\uparrow> .| < S_z;\uparrow |S_y;\uparrow> |.e^{i\alpha} + |S_z;\downarrow> .| < S_z;\downarrow |S_y;\uparrow> |.e^{i\beta}, \tag{29}$$

where, α is a discrete and random variable depending on the nature of source. Notice that, different $|S_y;\uparrow>$ states can be characterized either by α or β, because, α and β are always related as shown in section-II; here, α is chosen. The following set of vectors,

$$P_H = \{|S_y(\alpha);\uparrow> |\alpha \in [0, 2\pi]\}, \tag{30}$$

can be plotted on a complex-plane as shown in FIG. 3(a). The tips of all vectors lie on the circumference of a circle of unit radius, since, $|S_y(\alpha);\uparrow>$ is normalized to unity. Therefore, any vector belonging to P_H always passes through the F_{Uy}. In other words, in the perspective of quantum particle, our perspective of single direction in F_{Uy} appears as a hole (P_H). In a nutshell, the unit vector $|S_y;\uparrow>$ is actually a phase-hole, P_H, for the quantum particle. In reality, there is nothing special about the vector $|S_y;\uparrow>$. Hence, any arbitrary state vector encountered by a quantum particle can always be regarded as a corresponding phase-hole associated with that vector.

B. Superposition of Eigenstates with Equal Amplitudes

Consider $|S_y(\alpha);\uparrow>$ in Eq. (29) as a superposition of $\hat{S}_z$'s eigenstates with equal amplitudes as given below:

$$|S_y(\alpha);\uparrow>= \left(\frac{1}{2}\right)^{\frac{1}{2}} e^{i\alpha}|S_z;\uparrow> + \left(\frac{1}{2}\right)^{\frac{1}{2}} e^{i\beta}|S_z;\downarrow> . \tag{31}$$

However, according to non-duality, as already shown earlier, $|S_y(\alpha);\uparrow>$ lies in a circular phase-hole, P_H, of unit radius. By the same token, $|S_z;\uparrow>$ and $|S_z;\downarrow>$ can also be said to lie in the corresponding circular phase-holes, say P_{UH} and P_{DH}, each with $(\frac{1}{2})^{\frac{1}{2}}$ as radius; here, P_{UH} and P_{DH} correspond to up-phase-hole and down-phase-hole as shown in the FIG. 3(b), respectively. Notice that, $P_{UH} \cap P_{DH} = \{\}$, because, any vector from P_{UH} is orthogonal to any vector in P_{DH}. As the particle moves, P_H sweeps a tube, say P_T, which branches into P_{UT} and P_{DT}; here, P_{UT} and P_{DT} are phase-tubes generated by P_{UH} and P_{DH}, respectively. Also notice that, every particle state in P_{UT} has a corresponding empty state in P_{DT} and vice versa (see FIG. 3(b) & 3(c)).

When a huge number of particles, say N, enters P_T, then some of them, say N_U, moves through P_{UT} and the remaining, say N_D, through P_{DT}. Obviously, one has $N = N_U + N_D$. Also, $N_U = (A_U/A)N$ and $N_D = (A_D/A)N$; here, A, A_U and A_D are the areas of cross-section of P_T, P_{UT} and P_{DT}, respectively. Therefore, one has,

$$\frac{N_U}{N} + \frac{N_D}{N} = \frac{A_U}{A} + \frac{A_D}{A} = 1 = R_U + R_D, \tag{32}$$

where, $R_i = N_i/N = A_i/A$, corresponds to the relative frequency of detection or Born's probability; here, $i = U, D$. Therefore, it's clear that, the conservation of total number of particles implies the conservation of the total of area of cross-sections of the phase-tubes, which yields the Born rule in Eq. (32). Hence, one has,

$$A = A_U + A_D \implies \pi = \frac{\pi}{2} + \frac{\pi}{2}. \tag{33}$$

The above equation implies the splitting of the interval, $[0, \pi]$, as,

$$[0, \pi] = [0, \pi/2] \cup [\pi/2, \pi], \tag{34}$$

and the physical phenomenon in the interval, $[\pi, 2\pi]$, is exactly identical to the one in $[0, \pi]$. Therefore, depending on whether $|\alpha| \in [0, \pi/2]$ or $|\alpha| \in [\pi/2, \pi]$, the quantum particle enters into either P_{UT} or P_{DT}, respectively.

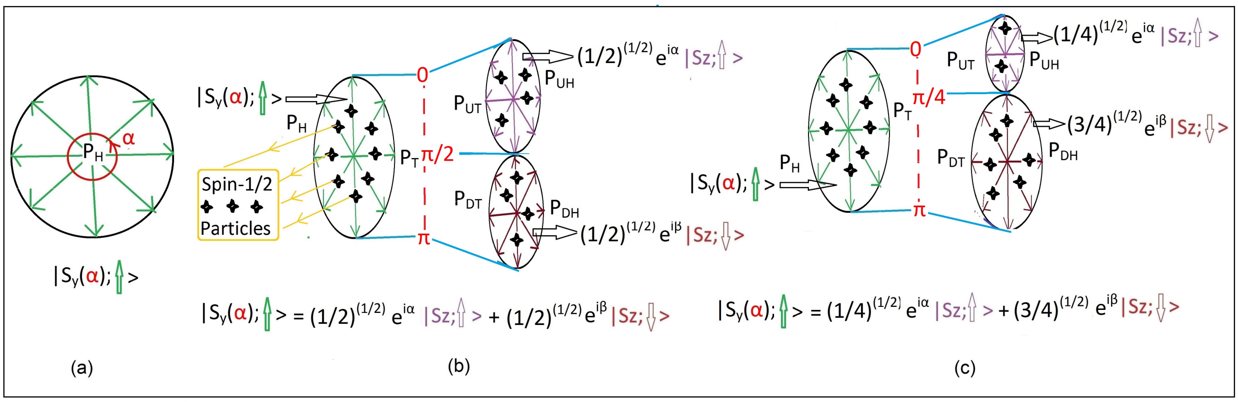

FIG. 3. **Schematic Diagram of Phase-Tubes**: **(a)** All initial states, $|S_y(\alpha);\uparrow>$, are plotted with a common origin on a complex plane. The tips of all vectors lie on a circle of unit radius, which is named as 'Phase-Hole', P_H; here, α occurs discretely and randomly. **(b) & (c)** P_H sweeps a 'Phase-Tube', P_T, in the direction of particle's motion. P_T branches into 'up-phase-tube', P_{UT}, and 'down-phase-tube', P_{DT}, because, any vector from P_{UH} is orthogonal to any vector in P_{DH}; here, P_{UH} and P_{DH} are up-phase-hole and down-phase-hole, respectively. For convenience, the state vectors are drawn symmetrically, which is not true in reality due to the nature of α. See main text for the details of equations.

C. Superposition of Eigenstates with Unequal Amplitudes

Consider $|S_y(\alpha);\uparrow>$ in Eq. (29) as a superposition of $\hat{S}_z$'s eigenstates with unequal amplitudes as given below:

$$|S_y(\alpha);\uparrow>= \left(\frac{1}{4}\right)^{\frac{1}{2}} e^{i\alpha}|S_z;\uparrow> + \left(\frac{3}{4}\right)^{\frac{1}{2}} e^{i\beta}|S_z;\downarrow> . \tag{35}$$

All phase-tube details of the above equation is identical to the one given in section-II(b) for Eq. (31), except for how the interval, $[0,\pi]$, splits. Notice that, the phase-tube structure given in FIG. 3(c) can be obtained from the one in FIG. 3(b) by uniformly shrinking and stretching the P_{UT} and P_{DT}, respectively.

By making use of the conservation of total cross-sectional area, one has from Eq. (35),

$$A = A_U + A_D \implies \pi = \frac{1}{4}\pi + \frac{3}{4}\pi, \tag{36}$$

implying the splitting of $[0,\pi]$ as,

$$[0,\pi] = [0,\pi/4] \cup [\pi/4,\pi], \tag{37}$$

Therefore, depending on whether $|\alpha| \in [0, \pi/4]$ or $|\alpha| \in [\pi/4, \pi]$, a quantum particle enters into either P_{UT} or P_{DT}, respectively.

D. General Case of Superposition of $\hat{S}_z$'s Eigenstates

The above analysis can be straightforwardly applied to the generic case given in Eq. (29),

$$|S_y(\alpha); \uparrow>= |S_z; \uparrow> .| < S_z; \uparrow |S_y; \uparrow> |.e^{i\alpha} + |S_z; \downarrow> .| < S_z; \downarrow |S_y; \uparrow> |.e^{i\beta}, \tag{38}$$

as follows:

By making use of the conservation of total cross-sectional area, one has,

$$A = A_U + A_D \implies \pi = | < S_z; \uparrow |S_y; \uparrow> |^2\pi + | < S_z; \downarrow |S_y; \uparrow> |^2\pi = R_U\pi + R_D\pi, \tag{39}$$

where, $R_U = | < S_z; \uparrow |S_y; \uparrow> |^2$ and $R_D = | < S_z; \downarrow |S_y; \uparrow> |^2$, implying the splitting of $[0, \pi]$ as,

$$[0, \pi] = [0, R_U\pi] \cup [R_U\pi, \pi], \tag{40}$$

Hence, depending on whether $|\alpha| \in [0, R_U\pi]$ or $|\alpha| \in [R_U\pi, \pi]$, the quantum particle enters into either P_{UT} or P_{DT}, respectively.

Consider the detection of a single particle in the SG_z apparatus for the case $|\alpha| \in [0, R_U\pi]$. According to non-duality, the state $|S_y(\alpha); \uparrow>$ induces its dual-state and interacts at the detector screen according to the inner-product:

$$\begin{aligned} < S_y(\alpha); \uparrow |S_y(\alpha); \uparrow> &= | < S_z; \uparrow |S_y; \uparrow> |^2 + | < S_z; \downarrow |S_y; \uparrow> |^2 \\ &\xrightarrow[|\alpha| \in [0, R_U\pi]]{\text{Detection}} | < S_z; \uparrow |S_y; \uparrow> |^2, \end{aligned} \tag{41}$$

resulting in the detection of eigenvalue, $+\frac{1}{2}$; the particle itself contributes a point to $| < S_z; \uparrow |S_y; \uparrow> |^2$, while $| < S_z; \downarrow |S_y; \uparrow> |^2$ receives zero contribution [see FIG. 2]. When a large number of particles are sent through F_{Uy}, either one at a time or all at once, then the particles from both intervals in Eq. (40) contribute:

$$< S_y(\alpha); \uparrow |S_y(\alpha); \uparrow>= | < S_z; \uparrow |S_y; \uparrow> |^2 + | < S_z; \downarrow |S_y; \uparrow> |^2, \tag{42}$$

which is the same result as in Eq. (39) modulo π.

V. DERIVATION OF BORN'S RULE USING INDIVIDUAL QUANTUM EVENTS

The results obtained in the sections-III & IV are generalized in this section.

A. Observable with Discrete Eigenstates: Minimum Phase and Quantum Jump

If $|\psi>$ encounters a CVS spanned by the discrete orthogonal eigenstates, $|a_i>$; $i = 1, 2, 3, \cdots$, of an operator, $\hat{A}$:

$$|\psi>=\sum_i |a_i><a_i|\psi>, \tag{43}$$

then, as shown in the sections-III & IV, the particle enters into one of the eigenstate, say $|a_p>$, having the minimum phase with respect to $|\psi>$, i.e., phase$\{<a_p|\psi>\}$ < phase$\{<a_i|\psi>\}\forall i \neq p$. Due to the interaction of $|\psi>$ with its dual as shown in Eq. (9), an observation yields,

$$<\psi|\psi>=\sum_i <\psi|a_i><a_i|\psi>\longrightarrow |<a_p|\psi>|^2, \tag{44}$$

and the particle will be naturally found in $|a_p>$ with an eigenvalue a_p, because, all other ontological orthogonal states are empty. The particle contributes a point to the function $|<a_p|\psi>|^2$. Repeated measurements on a large number of identical state vectors, each one of them differing from the other only by the initial phase, yield different eigenvalues of $\hat{A}$. Notice that, as shown by an explicit example in sections-III & IV, the range of the set of initial phases can be divided into various subsets, such that, each subset is related to a particular eigenvalue. Normalizing the number of particles found in $|a_p>$ with respect to the total number of particles yields the RFD. As it can be easily seen from Eq. (44), in the limit of infinite number of particles, the RFD coincides with $|<a_p|\psi>|^2$:

$$<\psi|\psi>=\sum_p |<a_p|\psi>|^2 = 1, \tag{45}$$

which is the well-known Born's rule. Therefore, QM itself is not about probabilities, because, it can be deterministically described in a CVS at a single quantum level. Nevertheless, the unavailability of the information about the initial phase of $|\psi>$ due to inner-product forces experiments to observe only RFD which, anyhow, yields Born's rule as a limiting case.

B. Bohr's Complementarity at a Single-Quantum Level

Suppose that, instead of $\hat{A}$, $|\psi>$ encounters a different observable, $\hat{B}$, whose CVS is spanned by the eigenstates, say $|b_i>$:

$$|\psi>=\sum_i |b_i><b_i|\psi> . \tag{46}$$

The particle will be present in some eigenstate, $|b_p>$, making a minimum phase with $|\psi>$. The inner-product at the detector B is,

$$<\psi|\psi>=\sum_i <\psi|b_i><b_i|\psi>\longrightarrow |<b_p|\psi>|^2, \tag{47}$$

yielding the eigenvalue b_p and the particle contributes a point to $|<b_p|\psi>|^2$. Therefore, it's the measuring device, either A or B, where the inner-product interaction occurs, decides which property, either a_p or b_p, of the particle to be observed. This is Bohr's principle of complementarity [8, 29, 30], but, at a single-quantum level, provided $\hat{A}$ and $\hat{B}$ are non-commuting observables. However, notice that, *the non-dualistic picture of a particle flying in its IRSM is further irreducible and is independent of any measuring device.* Therefore, the wave and particle natures are not complementary to each other in QM like in the CM, albeit the position and momentum of a quantum are. Finally, notice that, due to the principle of minimum phase and the inner-product interaction, the *measurement problem* is absent in QM.

C. Sequential Selective Measurements

Consider three sequential detectors A, B and C corresponding to the observables $\hat{A}$, $\hat{B}$ and $\hat{C}$, whose eigenstates and eigenvalues are $|a_i>$, $|b_j>$ and $|c_k>$ and a_i, b_j and c_k, respectively [27]; here, $i, j, k = 1, 2, 3, \cdots$. Let A, B and C select some particular states $|a'_i>$, $b'_j>$ and $|c'_j>$ and reject the rest, respectively. Then, in A's CVS,

$$|\psi>=\sum_i <a_i|\psi>|a_i>, \tag{48}$$

and only $|a'_i>$ component comes out:

$$|\psi>=\sum_i <a_i|\psi>|a_i>\longrightarrow <a'_i|\psi>|a'_i>\equiv |\tilde{a}_i> \tag{49}$$

If $|\psi>$ makes a minimum phase with respect to $|a'_i>$, then the particle enters into B's CVS:

$$|\tilde{a}_i>=\sum_j <b_j|\tilde{a}_i>|b_j>\longrightarrow <b'_j|\tilde{a}_i>|b'_j>, \tag{50}$$

because, B allows only $|b'_j>$, which in turn encounters C's CVS,

$$<b'_j|\tilde{a}_i>|b'_j>=<b'_j|\tilde{a}_i>\sum_k <c_k|b'_j>|c_k>\longrightarrow <b'_j|\tilde{a}_i><c'_k|b'_j>|c'_k> \ . \tag{51}$$

because, only $|c'_k>$ comes out and the inner-product interaction results in a RFD,

$$\mathrm{RFD}_C = |<b'_j|\tilde{a}_i>|^2|<c'_k|b'_j>|^2. \tag{52}$$

Suppose, B allows all $\{|b_j>|j\in N\}$ to pass through. Then C will encounter a superposition,

$$\sum_j <b_j|\tilde{a}_i>|b_j>=|\tilde{a}_i>, \tag{53}$$

In B's CVS, though only $|b'_j>$ contains the particle, but all other empty modes do exist ontologically and if unblocked, they contribute at C:

$$\begin{aligned} |\tilde{a}_i> &= \sum_k\sum_j <b_j|\tilde{a}_i><c_k|b_j>|c_k> \\ &= \sum_k <c_k|\tilde{a}_i>|c_k>\longrightarrow <c'_k|\tilde{a}_i>|c'_k> \\ \text{yielding,} \qquad \mathrm{RFD}_C &= |<\tilde{a}_i|c'_k>|^2, \end{aligned} \tag{54}$$

which is entirely different from Eq. (52). Therefore, ontological empty modes produce physically observable effect.

In Eq. (52), if $|<b'_j|\tilde{a}_i>|^2$ is regarded as a probability for the particle to go through the $|b'_j>$ route in B and $|<c'_k|b'_j>|^2$ as a probability of finding the same at C, then they do obey the usual rule of probability multiplication. Here, if the probability is really in play, then its total, say $P(c'_k)$, for the particle to arrive at C through all possible routes in B,

$$P(c'_k) = \sum_j |<b'_j|\tilde{a}_i>|^2|<c'_k|b'_j>|^2, \tag{55}$$

must be the same as without B. But, in the absence of B, Eq. (54) gives the total probability of finding the particle at C, which is not the same as Eq. (55). This is a clear proof for the absence of probability in quantum mechanics. Only the RFD arises at a detector when repeated measurements are made on identical states, differing only by initial phases. If the presence of a particle is inferred by a probability, that too, in the absence of observation, then it will not yield the correct picture of a single-quantum.

D. Observable with Continuous Eigenstates and no Quantum Jump

In the case of an observable with continuous eigenvalues, there will always be an eigenstate whose phase with respect to $|\psi>$ will be the same as the initial phase of $|\psi>$ itself. Instead of $\hat{A}$ in the Eq. (43), consider the position operator, $\hat{\mathbf{r}}$, with orthogonal eigenstates, $|\mathbf{r}>$ and continuous eigenvalues, $\mathbf{r} = (x, y, z)$, spanning the 3DES:

$$|\psi>= \iiint d^3\mathbf{r}|\mathbf{r}><\mathbf{r}|\psi> . \tag{56}$$

The particle naturally enters into a position eigenstate, say $|\mathbf{r}_p><\mathbf{r}_p|\psi>$, without any quantum jump, such that phase$\{<\mathbf{r}_p|\psi>\}$ = phase$\{|\psi>\}$ (also, see section-II). Therefore, the interaction of $|\psi>$ with its induced dual is,

$$<\psi|\psi>= \iiint d^3\mathbf{r} <\psi|\mathbf{r}><\mathbf{r}|\psi> \longrightarrow |<\mathbf{r}_p|\psi>|^2, \tag{57}$$

because, except $|\mathbf{r}_p><\mathbf{r}_p|\psi>$, the remaining orthogonal states, $|\mathbf{r}><\mathbf{r}|\psi>$, are empty. The RFD in the limit of infinite number particles is,

$$<\psi|\psi>= \iiint d^3\mathbf{r}_p|<\mathbf{r}_p|\psi>|^2 = 1, \tag{58}$$

which is the Born rule. Therefore, if the position variable in Schrödinger's wave equation is identified with $\mathbf{r}_p$, then the unavoidable inference is that the particle should be present in multiple locations at the same time, which, in turn demands the 'collapse of the wave function' upon observation [9–11, 16, 17].

VI. PATH OF A QUANTUM PARTICLE THROUGH ITS IRSM

In section-II, it's shown that a particle moves in its IRSM ($|\psi>$), but nothing was said about its motion along some trajectory, if exists. In order to uncover the same, the propagators [27] are derived in a new way using the Heisenberg's equations of motion, because, the time-dependent Schrödinger's wave equation is not explicitly considered in the present article. Application of the notion of minimum phase, as given in subsection 5(D), to the propagator results in a path of least action as shown below:

Substitution from Eq. (3) into the second part of Eq. (6) results,

$$(\hat{x}(0) + \frac{t}{m}\hat{p}(0))|x(t)>= x(t)|x(t)>, \tag{59}$$

which can be expressed as a first order partial differential equation by making use of the unit operator, $\int dx(0)|x(0)><x(0)|$,

$$\left(-i\hbar\frac{t}{m}\frac{\partial}{\partial x(0)}+x(0)-x(t)\right)<x(0)|x(t)>=0, \tag{60}$$

whose solution can be found to be,

$$<x(0)|x(t)>=\exp\left\{-\frac{im}{2\hbar t}[x^2(0)-2x(0)x(t)+\alpha]\right\}, \tag{61}$$

where, $-\frac{im}{2\hbar t}\alpha$ is an integration constant. Similarly, making use of the identity operator, $\int dx(t)|x(t)><x(t)|$, in the first part of Eq. (6) along with a substitution from Eq. (3), results in the equation,

$$\left(i\hbar\frac{t}{m}\frac{\partial}{\partial x(t)}+x(t)-x(0)\right)<x(t)|x(0)>=0, \tag{62}$$

having a solution,

$$<x(t)|x(0)>=\exp\left\{\frac{im}{2\hbar t}[x^2(t)-2x(0)x(t)+\beta]\right\}, \tag{63}$$

where, $\frac{im}{2\hbar t}\beta$ is another constant of integration. Using the property, $C=C^\star$, of a complex number, C, in Eqs. (61) and (63) yields,

$$x^2(t)-2x(0)x(t)+\beta=x^2(0)-2x(0)x(t)+\alpha^\star, \tag{64}$$

whose solutions are,

$$\beta=\sigma+x^2(0)\quad\text{and}\quad\alpha^\star=\sigma+x^2(t), \tag{65}$$

where, σ is a constant. Hence, Eq. (63) can be rewritten as,

$$<x(t)|x(0)>=\exp\left\{\sigma'+\frac{im}{2\hbar t}\left(x(t)-x(0)\right)^2\right\}, \tag{66}$$

with $\sigma'=\frac{im}{2\hbar t}\sigma$. From the requirement,

$$\lim_{t\to 0}<x(t)|x(0)>=\delta(x(t)-x(0)), \tag{67}$$

an inference, $e^{\sigma'}=\sqrt{\frac{m}{2\pi i\hbar t}}$, can be made, but it works only for a free particle case. The following is a general procedure:

Using the identity operators in the position basis at time t and at $t=0$ as,

$$\begin{aligned}\hat{I}(t)&=\int dx(t)|x(t)><x(t)|\\&=\iiint dx'(0)dx''(0)dx(t)|x'(0)>F(x(t),x'(0),x''(0))<x''(0)|,\end{aligned} \tag{68}$$

where,

$$\begin{aligned} F(x(t),x'(0),x''(0)) &\equiv < x'(0)|x(t) >< x(t)|x''(0) > \\ &= e^{\left\{\sigma'+\sigma'^{\star}+\frac{im}{\hbar t}[x'(0)-x''(0)]x(t)+\frac{im}{2\hbar t}(x'(0)-x''(0))^2\right\}} \end{aligned} \tag{69}$$

such that,

$$\int dx(t)F(x(t),x'(0),x''(0)) = e^{2\sigma'_R}\frac{2\pi\hbar t}{m}\delta(x'(0)-x''(0)), \tag{70}$$

yielding, $e^{\sigma'_R} = \sqrt{\frac{m}{2\pi\hbar t}}$; here, $\sigma'_R = (\sigma' + \sigma'^{\star})/2 = \mathrm{Re}\{\sigma'\}$ is the real part of σ'. Now, Eq. (66) becomes,

$$< x(t)|x(0) >= e^{i\sigma'_I}\sqrt{\frac{m}{2\pi\hbar t}}\exp\left\{\frac{im}{2\hbar t}\left(x(t)-x(0)\right)^2\right\}, \tag{71}$$

where, $\sigma'_I = (\sigma' - \sigma'^{\star})/(2i) = \mathrm{Im}\{\sigma'\}$ is the imaginary part of σ', which can be evaluated from the requirement given in Eq. (67):

$$\begin{aligned} \delta(x(t)-x(0)) &= \lim_{t\to 0} < x(t)|x(0) > \\ &= \lim_{t\to 0} e^{i\sigma'_I}\sqrt{\frac{m}{2\pi\hbar t}}\exp\left\{\frac{im}{2\hbar t}\left(x(t)-x(0)\right)^2\right\} \\ &= e^{i\sigma'_I} i^{\frac{1}{2}}\delta(x(t)-x(0)), \end{aligned} \tag{72}$$

implying $e^{i\sigma'_I} i^{\frac{1}{2}} = 1$. Hence,

$$< x(t)|x(0) >= \sqrt{\frac{m}{2\pi i\hbar t}}\exp\left\{\frac{im}{2\hbar t}\left(x(t)-x(0)\right)^2\right\}. \tag{73}$$

Similar analysis can be carried out for a simple harmonic oscillator:

$$< x(t)|x(0) >= \sqrt{\frac{m\omega}{2\pi i\hbar\sin(\omega t)}}\,.\exp\left\{\frac{im\omega}{2\hbar\sin(\omega t)}G(x(t),x(0))\right\}, \tag{74}$$

where, $G(x(t),x(0)) \equiv (x^2(t)+x^2(0))\cos(\omega t) - 2x(t)x(0)$.

When, $t = \Delta t \to 0$, both Eq. (73) and (74) can be written as

$$\begin{aligned} \lim_{\Delta t\to 0} < x(t)|x(0) > &= K\exp\left\{\frac{i\Delta t}{\hbar}\left[\frac{m}{2}\left(\frac{x(t)-x(0)}{\Delta t}\right)^2 - \frac{1}{2}\left[V(x(t))+V(x(0))\right]\right]\right\} \\ &= K\exp\left\{\frac{i}{\hbar}\int_0^t dtL(\dot{x}(t),x(t))\right\}. \end{aligned} \tag{75}$$

where, $K \equiv \sqrt{\frac{m}{2\pi i\hbar\Delta t}}$. Now, consider the energy eigenstate,

$$\begin{aligned} |\psi > &= \int dx(0)|x(0)>< x(0)|\psi > \\ &= \iint dx(0)dx(t)|x(0)>< x(0)|x(t)>< x(t)|\psi > . \end{aligned} \tag{76}$$

The particle will be present at some particular eigenstates $|x_p(0)>$ whose phase ph$\{<x_p(0)|\psi>\}$ at time $t = 0$ is the same as ph$\{|\psi>\}$ (see subsection-V(D)). This criterion yields the following relation from Eq. (76):

$$\begin{aligned}\mathrm{ph}\{|\psi>\} = \mathrm{ph}\{<x_p(0)|\psi>\} &= \mathrm{ph}\{<x_p(0)|x_p(t)><x_p(t)|\psi>\}\\ &= \mathrm{ph}\{<x_p(0)|x_p(t)>\} + \mathrm{ph}\{<x_p(t)|\psi>\},\end{aligned} \tag{77}$$

where, ph$\{<x_p(t)|\psi>\}$ is the phase of the particle state at t. These phases of the particle states at $t = 0$ and t need not be the same, i.e.,

$$\mathrm{ph}\{<x_p(0)|\psi>\} \neq \mathrm{ph}\{<x_p(t)|\psi>\}, \tag{78}$$

but, any infinitesimal variation of phase at $t = 0$ results in the corresponding variation of phase at t:

$$\delta\{\mathrm{ph} <x_p(0)|\psi>\} = \delta\{\mathrm{ph} <x_p(t)|\psi>\}. \tag{79}$$

Applying the above condition to Eq. (77) results,

$$\delta\{\mathrm{ph} <x_p(0)|x_p(t)>\} = 0, \tag{80}$$

which, in turn, by use of Eq. (75), yields the classical least action principle,

$$\delta \int_0^t dt L(\dot{x}_p(t), x_p(t)) = 0. \tag{81}$$

The above equation explicitly shows that the position eigenvalues of a particle state always, as a function of time, lie on a classical path. It can be straightforwardly verified that the same result can be obtained even for the case of 3DES. Also, this result is independent of whether the physical system is microscopic or macroscopic and proves that the time parameter entering both QM and CM is one and the same. Even though the result in Eq. (81) is proved here for free particle and harmonic oscillator, its general validity can be verified by noting the additive property of phase in Eq. (77) and time-interval independence of Eq. (80).

Keeping in mind the particle trajectories observed in particle detectors like Wilson's chamber, consider a special type of scattering process: Let $t_1 < t_2 < \cdots < t_i < \cdots < t_N$ be the time sequence and the elements of the following set,

$$R_p(t) \equiv \{r_p(t_1), r_p(t_2), \cdots, r_p(t_N)\} \subset \mathbf{R}^3,$$

be the locations of some point-scatterers. A moving particle gets scattered by those scatterers. Let the initial state vector, say $|\psi_1>$, gets scattered into $|\psi_2>$ at $r_p(t_1)$, $|\psi_2>$ into $|\psi_3>$ at $r_p(t_2)$, etc.,. Now, by using Eq. (10), one has,

$$|\psi_1> \longrightarrow \prod_{i=1}^{N} |<r_p(t_i)|\psi_i>|^2 . |\psi_{N+1}> . \tag{82}$$

If the loss of energy and change in momentum of the particle is extremely small when compared to its actual energy and momentum at each scatterer, then, it can be concluded from Eq. (81) that the elements of the set, $R_p(t)$, will almost lie on a classical trajectory for an appropriately chosen time interval, (t_1, t_N).

VII. YOUNG'S DOUBLE-SLIT EXPERIMENT: WHAT'S REALLY HAPPENING?

Consider the Young's double-slit (YDS) experiment (FIG. 4) with a single-particle source. Each particle is shot onto the screen through the YDS, only after the registration of the previous one. Classically, the particles were expected to leave a pattern of two strips on the screen, as some of them pass through slit-1 and the others through slit-2, because, they were infered to be moving in the 3DES. But, according to WPND, each particle actually moves in its own IRSM, i.e., Schrödinger's wave function, and hence an interference pattern occurs.

The state vector, $|\psi_0>$, of a particle emitted at the source is projected through YDS as $|\psi>$ onto the screen:

$$|\psi> = |\psi_1> + |\psi_2>, \tag{83}$$

where, $|\psi_1>$ and $|\psi_2>$ are the IRSMs through slit-1 and slit-2, respectively. As explained in the section-II(A), $|\psi>$ interacts with its induced dual in the screen:

$$<\psi|\psi> = <\psi_1|\psi_1> + <\psi_2|\psi_2> + <\psi_1|\psi_2> + <\psi_2|\psi_1> . \tag{84}$$

Notice that, the above inner-product interaction happens instantaneously the moment a particle is emitted, but its effect remains unfelt until the hit of the particle on the screen. Using Eq. (77), it can be seen that, depending on the initial phase of $|\psi_0>$, the particle flies from the source through either slit-1 or slit-2, towards the screen excluding the regions of dark fringes. As given in the section-II - if particle's momentum changes due to either absorption or scattering at the screen, then the IRSM disappears in such a way that the particle contributes a point to $< \psi|\psi >$.

The next particle appears at the source along with its IRSM whose initial phase will be different from the previous one. However, its interaction region, $< \psi|\psi >$, being independent of the initial phase, is the same as all previous ones. The hits of particles on the screen occur randomly at different locations due to different initial phases. This randomness in the phase is due to its dependence on the detailed nature of the source. After a large collection of particles, an interference pattern emerges out, which is nothing but the construction of the function $| < \mathbf{r}_p(t_a)|\psi > |^2$ with individual points; here, the set of position eigenvalues, $\{\mathbf{r}_p(t_a)\}$, span the detector screen and t_a is the arrival time which will be different for different particles (see Eq. (77)). No particle will be found in the regions of dark fringes because, $< \psi|\psi >$ vanishes there, which in turn implies that no classical paths, formed by the position eigenvalues of the particle states, are available from any slit to any dark fringe. Therefore, a moving particle itself never behaves like a wave though it is associated with the wave nature (IRSM). Therefore, the interference pattern obtained with macroscopic molecules of definite internal structure [4, 5, 7] can also be explained unambiguously.

If slit-1 (slit-2) is blocked, then the diffraction due to slit-2 (sli1-1) is produced as an approximate clump pattern given by $< \psi_2|\psi_2 >$ ($< \psi_1|\psi_1 >$). Wave-particle duality attributes single-slit diffraction to the particle nature, while the double-slit interference to the wave nature. But, according to WPND, particle always moves in its IRSM irrespective of single slit or double-slit. That's why the which-path detectors, D_1 and D_2, always find the particle as going through either slit-1 or slit-2. As shown in the section-II and also in Eq. (82), the scattering of detector's probe results in the disappearance of $|\psi >$, which had two origins, one at each slit. A new IRSM, either $|\psi_1' >$ or $|\psi_2' >$, appears with a single origin where the scattering took place in the vicinity of the respective slit. Its inner-product interaction with the detector screen is given by either $< \psi_1'|\psi_1' >$ or $< \psi_2'|\psi_2' >$: the RFD is,

$$< \psi|\psi > \longrightarrow < \psi_1'|\psi_1' > + < \psi_2'|\psi_2' > . \tag{85}$$

Therefore, in the presence of detectors, clump patterns occur and in their absence, the interference pattern (Eq. (84)) comes back.

The de Broglie wave length of a macroscopic object is extremely small when compared to its own size, the dimensions of slits and their separation, yielding the clump patterns in accordance with the prediction of CM. In this case, Eq. (75) shows the diminishing of wave nature and hence, the particle nature, which is always present as given in Eq. (81), becomes apparent. Eq. (82) also shows the classical behavior when the macroscopic object is not isolated from its environment.

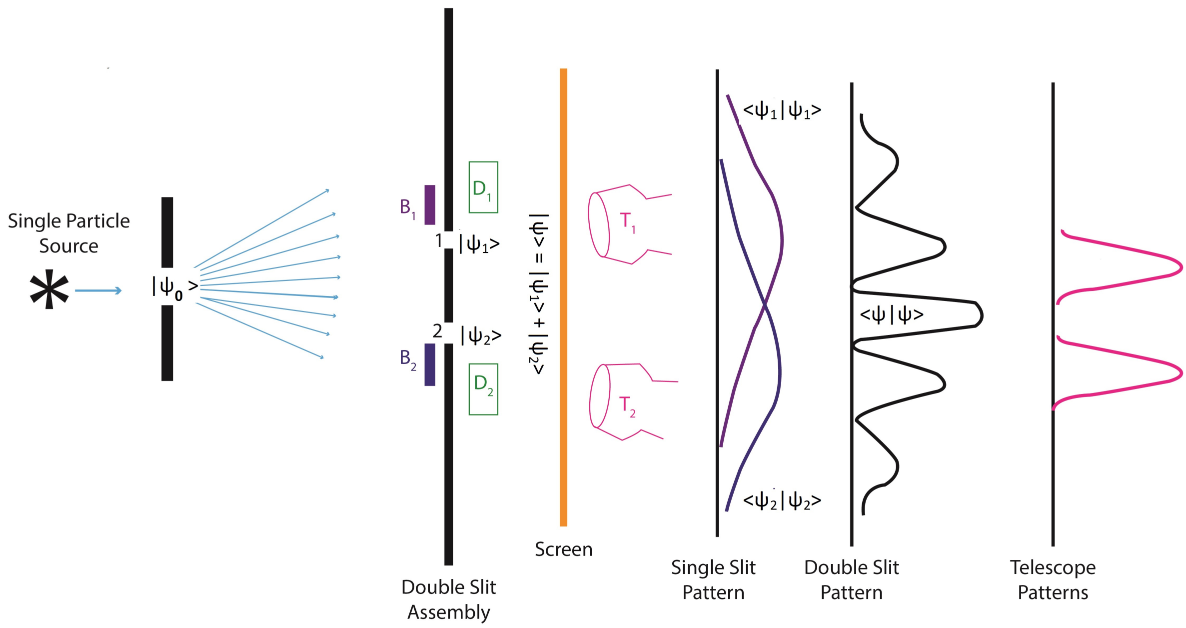

FIG. 4. **Young's double-slit experiment**: A source shoots particles, one at a time, towards a double-slit assembly. State vectors $|\psi_1>$ and $|\psi_2>$ from slits 1and 2 get superposed as $|\psi> = |\psi_1> + |\psi_2>$. B_1 and B_2 are two blockers which can block either slit 1 or slit 2 at any time. D_1 and D_2 are which-path detectors and T_1 and T_2 are telescopes, tightly focused on slit-1 and slit-2, respectively. Particles' distributions at the screen and telescopes were given at the right hand side. If slit 2 (1) is blocked, then the distribution is $<\psi_1|\psi_1>$ ($<\psi_2|\psi_2>$).

VIII. CAUSALITY IN WHEELER'S DELAYED-CHOICE EXPERIMENT

According to Bohr's principle of complementarity [8, 11, 29, 30], observation of wave nature excludes the simultaneous observation of particle nature and vice versa, depending on the experimental configuration. In the YDS experiment (FIG. 4), the presence of a screen or a twin-telescopes, T_1 and T_2, corresponds to observe the wave or particle behavior, respectively. Alternatively, the same can also be viewed as a decision taken by a photon which ‘somehow’ senses the configuration of the measuring device and behaves accordingly like a wave or a particle [10, 11]. This later view-point was examined in Wheeler’s delayed-choice experiment [8] (WDCE): after the photon has already passed through the YDS, the screen is quickly removed exposing the twin-telescopes. The expected interference pattern on the screen is lost and two clump patterns, one at each telescope, are formed. According to duality, the photon retroactively rearranges its past history of simultaneously passing through both the slits like a wave to that of passing through any one slit like a particle, resulting in the clump patterns.

So far, only the non-relativistic QM is considered by WPND, which, for the relativistic case, will be considered separately elsewhere. But, notice that, a wave function yielding the photon’s dispersion relation is sufficient for the present purpose of a causal explanation of all the observations in WDCE. By considering such a wave function or a state vector as an IRSM, the WPND can be applied to the case of a single-photon.

As mentioned above, if the screen is quickly removed, then the state vector encountered jointly by T_1 and T_2, is

$$\begin{aligned}(\hat{T}_1 + \hat{T}_2)|\psi >= (\hat{T}_1 + \hat{T}_2)(|\psi_1 > +|\psi_2 >) &= \hat{T}_1|\psi_1 > +\hat{T}_2|\psi_2 > \\ &= |\tilde{\psi}_1 > +|\tilde{\psi}_2 > \equiv |\tilde{\psi} >, \end{aligned} \tag{86}$$

where, $\hat{T}_1$ and $\hat{T}_2$ are projection operators associated with the telescopes and $\hat{T}_1|\psi_2 >= \hat{T}_2|\psi_1 >= 0$, because, T_1 and T_2 are tightly focused on slit-1 and slit-2, respectively.

Let $|\Phi >$ be the photon’s eigenstate:

$$[|\Phi >]^{\rm FBC=DS}_{\rm IBC=PO} \equiv |\psi > \quad ; \quad [|\Phi >]^{\rm FBC=TT}_{\rm IBC=PO} \equiv |\tilde{\psi} >, \tag{87}$$

where, IBC and FBC are initial and final boundary conditions; PO stands for photon’s origin; DS and TT are detector screen and twin-telescopes, respectively. Both $|\psi >$ and $|\tilde{\psi} >$ have the same IBC and they differ only by FBC. Notice that, changing the FBC is equivalent to changing the representation of detector’s CVS. Initially, the photon is flying in $|\psi >$. When it is in the mid-flight after crossing the YDS, $|\psi >$ is quickly replaced by $|\tilde{\psi} >$. Wherever be the photon at the moment of replacement - from there - it continues flying in $|\tilde{\psi} >$. According to WPND, the physical nature of $|\psi >$ and $\tilde{\psi} >$ is an IRSM and hence, one can be replaced by the other instantaneously. This is equivalent to a photon flying from the source to YDS and then, from the YDS to screen, because, the photon feels a sudden change of its IRSM from $|\psi_0>$ to $|\psi>$ at the YDS. Therefore, *photon's motion is always continues, preserving the causality, even though the final boundary condition changes suddenly.* Therefore, the observed RFD at T_1 and T_2 is,

$$< \tilde{\psi}|\tilde{\psi} >=< \tilde{\psi}_1|\tilde{\psi}_1 > + < \tilde{\psi}_2|\tilde{\psi}_2 >, \tag{88}$$

which corresponds to clump patterns.

Again consider the same YDS experiment with horizontal and vertical polarization filters H and V fitted to the slits 1 and 2, respectively. In this case, the state vector is,

$$|\psi >>= |\psi_1 > |H > +|\psi_2 > |V >, \tag{89}$$

where, $|H>$ and $|V>$ are horizontal and vertical polarization states of a photon, respectively. Insertion of a 45^o polarization rotator (PR), with an unit operator, $\hat{I}_{pr} = |\bar{H} >< \bar{H}| + |\bar{V} >< \bar{V}|$, just before the screen, changes the representation of IRSM:

$$|\psi >>= |\bar{H} >< \bar{H}|\psi >> +|\bar{V} >< \bar{V}|\psi >>= |\bar{H} > |\bar{\psi}_1 > +|\bar{V} > |\bar{\psi}_2 >, \tag{90}$$

$$\begin{aligned}
\text{where,}\quad & |\bar{H} > = \frac{1}{\sqrt{2}}(|H > +|V >) \quad ; \quad |\bar{V} >= \frac{1}{\sqrt{2}}(-|H > +|V >) \\
& |\bar{\psi}_1 > = < \bar{H}|\psi >>= \frac{1}{\sqrt{2}}(|\psi_1 > +|\psi_2 >) \\
\text{and}\quad & |\bar{\psi}_2 > = < \bar{V}|\psi >>= -\frac{1}{\sqrt{2}}(|\psi_1 > -|\psi_2 >).
\end{aligned} \tag{91}$$

It's clear from Eq. (91) that the photon passing through slit-1 will be present in either $|\bar{\psi}_1>$ or $|\bar{\psi}_2>$ and the same can be said if it passes through slit-2 as well.

Now, insertion of a Wollaston prism (WP), with unit operator, $\hat{I}_{WP} = |H >< H|+|V >< V|$, between PR and the screen, changes again the representation of the photon state:

$$\begin{aligned}
|\psi >> &= (< H|\bar{H} > |\bar{\psi}_1 > + < H|\bar{V} > |\bar{\psi}_2 >)|H > \\
&\quad +(< V|\bar{H} > |\bar{\psi}_1 > + < V|\bar{V} > |\bar{\psi}_2 >)|V > \\
&= |\psi_1 > |H > +|\psi_2 > |V >,
\end{aligned} \tag{92}$$

whose interaction with its dual at the screen is given by,

$$
\begin{aligned}
<<\psi|\psi>> &= \frac{1}{2}\sum_{i,j=1}^{2}(-1)^{i+j}<\bar{\psi}_i|\bar{\psi}_j><H|H> \\
&+\frac{1}{2}\sum_{i,j=1}^{2}<\bar{\psi}_i|\bar{\psi}_j><V|V> \\
&= <\bar{\psi}_1|\bar{\psi}_1>+<\bar{\psi}_2|\bar{\psi}_2>=<\psi_1|\psi_1>+<\psi_2|\psi_2>, \qquad (93)
\end{aligned}
$$

which shows that the clump patterns are intact even in the presence of both PR and WP. Note that, the above equation never implies $<\psi_i|\psi_i>=<\bar{\psi}_i|\bar{\psi}_i>$.

The RFDs of two orthogonal components, $|H>$ and $|V>$, of WP can be detected by two independent detectors, say D_H and D_V:

$$RFD_{D_H} = \frac{1}{2}\sum_{i,j=1}^{2}(-1)^{i+j}<\bar{\psi}_i|\bar{\psi}_j>=<S_1|S_1> \qquad (94)$$

and

$$RFD_{D_V} = \frac{1}{2}\sum_{i,j=1}^{2}<\bar{\psi}_i|\bar{\psi}_j>=<S_2|S_2>. \qquad (95)$$

Therefore, a photon present in the $|H>$ component of the WP will contribute a point to the anti-interference pattern given in Eq. (94) and the one in $|V>$, to the interference pattern in Eq. (95). Also, these equations predict that the photon initially entered through slit-1 or slit-2 of YDS will be detected by D_H or D_V, respectively. In the absence of PR, the usual clump patterns corresponding to YDS will be formed at D_H and D_V. The role of PR is simply to replace the clump patterns by the respective anti-interference and interference structures. If the screen is used for detection instead of D_H and D_V, then these structures disappear into each other yielding the clump patterns as given by Eq. (93).

Notice that, the PR can be randomly introduced or removed before a photon passes through the same, because, as already shown, even during the random changes of representations or boundary conditions for the IRSM, the photon flies continuously. This technique was used in an experiment by Jacques et al. [31, 32], where the Mach-Zehnder interferometer is used instead of YDS assembly. Similar experiments using single atoms [33] and its quantum versions [34–36] were also done.

IX. EXPERIMENTAL PROPOSAL TO VERIFY THE IRSM

The WPND has shown that the physical nature of Schrödinger's wave function is an IRSM. To verify this, a modified Mach-Zehnder interferometer (mMZI) experimental set up, as given in FIG. (5), can be used, where, BS is a 50 : 50 beam splitter resolving a single-photon's state vector into refracted and reflected components along the Path_1 and Path_2, respectively - which are recombined by the inverse beam splitter, IBS. The path difference $= \mathrm{Path}_2 - \mathrm{Path}_1 = \delta$ is to be chosen such that the recombined components at IBS interfere destructively and constructively towards the photon detectors D_1 and D_2, respectively.

A large number of single-photons are fired into mMZI such that the time interval between any two consecutive ones is chosen to be sufficiently greater than the time of flight of a photon along the Path_2 to either D_1 or D_2 (if D_1 and D_2 are placed at the same distance from IBS). This guarantees that there will never be more than one photon inside the mMZI at a given time. If the wave nature associated with the photon is really propagating like a classical wave, then the refracted and reflected components will not be recombined by the IBS where they arrive at different times and hence, the interference condition becomes invalid; the reflected component along Path_2 lags behind the refracted one along Path_1 (see FIG. 5). Therefore, each one of D_1 and D_2 will detect 50% of the total number of photons.

Prediction by the WPND: If the wave function is an IRSM in accordance with the WPND, then the interference condition is automatically satisfied and D_2 will register 100% of all the photons entered into mMZI. This is because, the moment a photon appears, its IRSM gets refracted and reflected by the BS and recombined at the IBS, forming destructive and constructive interferences towards D_1 and D_2, respectively - all at once. Depending on the initial phase of the IRSM, the photon will enter into either the Path_1 or Path_2 and always emerges out of IBS towards D_2. If a similar experiment is done with slow-moving single-electrons, particularly single-atoms or single-molecules, then not only the results will be better obtained but also the instantaneous nature of the wave function becomes very apparent.

Let T_1 and T_2 be the times of flight of the photon along Path_1 and Path_2, respectively. If the difference, $T_2 - T_1$, is sufficiently larger and also very greater than all possible experimental errors involved in determining the initial time of production and the final time of detection of the photon, then half of the total number of photons detected by D_2 will have

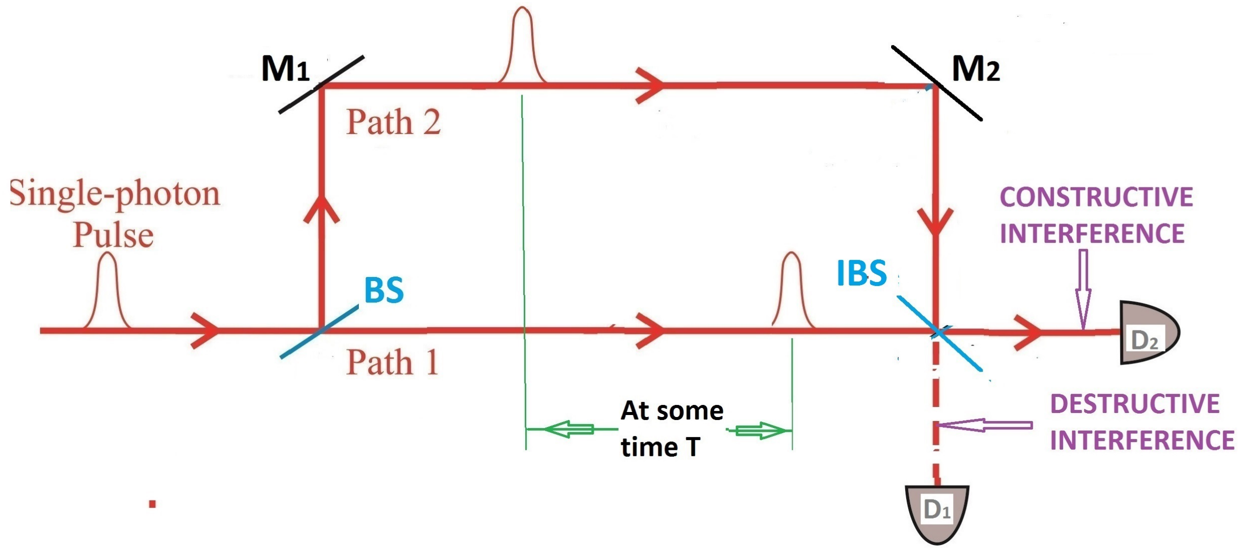

FIG. 5. **Modified Mach-Zehnder Experiment**: BS and IBS are 50:50 beam splitter and inverse beam splitter. M1 and M2 are 100% reflecting mirrors and D1 and D2 are single-photon detectors. A single-photon pulse, entering BS gets partially refracted and partially reflected along Path_1 and Path_2, respectively. At the moment when the refracted pulse reaches IBS, the reflected one, along the Path_2, lags behind by a path difference $= \text{Path}_2 - \text{Path}_1 = \delta$ which is chosen to yield the destructive and constructive interferences towards the detectors D_1 and D_2, respectively. Also, the pulse width should be much smaller than the path difference. (If a ripple-packet produced for a brief time by dropping a single small stone on the surface of water is considered in the places of the refracted and reflected pulses, then their wave-fronts will never be recombined at IBS).

arrival time T_2 and the remaining half will have T_1. Therefore, in this particular experiment, 'which path information' can be obtained by merely measuring T_1 and T_2.

X. CONCLUSIONS AND DISCUSSIONS

The physical nature of Schrödinger's wave function is shown to be an *instantaneous resonant spatial mode* (IRSM) in which a particle flies akin to the case of a test particle in the curved space-time of the general theory of relativity. The inseparable nature of IRSM and its particle, which is like the eigenstate and eigenvalue, is named as wave-particle non-duality. The state vector interacts, according to the inner-product, with its induced dual in a measuring device. Collection of these interactions for a large number of particles yields the relative frequency of detection and hence, the Born rule; here, the unavoidable initial phase of the state vector is shown to be responsible for the outcome of a definite eigenvalue of an observable. At this moment, it may worth recollecting a philosophical saying, "*It is necessary for the very existence of science that the same conditions always produce the same result*" - which seems to be in perfect agreement even in the quantum domain, because, all initially prepared identical states are not actually identical with respect to their initial phases.

It's shown that the eigenvalues of a particular position eigenstate, where the particle resides, always lie on a classical path of least action. The equality of quantum mechanical time to the classical time and also, the emergence of classical world from the underlying quantum world is explicitly shown (in the case of non-relativistic quantum mechanics). 'What's really going on?' in the Young's double-slit experiment at a single quantum level is unambiguously explained. Also, a causal explanation of Wheeler's delayed-choice experiment is provided for the first time. With respect to non-duality, the measurement problem does not exist and the quantum mechanics is indeed a classical mechanics, but in a complex vector space. Finally, an interference experiment is proposed to verify the instantaneous nature of the wave function.

In the relativistic case, the IRSM is such that, apart from obeying the usual quantum mechanical commutation relations, it takes care of the cosmic speed limit of its resonant particle, though it itself can change instantaneously - which will be reported elsewhere. Another mystery of the quantum world, untouched in the present article, is Einstein's spooky action-at-a-distance among two or more entangled particles. It's worth mentioning that the non-duality is capable of providing the physical mechanism for spooky action by making use of the nature of IRSM and will be reported elsewhere. Also, the explanations of Young's double-slit experiment using the entanglement with which-path detectors' probes, quantum erasure, entanglement swapping both in space and time and the well-known paradoxes like Schrödinger's cat, Wigner's friend etc., will be reported elsewhere. Undoubtedly, non-duality will further enhance the deeper understanding of Nature's working at the most fundamental level.

Appendix A: Generalized Representation for the $SU(2)$ Algebra

According to the requirement of non-duality to describe a single-quantum behavior, a generalized representation for the $SU(2)$ algebra respecting the Eqs. (21), (25), (26) and (27) is explicitly worked out below:

Writing down the other operators,

$$\begin{aligned}\hat{S}_x &= \frac{1}{2}(|S_x;\uparrow>< S_x;\uparrow| - |S_x;\downarrow>< S_x;\downarrow|) \\ &= \frac{R^2}{2}(C_x|S_z;\uparrow>< S_z;\downarrow| + C_x^*|S_z;\downarrow>< S_z;\uparrow|),\end{aligned} \tag{A1}$$

$$\begin{aligned}\hat{S}_y &= \frac{1}{2}(|S_y;\uparrow>< S_y;\uparrow| - |S_y;\downarrow>< S_y;\downarrow|) \\ &= \frac{R^2}{2}(C_y|S_z;\uparrow>< S_z;\downarrow| + C_y^*|S_z;\downarrow>< S_z;\uparrow|),\end{aligned} \tag{A2}$$

where, $C_x = e^{i(\gamma-\delta)} - e^{i(\gamma'-\delta')}$ and $C_y = e^{i(\alpha-\beta)} - e^{i(\alpha'-\beta')}$ and $|< S_z;\uparrow|S_x;\uparrow>| = |< S_z;\downarrow|S_x;\uparrow>| = |< S_z;\uparrow|S_x;\downarrow>| = |< S_z;\downarrow|S_x;\downarrow>| = R$. It can be shown that,

$$< S_x;\downarrow|S_x;\uparrow> = 0 \Longrightarrow (\gamma-\gamma') - (\delta-\delta') = \pm\pi \tag{A3}$$

$$< S_y;\downarrow|S_y;\uparrow> = 0 \Longrightarrow (\alpha-\alpha') - (\beta-\beta') = \pm\pi \tag{A4}$$

As it's well known, the sign ambiguity in the above equations is related to the two possible ways of writing the commutation relations viz., $[\hat{S}_x\ ,\ \hat{S}_y] = i\hat{S}_z$ or $[\hat{S}_y\ ,\ \hat{S}_x] = i\hat{S}_z$ due to the rotational symmetry about Z-axis, which can be fixed using the $SU(2)$ algebra:

$$[\hat{S}_x\ ,\ \hat{S}_y] = \frac{R^4}{4}(A_{xy}|S_z;\uparrow>< S_z;\uparrow| + A_{xy}^*|S_z;\downarrow>< S_z;\downarrow|) = i\hat{S}_z, \tag{A5}$$

$$[\hat{S}_z\ ,\ \hat{S}_x] = \frac{R^2}{2}(C_x|S_z;\uparrow>< S_z;\downarrow| - C_x^*|S_z;\downarrow>< S_z;\uparrow|) = i\hat{S}_y, \tag{A6}$$

$$[\hat{S}_y\ ,\ \hat{S}_z] = \frac{R^2}{2}(C_y^*|S_z;\uparrow>< S_z;\downarrow| - C_y|S_z;\downarrow>< S_z;\uparrow|) = i\hat{S}_x, \tag{A7}$$

where, $A_{xy} = C_x C_y^* - C_x^* C_y$. The above commutation relations yield $C_x = iC_y$ and $\frac{R^4}{4}A_{xy} = \frac{i}{2}$, which result in the following unique relations:

$$(\gamma-\delta) - (\alpha-\beta) = (\gamma'-\delta') - (\alpha'-\beta') = \frac{\pi}{2}, \tag{A8}$$

$$\text{and} \qquad (\alpha-\alpha') - (\beta-\beta') = +\pi\ ;\ (\gamma-\gamma') - (\delta-\delta') = -\pi\ ;\ R = \frac{1}{\sqrt{2}}, \tag{A9}$$

which are sufficient to satisfy the other aspects of $SU(2)$ algebra, viz.,

$$\{\hat{S}_i\ ,\ \hat{S}_j\} = \frac{1}{2}\delta_{ij}\ ;\ \hat{S}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2 = \frac{3}{4}\hat{I}\ ;\ [\hat{S}^2\ ,\ \hat{S}_i] = 0 \tag{A10}$$

where, i and j run over x, y and z and $\{\quad ,\quad \}$ stands for the anti-commutator, δ_{ij} is the Kronecker delta and $\hat{I}$ is the identity operator.

It's straightforward to check the special case by setting $\alpha = \alpha' = \gamma = \gamma' = 0$ in Eqs. (A8) and (A9), yielding the well-known representation of $SU(2)$ algebra available in any text book of QM. This special case does not admit the notion of minimum phase and is good only for the probabilistic description.

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