Distribution of Integrals of Wiener Paths

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Abstract

We show that the normal distribution with mean zero and variance 1/3 is the
distribution of the integrals \( \int_{[0,1]} W_t \, dt \) of the sample paths of Wiener process \( W \in C([0,1], \mathbb{R}) \).

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1 Introduction

In contrast with the notion of “martingale (stochastic) integration” associated with
Wiener measure, attention is less directed to the integrals of the sample paths of Wiener
process \( W \) in \( C([0,1], \mathbb{R}) \). Since every realization of \( W \) is a continuous function on
a compact interval, it always makes sense to speak of the integral of a Wiener path;
investigating the integrals of Wiener paths, in particular the distribution of such integrals
(which is evidently possible and is justified in what follows), is then a natural move.

In the present short communication, we prove

Theorem *. If \( W \) is Wiener process in \( C([0,1], \mathbb{R}) \), then

\[
\int_{[0,1]} W_t \, dt \sim N(0, 1/3).
\]

2 Proof

Throughout, let \( C_w \) be the metric space \( C([0,1], \mathbb{R}) \) equipped with the uniform metric;
and let \( W \) be Wiener process in \( C_w \).

We now give

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Proof (of Theorem *). For all $f, g \in C_w$, we have

$$\left| \int (f - g) \right| \leq \sup_t |f(t) - g(t)|;$$

so the integration operator $\int$ is (uniformly) continuous on $C_w$.

If $X_1, X_2, \ldots$ are independent identically distributed standard normal random variables, let $\hat{W}^n$ be for each $n \in \mathbb{N}$ the “Donsker process” obtained by linear interpolation between the $\frac{1}{\sqrt{n}}$-scaled cumulative sums of $X_1, \ldots, X_n$ such that the resulting process fixes the origin, so that the sequence $(\hat{W}^n)_{n \in \mathbb{N}}$ satisfies the assumptions of Donsker’s theorem (Theorem 8.2 in Billingsley [1], for concreteness). The continuous mapping theorem and Donsker’s theorem then jointly imply the weak convergence

$$\int \hat{W}^n_t \, dt \rightsquigarrow \int W_t \, dt. \quad (1)$$

Let $S_0 := 0$; and let $S_j := \sum_{i=1}^{j} X_i$ for all $1 \leq j \leq n$ and all $n \in \mathbb{N}$. If $n \in \mathbb{N}$, then we have

$$\int \hat{W}^n_t \, dt = \sum_{j=1}^{n} \frac{j}{n} \hat{W}^n_{j/n} \, dt,$$

and we have $\hat{W}^n_{j/n} = S_j / \sqrt{n}$ for each $0 \leq j \leq n$. Given any $1 \leq j \leq n$, we have

\[
\int_{(j-1)/n}^{j/n} \hat{W}^n_t \, dt = \frac{1}{\sqrt{n}} \int_{(j-1)/n}^{j/n} \tau S_j + (1 - \tau)S_{j-1} \, d\tau = \frac{1}{\sqrt{n}} \left( S_j \frac{\tau^2}{2} \right)_{(j-1)/n}^{j/n} + \frac{1}{n} S_{j-1} - S_{j-1} \frac{\tau^2}{2} \right)_{(j-1)/n}^{j/n}. \]

Summing the last term above over each $1 \leq j \leq n$ gives

$$\int \hat{W}^n_t \, dt = \frac{1}{n^{3/2}} \left( nX_1 + (n-1)X_2 + \cdots + X_n \right) - \frac{1}{2n^{5/2}} S_n. \quad (2)$$

The last term in (2) vanishes in probability by the continuous mapping theorem and the usual weak law of large numbers.

If $n \in \mathbb{N}$, the sum of the independent normal random variables $(n - j + 1)X_j$ with $1 \leq j \leq n$ in (2) is the normal random variable with mean zero and variance $1^2 + 2^2 + \cdots + n^2 = n(n+1)(2n+1)/6$. If $\kappa := 2^{3/2}\Gamma(2)/\sqrt{\pi}$, then

$$\sum_{j=1}^{n} \mathbb{E}[(n-j+1)X_j]^3 = \kappa \sum_{j=1}^{n} j^3 = \kappa \frac{n^2(n+1)^2}{4},$$

which grows more slowly than $(n(n+1)(2n+1)/6)^{3/2}$ as $n \to \infty$. The classical Lyapunov central limit theorem (e.g. p. 332, Shiryaev [2], for concreteness) and the continuous
mapping theorem together imply that
\[
\frac{1}{n^{3/2}} \left( nX_1 + (n - 1)X_2 + \cdots + X_n \right)
= \sqrt{\frac{n(n+1)(2n+1)}{6}} \left( \frac{n(n+1)(2n+1)}{6} \right)^{-1} \left( nX_1 + (n - 1)X_2 + \cdots + X_n \right)
\sim N(0, 1/3).
\]

Upon applying the continuous mapping theorem once more, we have
\[
\int \hat{W}^n_t \, dt \sim N(0, 1/3)
\]
from (2). But then from [1] and the uniqueness of weak limit it follows that
\[
\int W_t \, dt \sim N(0, 1/3)
\]
as desired.

\[\square\]

References
