On a nonlinear differential equation of Lienard type

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Abstract

We present, in this paper, an exceptional Lienard differential equation. In spite of the presence of strong and high order nonlinearity term, the proposed equation is explicitly integrable. The general solution is expressed in terms of trigonometric functions. Also the general solutions of related quadratic Lienard type equations are periodic and may exhibit harmonic oscillations. The presented equation includes many nonlinear equations like the cubic-quintic, and cubic Duffing equations, Mickens truly nonlinear equations and the Ermakov-Pinney equation as special cases so that the explicit general solution of several nonlinear equations may be easily obtained for the first time.

Keywords: Lienard equations, general periodic solutions, Ermakov-Pinney equation, Duffing equations, Mickens truly nonlinear equations.

Introduction

The Lienard equation

$$\ddot{x} + f(x) = 0 \tag{1}$$

where $f(x)$ is a nonlinear function of x, and the overdot designates a differentiation with respect to time, is one of the most investigated equation in mathematics and physics for specific expressions of $f(x)$. In physics, the equation of the form (1) may describe the dynamics of conservative systems. It is also known that nonlinear evolution equations may reduce to the form (1) under traveling wave transformations [1]. The special class of (1), that is

$$\ddot{x} + a\,x + g(x) = 0 \tag{2}$$

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where $g(x)$ is a nonlinear function of x, has been and continues to be the object of an intensive study in the literature, since several equations like the celebrated Duffing oscillator equation [2], the famous Ermakov-Pinney oscillator equation [3] and the exceptional Mickens truly nonlinear oscillator of the form [4]

$$\ddot{x} + x - x^{1/3} = 0 \tag{3}$$

belong to this class (2). These equations are very exceptional among the class of nonlinear differential equations as they have exact and explicit general solutions, when we know that this fact is not usual in the world of nonlinear differential equations. Another celebrated equation, but belonging to the class of quadratic Lienard type equations

$$\ddot{x} + h(x)\dot{x}^2 + f(v(x)) = 0 \tag{4}$$

is the Mathews-Lakshmanan equation highlighted in 1974 [5]. Later, Akande et al. [6] have shown the existence of several equations of the form (4) that may exhibit oscillations of the harmonic form as the Mathews-Lakshmanan oscillator. The problem now in this paper is to show the existence of a Lienard equation of the form (2), explicitly integrable with a general solution expressed in terms of trigonometric functions, including the Mickens truly nonlinear oscillator equation, the Duffing equations and the Ermakov-Pinney equation as special cases, and leading the related quadratic Lienard type equations to exhibit periodic solutions of the harmonic form, as implications. To do so, we state the general theory showing the existence of such an equation (section 2) and discuss its implications (section 3). A conclusion of the work is finally carried out.

2 General theory

2.1 Proposed equation

Let us consider the Lienard differential equation stated in [4, 7]

$$\ddot{x} + \frac{1}{2}(\alpha - q)a\,x^{\alpha-q-1} + \frac{qb}{2}x^{-q-1} = 0 \tag{5}$$

corresponding to the first-order differential equation

$$\dot{x}^2 x^q + a\,x^\alpha = b \tag{6}$$

Substituting $\alpha = q+2$, and $b = \frac{a(q+2)}{4}$, into (5), yields as equation

$$\ddot{x}+a\,x+\frac{aq\left(q+2\right)}{8}x^{-q-1}=0 \tag{7}$$

where a, b, α and q are arbitrary parameters.

The equation (7) is the proposed Lienard nonlinear differential equation. Now we may integrate (7) to give the general solution.

2.2 General solution

The use of (6) leads to the quadrature defined by

$$\int\frac{x^{\frac{q}{2}}dx}{\sqrt{1-\frac{4}{q+2}x^{q+2}}}=\pm\sqrt{b}\left(t+K\right) \tag{8}$$

where K is an arbitrary constant of integration, from which one may secure the exact and explicit general solution of (7) as

$$x(t)=\left[\frac{\sqrt{q+2}}{2}\sin\left(\pm\frac{q+2}{2}\sqrt{a}(t+K)\right)\right]^{\frac{2}{q+2}} \tag{9}$$

That being so, we may discuss of the implications of the equation (7).

3 Discussion

It is worth to notice that for $q>-2$, all the solutions (9) are periodic and expressed in terms of elementary functions, that is to say, in terms of trigonometric function. Some specific examples may be now given to illustrate the capacity of the equation (7) to include some well-known equations of the literature.

3.1 Illustrative examples

Substituting $q=2$, into (7) leads to the famous Ermakov-Pinney equation

$$\ddot{x}+a\,x+\frac{a}{x^{3}}=0 \tag{10}$$

which has been recently investigated in [7]. Its exact and explicit general solution obtained from (9) may read

$$x(t)=\left[\sin\left(\pm 2\sqrt{a}\,(t+K)\right)\right]^{\frac{1}{2}} \tag{11}$$

which is in agreement with the result found in [7]. The choice $q=-\frac{4}{3}$, reduces the equation (7) to the Mickens truly nonlinear oscillator equation

$$\ddot{x}+a\,x-\frac{a}{3}x^{1/3}=0 \tag{12}$$

where the exact and explicit general solution is

$$x(t)=\left[\frac{\sqrt{6}}{6}\sin\left(\pm\frac{1}{3}\sqrt{a}\,(t+K)\right)\right]^{3} \tag{13}$$

Choosing $q=-\frac{2}{3}$, leads to the Mickens oscillator of the form

$$\ddot{x}+a\,x-\frac{a}{9}x^{-1/3}=0 \tag{14}$$

where the general solution takes the expression

$$x(t)=\left[\frac{\sqrt{3}}{3}\sin\left(\pm\frac{2}{3}\sqrt{a}\,(t+K)\right)\right]^{3/2} \tag{15}$$

Consider now some Duffing type equations. For $q=-4$, the celebrated cubic Duffing equation is obtained as

$$\ddot{x}+a\,x+a\,x^{3}=0 \tag{16}$$

From the solution (9), we may obtain the exact and explicit general solution of the (16) in the form

$$x(t)=\left[i\frac{\sqrt{2}}{2}\sin\left(\pm\sqrt{a}\,(t+K)\right)\right]^{-1} \tag{17}$$

which is a complex-valued solution. Usually the solution of the cubic Duffing equation is expressed as a Jacobi elliptic function. However, in an earlier paper [8] Monsia and coworkers have shown that when the coefficient of the cubic term is negative, a general non-periodic hyperbolic solution may be secured for the cubic Duffing equation. Here we find also a non-periodic solution in the case where the coefficient of the cubic term is positive. The application of $q=-6$, allows one to obtain the quintic Duffing equation

$$\ddot{x} + a\,x + 3a\,x^5 = 0 \qquad (18)$$

known in the literature to have no exact and explicit general solution. Here we find, from (9), that its exact and explicit general solution may be written in the form

$$x(t) = \left[i \sin\left(\pm 2\sqrt{a}\,(t+K)\right)\right]^{-1/2} \qquad (19)$$

The general solutions of the quadratic oscillator equation, the heptic Duffing equation and of several others equations may be easily obtained using the present theory. Now the related quadratic Lienard type equations may be investigated.

3.2 Quadratic Lienard type equations

To derive the quadratic Lienard type equations related to (7) we have to use the point transformation

$$u = x^p \qquad (20)$$

where p is an arbitrary parameter. In this situation, the first and second derivatives of x in terms of u may be expressed as

$$\frac{dx}{dt} = \frac{1}{p} u^{\frac{1}{p}-1}\,\dot{u} \qquad (21)$$

and

$$\frac{d^2x}{dt^2} = \frac{1}{p}\left[\frac{1-p}{p} u^{\frac{1-p}{p}-1}\,\dot{u}^2 + u^{\frac{1-p}{p}}\,\ddot{u}\right] \qquad (22)$$

where $p \neq 0$. From this, we may ensure the quadratic Lienard type equation

$$\ddot{u} + \frac{1-p}{p}\frac{\dot{u}^2}{u} + pa\,u + \frac{paq\left(q+2\right)}{8} u^{\frac{p-q-2}{p}} = 0 \qquad (23)$$

which has the general solution

$$u(t) = \left[\frac{\sqrt{q+2}}{2}\sin\left(\pm\frac{q+2}{2}\sqrt{a}(t+K)\right)\right]^{\frac{2p}{q+2}} \qquad (24)$$

An interesting case of (23) is obtained when $\frac{2p}{q+2}=1$, that is to say, when $q=2p-2$. In this case the equation (23) reduces to

$$\ddot{u}+\frac{1-p}{p}\frac{\dot{u}^2}{u}+a\,p\,u+\frac{ap^2(p-1)}{2}\frac{1}{u}=0 \tag{25}$$

where the exact and explicit general solutions are

$$u(t)=\frac{\sqrt{2p}}{2}\sin\left[\pm\, p\sqrt{a}(t+K)\right] \tag{26}$$

For p positive, the solutions (26) are periodic and may exhibit harmonic oscillations. The angular frequency depends on the amplitude $A=\frac{\sqrt{2p}}{2}$, which is a characteristic of nonlinear oscillator equations.

Conclusion

A Lienard differential equation with strong and high order nonlinearity term is presented in this work. The proposed equation is exactly and explicitly integrable. The general solution is a power law of trigonometric functions. As implications, the general solution of several nonlinear differential equations may be easily obtained for the first time, and the related quadratic Lienard type equations are found to exhibit periodic solutions of the harmonic form.

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