# The Kepler two body problem in the language of geometric algebra

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This text is for the young people, as an extension to the book [3].

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*geometric algebra*, *Kepler's laws*, *Laplace-Runge-Lenz vector*, *eccentricity*, *energy conservation*, *angular momentum*

The Kepler problem can be nicely treated in the language of geometric algebra, without coordinates. From *Newton's laws*, we have (see **Fig. 1**)

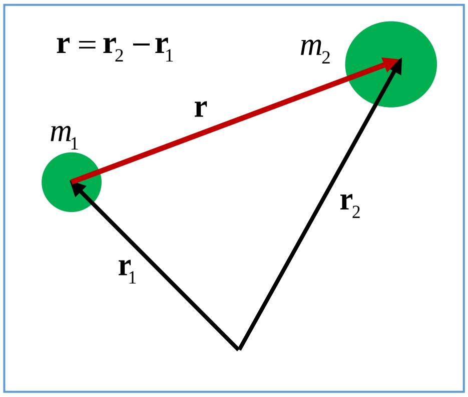


**Fig. 1** The Kepler problem

$$\mathbf{F}_{21}=-\mathbf{F}_{12}=\frac{Gm_1m_2}{r^2}\hat{\mathbf{r}},$$

$$\ddot{\mathbf{r}}_1=\frac{Gm_2}{r^2}\hat{\mathbf{r}},\ \ddot{\mathbf{r}}_2=-\frac{Gm_1}{r^2}\hat{\mathbf{r}},$$

($\hat{\mathbf{r}}$ is a unit vector)

$$\ddot{\mathbf{r}}=\ddot{\mathbf{r}}_2-\ddot{\mathbf{r}}_1=-\frac{G\left(m_1+m_2\right)}{r^2}\hat{\mathbf{r}}=-\frac{\mu}{r^2}\hat{\mathbf{r}},\ \mu\equiv G\left(m_1+m_2\right).$$

Denoting $\mathbf{v}=\dot{\mathbf{r}}$, $\mathbf{p}=m\dot{\mathbf{r}}$, $m=\dfrac{m_1m_2}{m_1+m_2}$ (the *reduced mass*), we define the *angular momentum* of the system

$$\mathbf{L}=\mathbf{r}\wedge\mathbf{p},\ \mathbf{l}=\mathbf{r}\wedge\dot{\mathbf{r}},$$

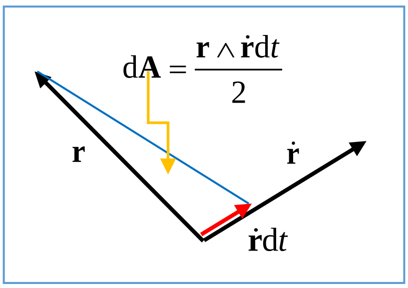


**Fig. 2** The area swapped

whence, due to $\ddot{\mathbf{r}}\parallel\hat{\mathbf{r}}$, we have

$$\dot{\mathbf{l}}=\dot{\mathbf{r}}\wedge\dot{\mathbf{r}}+\mathbf{r}\wedge\ddot{\mathbf{r}}=0\Rightarrow\dot{\mathbf{L}}=0\Rightarrow\mathbf{L}=const.$$

From **Fig. 2**, we see that the area swept by $\mathbf{r}$ in a small time is

$$\dot{\mathbf{A}}=\mathbf{l}/2=const$$

(*Kepler's second law*).

## Energy conservation

From

$$\hat{\mathbf{r}}\cdot\dot{\hat{\mathbf{r}}}=\frac{1}{2}\frac{\mathrm{d}\left(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}\right)}{\mathrm{d}t}=0\,,$$

($\hat{\mathbf{r}}\perp\dot{\hat{\mathbf{r}}}\Rightarrow\hat{\mathbf{r}}\wedge\dot{\hat{\mathbf{r}}}=\hat{\mathbf{r}}\dot{\hat{\mathbf{r}}}$) we have

$$\frac{1}{2}\frac{\mathrm{d}\left(\dot{\mathbf{r}}\cdot\dot{\mathbf{r}}\right)}{\mathrm{d}t}=\ddot{\mathbf{r}}\cdot\dot{\mathbf{r}}=-\frac{\mu}{r^2}\hat{\mathbf{r}}\cdot\dot{\mathbf{r}}=-\frac{\mu}{r^2}\hat{\mathbf{r}}\cdot\left(\dot{r}\hat{\mathbf{r}}+r\dot{\hat{\mathbf{r}}}\right)=-\frac{\mu}{r^2}\dot{r}=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mu}{r}\right),$$

that is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{v^2}{2}-\frac{\mu}{r}\right)\equiv\frac{\mathrm{d}E}{\mathrm{d}t}=0\Rightarrow E=const\,,$$

where $E$ is the *total energy*, $T$ is the *kinetic energy*, while $U$ is the *potential energy*

$$E=\frac{v^2}{2}-\frac{\mu}{r},\ T=\frac{v^2}{2},\ U=-\frac{\mu}{r}\,.$$

## Laplace-Runge-Lenz vector

From $\hat{\mathbf{r}}\dot{\hat{\mathbf{r}}}=-\dot{\hat{\mathbf{r}}}\hat{\mathbf{r}}$ and $\dot{\mathbf{l}}=0$, we have

$$\mathbf{l}=\mathbf{r}\wedge\dot{\mathbf{r}}=\left(r\hat{\mathbf{r}}\right)\wedge\left(\dot{r}\hat{\mathbf{r}}+r\dot{\hat{\mathbf{r}}}\right)=r^2\hat{\mathbf{r}}\dot{\hat{\mathbf{r}}}\,,$$

$$\mathbf{l}\ddot{\mathbf{r}}=r^2\hat{\mathbf{r}}\dot{\hat{\mathbf{r}}}\ddot{\mathbf{r}}=-\mu\hat{\mathbf{r}}\dot{\hat{\mathbf{r}}}\hat{\mathbf{r}}=\mu\dot{\hat{\mathbf{r}}}\,,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{l}\dot{\mathbf{r}}-\mu\hat{\mathbf{r}}\right)=0\,,$$

and we can define the constant vector

$$\mathbf{c}=\mathbf{l}\dot{\mathbf{r}}-\mu\hat{\mathbf{r}}\,,$$

also known as the *Laplace-Runge-Lenz vector*. Note how easily we came to an important result. Now we can write

$$\mathbf{l}\dot{\mathbf{r}}\mathbf{r}=\mathbf{c}\mathbf{r}+\mu\hat{\mathbf{r}}\mathbf{r}\Rightarrow$$

$$\mathbf{l}\left(\dot{\mathbf{r}}\cdot\mathbf{r}+\dot{\mathbf{r}}\wedge\mathbf{r}\right)=\mathbf{c}\cdot\mathbf{r}+\mathbf{c}\wedge\mathbf{r}+\mu r\Rightarrow$$

$$\mathbf{l}\left(\dot{\mathbf{r}}\cdot\mathbf{r}-\mathbf{l}\right)=\mathbf{c}\cdot\mathbf{r}+\mathbf{c}\wedge\mathbf{r}+\mu r\,,$$

whence, comparing grades, we have

$$\left(\dot{\mathbf{r}}\cdot\mathbf{r}\right)\mathbf{l}=\mathbf{c}\wedge\mathbf{r}\qquad(*)$$

$$-\mathbf{l}^2 = \mathbf{c}\cdot\mathbf{r} + \mu r \qquad (**)$$

From (*), we see that for $\dot{\mathbf{r}} \perp \mathbf{r}$ (a *major axis* for planetary motions) it follows $\mathbf{c} \parallel \mathbf{r}$, which means that the LRL vector is always parallel to the major axis.

## Trajectories

As $\mathbf{l}^2 < 0$, we define $h^2 = -\mathbf{l}^2$. Defining also $\mathbf{c}\cdot\mathbf{r} = cr\cos\theta$, $p = h^2/\mu$, and $e = c/\mu$, the relation (**) gives

$$r = \frac{p}{1+e\cos\theta}. \qquad (***)$$

This is an equation of a *conic*, where $e$ is the *eccentricity* and $p$ is the *semi-latus rectum*.

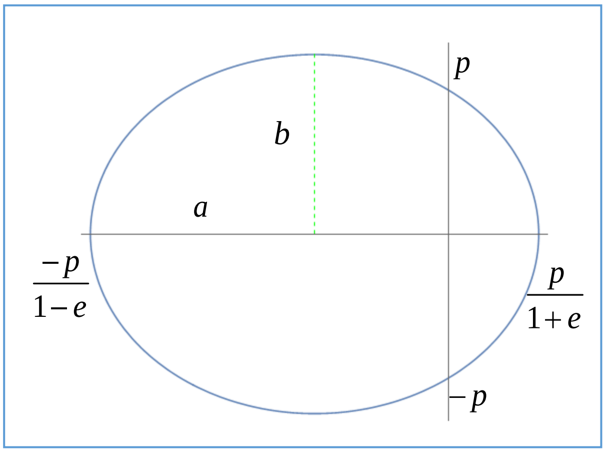


**Fig. 3** An ellipse

Note that for circular motion we have $e=0$, for $0<e<1$ we have an *ellipse*, for $e=1$ we have a *parabola*, while for $e>1$ we have a *hyperbola*. For $0<e<1$ (*ellipse*, *Kepler's first law*), we have (for $\theta=0$ and $\theta=\pi$)

$$a = \frac{1}{2}\left(\frac{p}{1+e} + \frac{p}{1-e}\right) = \frac{p}{1-e^2},$$

which is the *semi-major axis*.

For the *semi-minor axis*, we have $b=\sqrt{ap}$. From the expression for the surface of an ellipse, for the period $T$ we have

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{h}{2} \Rightarrow$$

$$ab\pi = \frac{hT}{2} \Rightarrow T^2 = \frac{4\pi^2}{\mu}a^3$$

(*Kepler's third law*).

## Vis-Viva Equation

Consider the expression for the energy once again. Defining a new vector $\mathbf{e} = \mathbf{c}/\mu$, we can write

$$\mathbf{l}\dot{\mathbf{r}} = \mu\left(\mathbf{e} + \hat{\mathbf{r}}\right).$$

From the property $\mathbf{l}\dot{\mathbf{r}} = \left(\mathbf{r} \wedge \dot{\mathbf{r}}\right)\dot{\mathbf{r}} = -\dot{\mathbf{r}}\left(\mathbf{r} \wedge \dot{\mathbf{r}}\right)$ (see [3]), we get

$$\left(\mathbf{l}\dot{\mathbf{r}}\right)^2 = -\left(\mathbf{r} \wedge \dot{\mathbf{r}}\right)\dot{\mathbf{r}}\dot{\mathbf{r}}\left(\mathbf{r} \wedge \dot{\mathbf{r}}\right) = -v^2\mathbf{l}^2 = v^2h^2,$$

$$v^2h^2 = \mu^2\left(\mathbf{e} + \hat{\mathbf{r}}\right)^2 = \mu^2\left(\mathbf{e}^2 + 2\mathbf{e}\cdot\hat{\mathbf{r}} + 1\right).$$

From (***), we have

$$\mathbf{e}\cdot\hat{\mathbf{r}} = \frac{h^2}{\mu r} - 1,$$

whence follows that

$$E = \frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu^2}{2h^2}\left(1 - e^2\right),$$

and we see that the sing of the total energy is related to the eccentricity $e$. From the previous definitions, we can also write

$$E = -\frac{\mu}{2a} \Rightarrow v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$$

(*Vis-Viva Equation*).

Finally, we can add that there is a much better approach to this problem, using *eigenspinors* (see [3], Sect. 2.8).

## References


[1] Arnold, V.I.: *Mathematical Methods of Classical Mechanics*, Springer, 1989

[2] Hestenes, D.: *New Foundations for Classical Mechanics*, Kluwer Academic, Dordrecht, 1999

[3] Josipović, M.: *Geometric Multiplication of Vectors - An Introduction to Geometric Algebra in Physics*, Birkhäuser, 2019