Extrinsic Method

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Abstract

[This] is another mathematical method to get answers to the so-called (E) question (see the intrinsic method). In opposition with the intrinsic one, the procedure brings a complete result; I mean: a main part and a residual part. Its disadvantages: the result is an approximation and the procedure itself must be counterchecked with a logical test. In fact, the (E) question receives reasonable answers when the methods are confronted. The confrontation realizes a calibration (see further developments).

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1 Basics

Definition 1.1 (Cube).

Per definition, in that theory, a cube is a mathematical object of which the components (i) are elements arbitrarily chosen in a set K and (ii) are situated at each knot of a three-dimensional Euclidean crystalline cubic structure.

Definition 1.2 (Deformed tensor products).

We consider a vector space $V = \{E(D, K), \otimes\}$ where $K$ is a commutative leaf and $D$ its dimension (in $\mathbb{N}^*$); and where $\otimes$ denotes the (classical and non deformed) tensor product. That product acts on pairs of elements arbitrarily chosen in $V$ and, as usual (see any scholar book):

$$\forall (q_1, q_2) \in V \times V : q_1 \otimes q_2 = q_1^\alpha \cdot q_2^\beta \cdot e_\alpha \otimes e_\beta$$

Nobody knows a priori if the result (the image) of a given tensor product is in $V$ (the source) again:

$$q_1 \otimes q_2 \in V?$$

A pragmatic manner to be certain that the tensor product acts as an inner product is to equip $V$ with a cube, $A$, and then to impose a relation of closure acting on the elements of its canonical basis $\Omega$: $(..., e_\alpha, ...)$; for the pedagogy and for the simplicity, let start the discussion with a cube containing $D^3$ components:

$$\forall e_\alpha \in \Omega : \exists A = \{A_{\alpha \beta}^\chi \in K | e_\alpha \otimes e_\beta = A_{\alpha \beta}^\chi \cdot e_\chi \in V\},$$

Effectively, in imposing that relation:

$$\forall (q_1, q_2) \in V \times V : q_1 \otimes q_2 = q_1^\alpha \cdot q_2^\beta \cdot A_{\alpha \beta}^\chi \cdot e_\chi \in V,$$

Per convention, we say that the tensor product has been deformed by the (action of a) cube $A$ and we relabel it as: $\otimes_A$.

$$\forall (q_1, q_2) \in V \times V : \otimes_A(q_1, q_2) = q_1^\alpha \cdot q_2^\beta \cdot A_{\alpha \beta}^\chi \cdot e_\chi \in V$$

Because of a natural isomorphism between $V$ and its dual $V^*$, it is also very usual and perhaps easier to work in $V^*$. This can be done in introducing the well-known Dirac’s convention (“bracket”). The deformed tensor product can also be understood as:

$$\forall (q_1, q_2) \in V \times V : | \otimes_A (q_1, q_2) >= |q_1^\alpha \cdot q_2^\beta \cdot A_{\alpha \beta}^\chi > \in M(D, 1)$$

where $M(D, 1)$ represents the set of matrices with one column and $D$ rows.
Definition 1.3. Decomposition

Inspired by the concept of division, an operation that every child is supposed to learn during the first years of his/her life if he/she get the chance to visit a primary school, that theory introduces a concept of division concerning the dual representation of deformed tensor products.

In the common language, realizing a division in staying inside the set of natural numbers, \( \mathbb{N} = \{0, 1, 2, 3, \ldots\} \), is yielding a subset of pairs \((M, R)\) in \( \mathbb{N}^2 \); the first argument, \( M \), is the main part of a given result whilst the second argument, \( R \), is the residual part of that same result. For example: if we were dividing 13 by 3, we would write 13 = (4 x 3) + 1 and decode the manoeuvre in saying that 4 is the main part whilst 1 is the residual one; hence, 13 : 3 \( \in \mathbb{N} \).

This basic idea is extrapolated and applied now to deformed tensor products. Per convention, the first argument intervening in such a product is called a projectile and the second is called a target. In dividing a given deformed tensor product by its target, we expect to find a main part and a residual part too. Here, recalling considerations loaned to the concept of torsion, we suspect that this division will now result in a kind of decomposition that may reasonably be written:

\[
(\text{projectile}, \text{target}) \in V^2 : \\
| \otimes_{A} (\text{projectile}, \text{target}) > = [\text{Main part}]. |\text{target} > + |\text{residual part} > \in M(D, 1) \sim K^D \\
[\text{Main part}] = [P] \in M(D, K), \text{ residual part} = z \in E(D, K)
\]

or more concisely:

\[
(q_1, q_2) \in V^2 : | \otimes_{A} (q_1, q_2) > = [P]. |q_2 > + |z >
\]

Remark 1.1. Postulate

Recalling that mathematics are a motor for physics and that experiments in physics permanently exhibit the fact that our measurements are rarely precise, we shall postulate that, in general, a decomposition is only realized approximately. That means that it is realized with an error \( \delta E \) such that:

\[
(q_1, q_2) \in V^2 : \exists \| \delta E > = | \otimes_{A} (q_1, q_2) > - \{[P]. |q_2 > + |z >
\]

Definition 1.4. Trivial decomposition

Per convention we say that a decomposition is trivial when the residual part vanishes.

Proposition 1.1. Existence of a trivial decomposition

There are mathematical circumstances such that each image in \( V^k \) of a given deformed tensor product accepts at least one trivial decomposition which we conventionally write:

\[
(q_1, q_2) \in V^2 : \exists [A\Phi(q_1), 0] : | \otimes_{A} (q_1, q_2) > = A\Phi(q_1). |q_2 > + |0 >
\]

Proof: Let write the deformed tensor product in extenso

\[
\forall (q_1, q_2) \in V \times V : \otimes_A(q_1, q_2) = q_1^\alpha \cdot q_2^\beta \cdot A_{\alpha\beta}^\chi \cdot e_\chi \in V
\]
Its image in $V^*$ is:

$$|⊗_A (q_1, q_2)^\chi| = |(q_1^\alpha \cdot q_2^\beta) \cdot A_{\alpha\beta}^\chi|$$

If $K$ is equipped with an associative and commutative multiplication, then:

$$|⊗_A (q_1, q_2)^\chi| = |q_1^\alpha \cdot (q_2^\beta \cdot A_{\alpha\beta}^\chi)| = |(q_1^\alpha \cdot A_{\alpha\beta}^\chi) \cdot q_2^\beta|$$

can be recondensed in:

$$|⊗_A (q_1, q_2) > = [A_{\alpha\beta}^\chi \cdot A_{1\beta}^\alpha] \cdot |q_2|$$

Hence, there always exists at least one trivial decomposition denoted $(A\Phi(q_1), 0)$ if $\{K, +, \cdot\}$ is a commutative ring.

**Remark 1.2. Coincidence**

Observing the postulates which have been proposed in remark 1.1, it is a fact that a decomposition is trivial when its error of realization coincides with its residual part.

$$(q_1, q_2) \in V^2, \delta E = z :$$

$$\downarrow$$

$$|⊗_A (q_1, q_2) > - [\{P\} \cdot |q_2| > + |z| >] = |\delta E| > = |z| >$$

$$\downarrow$$

$$|⊗_A (q_1, q_2) > - [P] \cdot |q_2| > = |0| >$$

**2 The procedure**

**2.1 Cubes and bilinear forms**

The first necessary prerequisite for the development of the so-called extrinsic method is the existence of at least one non-degenerated bi-linear form, $b$, acting on pairs of elements in a vector space. Here, it is represented by a matrix $[B]$ in $M(D, K)$ with $[B] \neq 0$ and we consider a vector space $W = \{E(D, K), ⊗_A, [B]\}$.

**Remark 2.1. Bilinear forms and metrics on $W$**

In that paragraph, I roughly initiate a discussion concerning a combinatorial link between a given cube and the existence of induced metrics.

Since $W$ is equipped with one cube, there are $[[01]; \S 13.7, p. 331; in the French language]$

$$A_D^{D^2} = \frac{D^{3!}}{(D^3 - D^2)!}; D \in \mathbb{N} - \{0, 1\}$$

theoretical possibilities to arrange, without repetition, $D^2$ components which have been extracted from the at most $D^3$ different components contained in the cube $A$. That means that we can find at most so many bi-linear forms for $W$. For example: if $D = 2$, then $A_8^2 = 8 \cdot 7 \cdot 6 \cdot 5 = 1 \, 680$ arrangements; if $D = 3$, then: $A_{27}^9 = 27 \cdot 26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 \cdot 19 = 1 \, 700 \, 755 \, 056 \, 000$ arrangements; if $D = 4$, then: $A_{64}^{16} = 64 \cdot 63 \cdot 62 \cdot 61 \cdot 60 \cdot 59 \cdot 58 \cdot 57 \cdot 56 \cdot 55 \cdot 54 \cdot 53 \cdot 52 \cdot 51 \cdot 50 \cdot 49 = 1,022,134,645,914,425. \ 10^{28}$ arrangements.

The last number is enormous. The concept of cube is illustrated in physics via the
The notion of connection. Due to internal symmetries, the effective number of different components can be smaller than the theoretical maximum; for example, the Levi-Civita connection is represented by a symmetric cube (that means that: \( \Gamma_{\alpha\beta}^\lambda = \Gamma_{\beta\alpha}^\lambda \)). Furthermore, effective models (example: isotropic metrics depending on the time) involve a set of at most eleven different expressions for the Christoffel's symbols of the second kind (see: star model in [[02]; chapter 44, p. 258; in the German language]) whilst the symmetric and diagonal metric itself has only at most four components. That means that, if we would like to extract bilinear forms (I did not say metrics) within that combinatorical context, we would have \( 11!4! = 11.10.9.8.7.6.5 = 1663 \) 200 arrangements. This is a quite smaller number than the theoretical maximum.

On one side, we know since a long time that the Levi-Civita connection is metric compatible and that the correspondence between that connection and the metric is unique. On the other side, as intuitively expected, the combinatoristic is yielding quite more bilinear forms than acceptable metrics. This is due to at least two well-identifiable facts: (i) not any bilinear form is a non-degenerated one; (ii) all components of a cube are not necessary different.

### 2.2 The scalar related to the projectile

Consider any non-degenerated bilinear form \( b \):

\[
(a, b) \in W^2 \overset{\text{b}}{\mapsto} b(a, b) = \langle a \cdot \{ [B] \cdot |b \rangle \rangle = \langle a, b \rangle_{[B]} \in K
\]

With that form, we can build an element of \( K \) (a scalar) which we say to be related to the projectile in calculating:

\[
(q_1, q_2) \in W^2 : \langle q_1, \delta E \rangle_{[B]} = \langle q_1, | \otimes A (q_1, q_2) \rangle - \{ [P] \cdot |q_2 \rangle + \{ |z \rangle \} \}_{[B]}
\]

Let suppose that there exists a function depending on the projectile and such that:

\[
\forall q_1 \in W, \exists P_1 \in F(W, K) : P_1(q_1) - P_1(0) = \langle q_1, \delta E \rangle_{[B]}
\]

This is implying the existence of a scalar which can be interpreted as a polynomial form of degree two depending on the components of the projectile:

\[
P_1(q_1) = \langle q_1, | \otimes A (q_1, q_2) \rangle - \{ [P] \cdot |q_2 \rangle + |z \rangle \} \}_{[B]} + P_1(0)
\]

Since, when \( K \) is a commutative ring (see proposition 1.1), there is always at least one trivial decomposition for any given deformed tensor product:

\[
P_1(q_1)
\]

\[
= \langle q_1 \cdot \{ [B] \cdot \{ \Phi(q_1) \cdot |q_2 \rangle \} \} - \langle q_1 \cdot \{ [B] \cdot \{ [P] \cdot |q_2 \rangle + |z \rangle \} \} + P_1(0)
\]

Let examine in which way that scalar may also be understood as the beginning of a Taylor - Mac Laurin development for the polynomial \( P_1 \) around the origin:

\[
P_1(q_1) = \frac{1}{2} \cdot \langle q_1 \cdot \{ [HessP_1(0)] \cdot |q_1 \rangle \rangle + \langle \text{Grad}_{q_1} P_1(0) \cdot |q_1 \rangle + P_1(0)
\]

That eventuality is only meaningful if three conditions are simultaneously realized:
• Terms with degree two:

\[
<q_1, [B] \cdot \{A\} \Phi(q_1), q_2 > = \frac{1}{2} \cdot <q_1, [Hess P_3(0)] \cdot q_1 >
\]

\[
q_1^\alpha \cdot (A^\alpha_\beta \cdot q_2^\beta) \cdot q_1^\alpha = \frac{1}{2} \cdot q_1^\alpha \cdot \frac{\partial^2 P_3(q_1 = 0)}{\partial q_1^\alpha \partial q_1^\alpha} \cdot q_1^\alpha
\]

\[
\forall q_1 : [A^\alpha_\beta \cdot q_2^\beta] = \frac{1}{2} \cdot [Hess q_1, P_3(q_1 = 0)]
\]

The left hand term must be managed carefully; although it is resembling a trivial decomposition with the target as argument, it is neither that trivial decomposition nor its transposed. The exact link with that trivial decomposition depends on the properties of the cube \( A \) at hand. To avoid confusion in further development, we shall write:

\[
[A^\alpha_\beta \cdot q_2^\beta] = A^\alpha_\beta(q_2) \neq A^\alpha_\beta(q_2)
\]

As a matter of evidence, both matrices coincide when the cube is symmetric; i.e.: \( A^\alpha_\beta = A^\beta_\alpha \).

• Terms with degree one:

\[
\text{Grad}_{q_1} P_1(0) = -[B] \cdot \{P\} \cdot q_2 > + |z >
\]

• Term with degree zero:

\[
0(3) = P_1(0)
\]

2.3 The scalar related to the target

With the form, \( b \), we can also build an element of \( K \) (a scalar) which we say to be related to the target in calculating:

\[
(q_1, q_2) \in W^2 : <q_2, \delta E > |B| = <q_2, \mid(4) (q_1, q_2) > - \{P\} \cdot q_2 > + |z > > |B|
\]

Let suppose that there exists a function depending on the targets and such that:

\[
\forall q_2 \in W, \exists P_2 \in F(W, K) : P_2(q_2) - P_2(0) = <q_2, \delta E > |B|
\]

This is implying the existence of a scalar which can be interpreted as a polynomial form of degree two depending on the components of a given target:

\[
P_2(q_2) = <q_2, \mid(4) (q_1, q_2) > - \{P\} \cdot q_2 > + |z > > |B| + P_2(0)
\]

Since there is always at least one trivial decomposition for any given deformed tensor product, it follows:

\[
P_2(q_2) = <q_2, [B] \cdot \{A\} \Phi(q_1) - [P] \cdot q_2 > - <q_2, [B] \cdot |z > + P_2(0)
\]

Let examine in which way that scalar may also be understood as the beginning of a Taylor - Mac Laurin development for the polynomial \( P_2 \) around the origin:

\[
P_2(q_2) = \frac{1}{2} \cdot <q_2, [Hess P_2(0)] \cdot q_2 > + <\text{Grad}_{q_2} P_2(0) \cdot q_2 > + P_2(0)
\]

That eventuality is only meaningful if three conditions are simultaneously realized:
• Terms of degree two:

\[ [B] \cdot \{[A] \Phi(q_1) - [P]\} = \frac{1}{2} \cdot [HessP_2(0)] \]

• Terms of degree one:

\[ \text{Grad}_{(q_2)} P_2(0) = -[B] \cdot [z >] \]

• Term of degree zero:

\[ 0(3) = P_2(0) \]

2.4 Results

If (i) the matrix [B] representing the bi-linear form at hand is not degenerated (i.e.: if \( [B] \neq 0 \)), and if (ii) the confrontation between the scalar related to a target and a Taylor - Mac Laurin development is meaningful, then the deformed tensor product involved in that procedure can be approximately decomposed. Indeed if:

\[ |B| \neq 0 \Rightarrow \exists [B]^{-1} \]

The three necessary conditions are:

• Terms of degree two:

\[ [P] = A \Phi(q_1) - \frac{1}{2} \cdot [B]^{-1} \cdot [HessP_2(0)] \]

• Terms of degree one:

\[ [z >] = -[B]^{-1} \cdot [\text{Grad}_{(q_2)} P_2(0)] > \]

• Term of degree zero:

\[ 0(3) = P_2(0) \]

This is yielding a set of plausible non-trivial decompositions ([P], [z]) such that:

\[ (q_1, q_2) \in W^2 : [\delta E] = | \otimes_A (q_1, q_2) > - \{[P] \cdot [q_2 > + [z >] \} \]

2.5 Comments

The polynomials \( P_1 \) and \( P_2 \), their gradients and their classical Hessian matrices are the main ingredients of these acceptable decompositions. Let add some complementary comments. For the sake of generality, these polynomials have been chosen as if they were different; but there are certainly some situations in physics allowing to repeat the method with a unique function:

\[ P_1(q_1) = L(q_1, q_2 = \text{invariant}) \]

\[ P_2(q_2) = L(q_1 = \text{invariant}, q_2) \]

Since the same decomposition ([P], [z]) should be reached with both scalars, a confrontation between the conditions which have been obtained is theoretically meaningful. It follows that that method can only be applied when:

\[ L(0, 0) = 0(3) \]

\[ \text{Grad}_{(q_2)} P_2(0) - \text{Grad}_{(q_1)} P_1(0) = \{(B) \cdot A \Phi(q_1) - \frac{1}{2} \cdot [HessP_2(0)] \} \cdot [q_2 > \]

The method is said to be “extrinsic” because, in all cases, it needs the intervention of a non-degenerated bilinear form which is an actor not intrinsically contained in the problematic dubbed as:
Definition 2.1 (The so-called (E) question).

\((q_1, q_2) \in W^2 : \exists \{[P], z\} : |\otimes_A (q_1, q_2) > = \{[P] \cdot |q_2 > + |z >\}\)

2.6 Logical test

All mathematical methods have a domain of validity; the extrinsic method too. The existence of an exact non-trivial decomposition annihilates the scalar related either to the projectile or to the target:

\[\exists ([P], z) : |\otimes_A (q_1, q_2) > = \{[P] \cdot |q_2 > + |z >\} = |0 >\]

\(\delta E = 0\)

\(i = 1, 2 : < q_i, \delta E >_{[B]} = 0\)

But conversely, the vanishing of the scalar, for example related to the target, is corresponding to three plausible and different logical configurations:

\(< q_2, \delta E >_{[B]} = 0\)

- **Configuration 1**: The target is null - the deformed tensor product is null - this situation is obviously meaningless here;

\(q_2 = 0\)

- **Configuration 2**: The non-trivial decomposition is exact:

\(\delta E = 0\)

- **Configuration 3**: The target and the error are orthogonal but none of them vanishes:

\(q_2 \neq 0, \delta E \neq 0, < q_2, \delta E >_{[B]} = 0\)

3 Applications - personal contributions


- The Heisenberg’s uncertainty principle (HUP) for the pair (energy, time): ISBN 978-2-36923-096-6, EAN 9782369230966

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4 Bibliography

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