The Kepler two body problem in the language of geometric algebra

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This text is for the young people, as an extension to the book [3].

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The Kepler problem can be nicely treated in the language of geometric algebra, without coordinates. From Newton's laws, we have (see Fig. 1)

\[ \mathbf{F}_{12} = -\mathbf{F}_{21} = \frac{G m_1 m_2}{r^2} \hat{\mathbf{r}}, \]

\[ \mathbf{f}_1 = \frac{G m_2}{r^2} \hat{\mathbf{r}}, \quad \mathbf{f}_2 = -\frac{G m_1}{r^2} \hat{\mathbf{r}}, \]

(\( \hat{\mathbf{r}} \) is a unit vector)

\[ \mathbf{f} = \mathbf{f}_2 - \mathbf{f}_1 = -\frac{G (m_1 + m_2)}{r^2} \hat{\mathbf{r}} = -\frac{\mu}{r^2} \hat{\mathbf{r}} , \quad \mu \equiv G (m_1 + m_2) . \]

Denoting \( \mathbf{v} = \hat{\mathbf{r}} , \quad \mathbf{p} = m \hat{\mathbf{r}} , \quad m = \frac{m_1 m_2}{m_1 + m_2} \) (the reduced mass), we define the angular momentum of the system

\[ \mathbf{L} = \mathbf{r} \wedge \mathbf{p} , \quad \mathbf{l} = \mathbf{r} \wedge \hat{\mathbf{r}} , \]

whence, due to \( \hat{\mathbf{r}} \parallel \hat{\mathbf{r}} \), we have

\[ \dot{\mathbf{l}} = \mathbf{r} \wedge \dot{\mathbf{r}} + \mathbf{r} \wedge \mathbf{r} = 0 \Rightarrow \mathbf{L} = 0 \Rightarrow \mathbf{L} = \text{const} . \]

From Fig. 2, we see that the area swept by \( \mathbf{r} \) in a small time is

\[ \dot{\mathbf{A}} = 1/2 = \text{const} \]

(Kepler's second law).
Energy conservation

From

\[ \mathbf{\hat{r}} \cdot \mathbf{\ddot{r}} = \frac{1}{2} \frac{d(\mathbf{\dot{r}} \cdot \mathbf{\dot{r}})}{dt} = 0, \]

(\( \mathbf{\hat{r}} \perp \mathbf{\dot{r}} \Rightarrow \mathbf{\hat{r}} \wedge \mathbf{\dot{r}} = \mathbf{\dddot{r}} \)) we have

\[ \frac{1}{2} \frac{d(\mathbf{\dot{r}} \cdot \mathbf{\dot{r}})}{dt} = \mathbf{\dot{r}} \cdot \mathbf{\ddot{r}} = -\frac{\mu}{r^2} \mathbf{\dot{r}} \cdot \mathbf{\ddot{r}} = -\frac{\mu}{r^2} \mathbf{\dot{r}} \left( \mathbf{\dot{r}} \mathbf{\hat{r}} + \mathbf{\dot{r}} \right) = -\frac{\mu}{r^2} \mathbf{\dot{r}} = \frac{d}{dt} \left( \frac{\mu}{r} \right), \]

that is

\[ \frac{d}{dt} \left( \frac{v^2}{2} - \frac{\mu}{r} \right) \equiv \frac{dE}{dt} = 0 \Rightarrow E = \text{const}, \]

where \( E \) is the total energy, \( T \) is the kinetic energy, while \( U \) is the potential energy

\[ E = \frac{v^2}{2} - \frac{\mu}{r}, \quad T = \frac{v^2}{2}, \quad U = -\frac{\mu}{r}. \]

Laplace-Runge-Lenz vector

From \( \mathbf{\dddot{r}} = -\mathbf{\dddot{r}} \) and \( \mathbf{\dot{r}} = 0 \), we have

\[ \mathbf{l} = \mathbf{r} \wedge \mathbf{\dot{r}} = (\mathbf{\dot{r}} \mathbf{\hat{r}}) \wedge \left( \mathbf{\dot{r}} \mathbf{\hat{r}} + \mathbf{\ddot{r}} \right) = r^2 \mathbf{\dddot{r}}, \]

\[ \mathbf{ll} = r^2 \mathbf{\dddot{r}} = -\mu \mathbf{\dddot{r}} = \mu \mathbf{\hat{r}}, \]

\[ \frac{d}{dt} \left( \mathbf{l} \mathbf{\dot{r}} - \mu \mathbf{\hat{r}} \right) = 0, \]

and we can define the constant vector

\[ \mathbf{c} = \mathbf{l} \mathbf{\dot{r}} - \mu \mathbf{\hat{r}}, \]

also known as the Laplace-Runge-Lenz vector. Note how easily we came to an important result. Now we can write

\[ \mathbf{l} \mathbf{rr} = \mathbf{cr} + \mu \mathbf{\hat{r}} \Rightarrow \]

\[ \mathbf{l}(\mathbf{\dot{r}} \cdot \mathbf{r} + \mathbf{\dot{r}} \wedge \mathbf{r}) = \mathbf{c} \cdot \mathbf{r} + \mathbf{c} \wedge \mathbf{r} + \mu \mathbf{r} \Rightarrow \]

\[ \mathbf{l}(\mathbf{\dot{r}} \cdot \mathbf{r} - \mathbf{l}) = \mathbf{c} \cdot \mathbf{r} + \mathbf{c} \wedge \mathbf{r} + \mu \mathbf{r}, \]

whence, comparing grades, we have

\[ (\mathbf{\dot{r}} \cdot \mathbf{r}) \mathbf{l} = \mathbf{c} \wedge \mathbf{r} \quad (\ast) \]
\[-l^2 = c \cdot r + \mu r \quad (**)
\]

From (\(*\)), we see that for \( \dot{\mathbf{r}} \perp \mathbf{r} \) (a major axis for planetary motions) it follows \( c \parallel \mathbf{r} \), which means that the LRL vector is always parallel to the major axis.

**Trajectories**

As \( l^2 < 0 \), we define \( h^2 = -l^2 \). Defining also \( c \cdot \mathbf{r} = cr \cos \theta \), \( p = h^2/\mu \), and \( e = c/\mu \), the relation (**) gives

\[
r = \frac{p}{1 + e \cos \theta}. \quad (***)
\]

This is an equation of a conic, where \( e \) is the eccentricity and \( p \) is the semi-latus rectum.

Note that for circular motion we have \( e = 0 \), for \( 0 < e < 1 \) we have an ellipse, for \( e = 1 \) we have a parabola, while for \( e > 1 \) we have a hyperbola. For \( 0 < e < 1 \) (ellipse, Kepler’s first law), we have (for \( \theta = 0 \) and \( \theta = \pi \))

\[
a = \frac{1}{2} \left( \frac{p}{1+e} + \frac{p}{1-e} \right) = \frac{p}{1-e^2},
\]

which is the semi-major axis.

For the semi-minor axis, we have \( b = \sqrt{ap} \). From the expression for the surface of an ellipse, for the period \( T \) we have

\[
\frac{dA}{dt} = \frac{h}{2} \Rightarrow
\]

\[
ab \pi = \frac{hT}{2} \Rightarrow T^2 = \frac{4\pi^2}{\mu} a^3
\]

(Kepler’s third law).
Vis-Viva Equation

Consider the expression for the energy once again. Defining a new vector $\vec{e} = \frac{\vec{c}}{\mu}$, we can write

$$\vec{l} \vec{r} = \mu (\vec{e} + \vec{r}).$$

From the property $\vec{l} \vec{r} = (\vec{r} \wedge \vec{r}) \vec{r} = -\vec{r} (\vec{r} \wedge \vec{r})$ (see [3]), we get

$$(\vec{l} \vec{r})^2 = -(\vec{r} \wedge \vec{r}) \vec{r} (\vec{r} \wedge \vec{r}) = -v^2 l^2 = v^2 h^2,$$

$$v^2 h^2 = \mu^2 (\vec{e} + \vec{r})^2 = \mu^2 (\vec{e}^2 + 2 \vec{e} \cdot \vec{r} + 1).$$

From (***)**, we have

$$\vec{e} \cdot \vec{r} = \frac{h^2}{\mu r} - 1,$$

whence follows that

$$E = \frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu^2}{2 h^2} (1 - e^2),$$

and we see that the sign of the total energy is related to the eccentricity $e$. From the previous definitions, we can also write

$$E = -\frac{\mu}{2a} \Rightarrow v^2 = \mu \left( \frac{2}{r} - \frac{1}{a} \right)$$

(Vis-Viva Equation).

Finally, we can add that there is a much better approach to this problem, using eigenspinors (see [3], Sect. 2.8).

References