Biquaternion based construction of the Weyl- and Dirac matrices and their Lorentz transformation operators.

E.P.J. de Haas

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Abstract The necessity of Lorentz transforming the Dirac matrices is an ongoing issue with contradicting opinions. The Lorentz transformation of Dirac spinors is clear but for the Dirac adjoint, the combination of a spinor and the 'time-like' zeroth gamma-matrix, the situation is fussy again. In the Feynman slash objects, the gamma matrix four vector connects to the dynamic four vectors without really becoming one itself. The Feynman slash objects exist in 4-D Minkowsky space-time on the one hand, the gamma matrices are often taken as inert objects like the Minkowski metric itself on the other hand. To be short, a slumbering confusion exists in RQM's roots. In this paper, first a Pauli-level biquaternion environment equivalent to Minkowski space-time is presented. Then the Weyl-Dirac environment is produced as a PT doubling of the biquaternion Pauli-environment. It is the production process from basic elements that produces some clarification regarding the mentioned RQM foundational fussiness.

Keywords Relativistic Quantum Mechanics · Electrons · Quaternions · Relativistic wave equations

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1 Introduction

The Lorentz transformation of Dirac matrices is an ongoing issue with contradicting opinions. For some, the Dirac gamma matrices are Lorentz inert objects, although written in a four-vector notation. For others, they transform as ordinary four-vectors. The transformation of Dirac spinors is clear but for the Dirac adjoint,

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E.P.J. de Haas
Nijmegen, The Netherlands
E-mail: haas2u@gmail.com

the combination of a spinor and the ‘time-like’ zeroth gamma-matrix, the situation is fussy again.

The gamma matrix four vector is ontologically located in between the dynamic objects, as for example the energy momentum four vector, on the one hand and the 4-D basis for Minkowski space on the other hand. In Special Relativity, the Minkowski metric is Lorentz transformation inert because the transformation happens inside this metric and affects the dynamic variables only. In the Feynman slash objects, the gamma matrix four vector connects to the dynamic four vectors without really becoming one itself. The Feynman slash objects exist in 4-D Minkowsky space-time on the one hand, the gamma matrices are often taken as inert objects like the Minkowski metric on the other hand. In some approaches, the Dirac gamma matrices span a vector space for themselves but then it is unclear how this vector-space relates to Minkowski space-time. To be short, a slumbering confusion exists in RQM’s roots.

In this paper I construct the Weyl and Dirac matrices based on a doubling specific biquaternion approach that is morphologically equal to Minskowski space-time. From there, I construct the Lorentz transformation matrix operators for the Weyl and Dirac matrices and the Feynman slash dynamic objects. The fact of the construction from basis elements, of the Feynman slash objects first and the related Lorentz transformation operators second, produces a plateau from which a perspective is possible that allows one to somewhat clarify the mentioned confusion.

In the literature, the Lorentz transformation matrix operator isn’t constructed but derived based on the assumed Lorentz covariance of the Dirac equation. Compared to this standardly presented highly complex derivation, the construction presented in this paper has a pedagogical simplicity that might be beneficial for educational purposes. The construction procedure also sheds a new perspective on the contradictory opinions regarding the Lorentz properties of the gamma matrices and there connection to Minkowski space-time.

In the remainder of the introduction, I relate to the literature on the Lorentz transformation operator and the Lorentz transformation status of the gamma four vector. In part two of this paper, I construct a Pauli-level biquaternion math-phys environment morphologically equal to the Minkowski one. In part three, I construct the Weyl-Dirac environment based on a PT parity time reversal doubling of the biquaternion Pauli environment of part two. This construction produces the announced clarification.

1.1 The derivation of the LT-operator

The Lorentz transformation of Dirac matrices and spinors is an ongoing issue with contradicting opinions. For some, the Dirac gamma matrices are Lorentz inert objects, although written in a four-vector notation: *Does that mean that the gamma matrices that appear in the Dirac equation transform as a vector? The answer is no [1, p. 63]*. For others, they transform as ordinary four-vectors: *the covariance of the Dirac equation implies that* $\boldsymbol{\gamma}$ *transforms like a four-vector [2, p. 17].*

What is consistent in all presentations of the Lorentz properties of the gamma matrices, is the key equation $S\gamma^\nu S^{-1} = \Lambda_\mu{}^\nu \gamma^\mu$, as derived from the requirement of the Lorentz covariance of the Dirac equation. In this equation, the operator S is the Lorentz transformation operator, although the notation may vary from author to author.

In [3, p. 73], Stone wrote: *The Lorentz covariance of the Dirac equation is guaranteed if there exists a matrix representation* $S(L)$ *of the Lorentz group so that for any Lorentz transformation* $L^\mu{}_\nu$ *there exists a matrix* $S(L)$ *such that* $S(L)\gamma^\mu S^{-1}(L) = (L^{-1})^\mu{}_\nu \gamma^\nu$. This matrix S is defined by this equation and has to be found through it, see Greiner in [4, p. 138, Eqn. 3.34]: *This is the fundamental relation determining the operator* $\hat{S}$*: To find* $\hat{S}$ *means solving [*$\hat{S}(\hat{a})\gamma^\nu \hat{S}^{-1}(\hat{a}) = a_\mu{}^\nu \gamma^\mu$*].* A similar reasoning is given in [5, p. 147, Eqn. 5.102], [6, p. 42] and in [7, p. 93]. This usual textbook approach is critically analyzed and alternatively presented in [8].

So in the usual textbook approach the authors have to solve $S\Lambda S^{-1}\beta^\mu S\Lambda^{-1}S^{-1} = \Lambda_\mu{}^\nu \beta^\mu$ for $S\Lambda S^{-1}$ without the insight that it is a product of three operations. In [5, p. 147, Eqn. 5.102], the " S" is a black box, whereas in my approach I opened the box and found " S" = $S\Lambda S^{-1}$, a relation that I constructed and then used to prove $S\Lambda S^{-1}\beta^\mu S\Lambda^{-1}S^{-1} = \Lambda_\mu{}^\nu \beta^\mu$ instead of assuming it first and solving it later. I do not assume and solve, I construct and prove instead. This was possible mainly because of its close connection to the Lorentz transformation approach in the biquaternion representation of the Pauli level physics.

In this paper I will demonstrate that the S in $S\gamma^\nu S^{-1} = \Lambda_\mu{}^\nu \gamma^\mu$ can be constructed out of three operations when the Weyl representation is used. First, one transforms the Dirac representation of γ^ν into the Weyl representation using S, then the Lorentz transformation is applied to the Weyl γ^ν using Λ, after which one transforms the γ'^ν back to the Dirac representation using S^{-1}. The result gives $\gamma'^\nu = S\Lambda S^{-1}\gamma^\nu S\Lambda^{-1}S^{-1} = \Lambda_\mu{}^\nu \gamma^\mu$. The Lorentz transformation of the Weyl representation of four-vectors is uncomplicated when using rapidities. So two matrices, S and Λ are enough to construct the matrix that has a very complicated derivation in the standard textbook approach.

In this paper, biquaternions are used to deal with relativist physics, including mechanics, electrodynamics and quantum mechanics. Such a (bi-)quaternion approach has a long history, see [9] and [10] for an extensive literature on the subject. The originality of this paper is twofold, first in the attempt to develop a *notation* that is as close as possible to both Relativistic Quantum Mechanics and the General Relativity standards and second in the *construction* of the Lorentz transformation matrices needed in relativistic quantum mechanics on the Dirac-Weyl level. The developed notation is indispensable for the construction. The goal is to reach out to physicists, not to mathematicians and as a consequence the emphasis is on the physics of the biquaternion synthesis, under negligence of the mathematical foundations, subtleties and conventions. The paper is divided in two parts, the first on the level of the Pauli spin matrices and the second on that of the Dirac matrices as a double version of the Pauli ones. The biquaternion basis is represented by two by two complex matrices in a dual minquat space-time and pauliquat spin-norm version.

The first part is rather familiar in the context of the many biquaternion approaches that have been proposed the last hundred or so years, see [9] and [10]. Slight differences are present, making the biquaternion expose on the Pauli matrices level interesting on its own. The result is applied to the Maxwell and Lorentz EM-equations, in order to show its mathematical and conceptual consistency.

In the second part, matrices and spinors are treated on the Dirac level in order to arrive at the relativistic core of quantum mechanics. The Dirac matrices are presented as dual versions of the Pauli ones. The Lorentz transformation of the Dirac matrices is being simplified due to the method developed in the first Pauli level part of the paper. This simplification allows for a much shorten and more transparent way to demonstrate the Lorentz invariance and covariance of the equations and products. The Lorentz transformation properties of spinors is critically assessed.

2 The Pauli spin level

2.1 A complex quaternion basis for the metric

Quaternions can be represented by the basis $(\hat{\mathbf{1}}, \hat{\mathbf{I}}, \hat{\mathbf{J}}, \hat{\mathbf{K}})$. This basis has the properties $\hat{\mathbf{I}}\hat{\mathbf{I}} = \hat{\mathbf{J}}\hat{\mathbf{J}} = \hat{\mathbf{K}}\hat{\mathbf{K}} = -\hat{\mathbf{1}}$ and $\hat{\mathbf{1}}\hat{\mathbf{1}} = \hat{\mathbf{1}}$; $\hat{\mathbf{1}}\hat{\mathbf{K}} = \hat{\mathbf{K}}\hat{\mathbf{1}} = \hat{\mathbf{K}}$ for $\hat{\mathbf{I}}, \hat{\mathbf{J}}, \hat{\mathbf{K}}$; $\hat{\mathbf{I}}\hat{\mathbf{J}} = -\hat{\mathbf{J}}\hat{\mathbf{I}} = \hat{\mathbf{K}}$; $\hat{\mathbf{J}}\hat{\mathbf{K}} = -\hat{\mathbf{K}}\hat{\mathbf{J}} = \hat{\mathbf{I}}$; $\hat{\mathbf{K}}\hat{\mathbf{I}} = -\hat{\mathbf{I}}\hat{\mathbf{K}} = \hat{\mathbf{J}}$. A quaternion number in its summation representation is given by $A = a_0\hat{\mathbf{1}} + a_1\hat{\mathbf{I}} + a_2\hat{\mathbf{J}} + a_3\hat{\mathbf{K}}$, in which the a_μ are real numbers . Biquaternions or complex quaternions are given by $C = A+\mathbf{i}B = c_0\hat{\mathbf{1}}+c_1\hat{\mathbf{I}}+c_2\hat{\mathbf{J}}+c_3\hat{\mathbf{K}}$ in which the $c_\mu = a_\mu+\mathbf{i}b_\mu$ are complex numbers and the a_μ and b_μ are real numbers.

This standard biquaternion basis $(\hat{\mathbf{1}}, \hat{\mathbf{I}}, \hat{\mathbf{J}}, \hat{\mathbf{K}})$ can be used to provide a basis for relativistic 4-D space-time. One way to do this is by making the time coordinate $c_0 = b_0\mathbf{i}$ complex only and the space coordinates $(c_1, c_2, c_3) = (a_1, a_2, a_3)$ real only, see [11]. Synge called these objects Minkowski quaternions or ‘minquats’, Silberstein called them ‘physical quaternions’ [11]. This however produces confusion regarding the time-like complex number as the physics gets more complicated. As Synge put it, *the intrusion of the imaginary element is not trivial* [11]. The main reason is that minquats do not form a closed algebra under addition and multiplication as a subspace inside the wider biquaternion space, due to the multiplication operation. The reason they are used nevertheless is given by Synge: *For the application of quaternions to Lorentz transformations it is essential to introduce Minkowskian quaternions* [11].

The use of minquats produces language conflicts with almost all of modern physics, that is Quantum Mechanics and Special and General Relativity, where the space-time coordinates always are a set of four real numbers. So for several reasons, I choose to insert the time-like complex number of $c_0 = b_0\mathbf{i}$ in the basis instead of in the coordinate. So by using $c_0\hat{\mathbf{I}} = b_0\mathbf{i}\hat{\mathbf{I}} = b_0\hat{\mathbf{T}}$ the space-time basis is then given by $(\hat{\mathbf{T}}, \hat{\mathbf{I}}, \hat{\mathbf{J}}, \hat{\mathbf{K}})$. In this way, the coordinates are always a set of real numbers $\in \mathbb{R}$. The space-time basis $(\hat{\mathbf{T}}, \hat{\mathbf{I}}, \hat{\mathbf{J}}, \hat{\mathbf{K}})$, (a disguised minquat basis) is not closed under multiplications, as already mentioned by Synge.

A set of four numbers $\in \mathbb{R}$ is given by

$$A^\mu = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix},$$

or by $A_\mu = [a_0, a_1, a_2, a_3]$. In this way, the raising or lowering of the index doesn’t change any sign. A^μ simply is the transpose of A_μ and vise versa. The biquaternion basis can be given as a set $\mathbf{K}_\mu = (\hat{\mathbf{T}}, \hat{\mathbf{I}}, \hat{\mathbf{J}}, \hat{\mathbf{K}})$. Then a biquaternion space-time vector can be written as the product

$$A = A_\mu \mathbf{K}^\mu = [a_0, a_1, a_2, a_3] \begin{bmatrix} \hat{\mathbf{T}} \\ \hat{\mathbf{I}} \\ \hat{\mathbf{J}} \\ \hat{\mathbf{K}} \end{bmatrix} = a_0\hat{\mathbf{T}} + a_1\hat{\mathbf{I}} + a_2\hat{\mathbf{J}} + a_3\hat{\mathbf{K}} \tag{1}$$

I apply this to the space-time four vector of relativistic bi-quaternion 4-space R with the four numbers $R_\mu = (r_0, r_1, r_2, r_3) = (ct, r_1, r_2, r_3)$, so with $r_0, r_1, r_2, r_3 \in \mathbb{R}$. Then I have the space-time four-vector as the product of the coordinate set and the basis $R = R_\mu \mathbf{K}^\mu = r_0\hat{\mathbf{T}} + r_1\hat{\mathbf{I}} + r_2\hat{\mathbf{J}} + r_3\hat{\mathbf{K}} = ct\hat{\mathbf{T}} + \mathbf{r} \cdot \mathbf{K}$. I use the three-vector analogue of $R_\mu \mathbf{K}^\mu$ when I write $\mathbf{r} \cdot \mathbf{K}$. In this notation I have $R^T = -r_0\hat{\mathbf{T}} + r_1\hat{\mathbf{I}} + r_2\hat{\mathbf{J}} + r_3\hat{\mathbf{K}} = -r_0\hat{\mathbf{T}} + \mathbf{r} \cdot \mathbf{K}$ for the time reversal operator and $R^P = r_0\hat{\mathbf{T}} - r_1\hat{\mathbf{I}} - r_2\hat{\mathbf{J}} - r_3\hat{\mathbf{K}} = r_0\hat{\mathbf{T}} - \mathbf{r} \cdot \mathbf{K}$ for the space point mirror or parity operator, with $R^P = -R^T$. In this notation, the transpose of a matrix will be given by the suffix ‘t’, so $R^t_\mu = R^\mu$. The complex transpose of spinors is given by the dagger symbol, as in $\psi^\dagger$. The complex conjugate of a spinor is given by ψ^*. In this language, the operators T and P take the role of raising and lowering of indexes in the General Relativity convention.

2.2 Matrix representation of the quaternion basis

Quaternions can be represented by 2x2 matrices. Several representations are possible, but most of those choices result in conflict with the standard approach in physics. Given the unit quaternion as $\mathbf{\hat{1}}$, my choice for the space-time four set is

$$\mathbf{\hat{T}} = \begin{bmatrix} \mathbf{i} & 0 \\ 0 & \mathbf{i} \end{bmatrix}, \mathbf{\hat{I}} = \begin{bmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{bmatrix}, \mathbf{\hat{J}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \mathbf{\hat{K}} = \begin{bmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{bmatrix}. \tag{2}$$

I can compare these to the Pauli spin matrices $\boldsymbol{\sigma}_P = (\sigma_x, \sigma_y, \sigma_z)$.

$$\boldsymbol{\sigma}_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \boldsymbol{\sigma}_y = \begin{bmatrix} 0 & -\mathbf{i} \\ \mathbf{i} & 0 \end{bmatrix}, \boldsymbol{\sigma}_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{3}$$

If I exchange the σ_x and the σ_z, I get $\mathbf{K} = \mathbf{i}\boldsymbol{\sigma}$ and $\mathbf{K}_\mu = \mathbf{i}(\mathbf{\hat{1}}, \boldsymbol{\sigma})$. So in my use of the Pauli matrices, I use $\boldsymbol{\sigma} \equiv (\sigma_I, \sigma_J, \sigma_K) = (\sigma_z, \sigma_y, \sigma_x)$. So also $\mathbf{\hat{I}} = \mathbf{\hat{T}}\boldsymbol{\sigma}_I, \mathbf{\hat{J}} = \mathbf{\hat{T}}\boldsymbol{\sigma}_J, \mathbf{\hat{K}} = \mathbf{\hat{T}}\boldsymbol{\sigma}_K$ and $\boldsymbol{\sigma}_I = -\mathbf{\hat{T}}\mathbf{\hat{I}}, \boldsymbol{\sigma}_J = -\mathbf{\hat{T}}\mathbf{\hat{J}}, \boldsymbol{\sigma}_K = -\mathbf{\hat{T}}\mathbf{\hat{K}}$.

With this choice of matrices, a four-vector R can be written as

$$R = r_0 \begin{bmatrix} \mathbf{i} & 0 \\ 0 & \mathbf{i} \end{bmatrix} + r_1 \begin{bmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{bmatrix} + r_2 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + r_3 \begin{bmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{bmatrix}. \tag{4}$$

This can be compacted into a matrix representation of R:

$$R = \begin{bmatrix} r_0\mathbf{i} + \mathbf{i}r_1 & r_2 + \mathbf{i}r_3 \\ -r_2 + \mathbf{i}r_3 & r_0\mathbf{i} - \mathbf{i}r_1 \end{bmatrix} = \begin{bmatrix} R_{00} & R_{01} \\ R_{10} & R_{11} \end{bmatrix} \tag{5}$$

with the numbers $R_{00}, R_{01}, R_{10}, R_{11} \in \mathbb{C}$.

2.3 Multiplication of vectors as matrix multiplication adds pauliquats to minquats

In general, the multiplication of two vectors A and B follows matrix multiplication, with $A_{ij}, B_{ij}, C_{ij} \in \mathbb{C}$.

$$AB = \begin{bmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{bmatrix} \begin{bmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{bmatrix} = \begin{bmatrix} C_{00} & C_{01} \\ C_{10} & C_{11} \end{bmatrix} = C. \tag{6}$$

So we have

$$C = AB = \begin{bmatrix} A_{00}B_{00} + A_{01}B_{10} & A_{00}B_{01} + A_{01}B_{11} \\ A_{10}B_{00} + A_{11}B_{10} & A_{10}B_{01} + A_{11}B_{11} \end{bmatrix} = \begin{bmatrix} C_{00} & C_{01} \\ C_{10} & C_{11} \end{bmatrix}. \tag{7}$$

Of course, vectors A, B and C can be expressed with their a_μ, b_μ, c_μ coordinates $\in \mathbb{R}$ and if we use them, after some elementary but elaborate calculations and rearrangements we arrive at the following result of the multiplication expressed in the a_μ, b_μ and c_μ as

$$\begin{aligned} c_0 &= -a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3 \\ c_{1K} &= a_2b_3 - a_3b_2 \\ c_{2K} &= a_3b_1 - a_1b_3 \\ c_{3K} &= a_1b_2 - a_2b_1 \\ c_{1\sigma} &= -a_0b_1 - a_1b_0 \\ c_{2\sigma} &= -a_0b_2 - a_2b_0 \\ c_{3\sigma} &= -a_0b_3 - a_3b_0 \end{aligned} \tag{8}$$

In short, if we use the three-dimensional Euclidean dot and cross products of Euclidean three-vectors in classical physics, this gives for the coordinates

$$c_0 = -a_0b_0 - \mathbf{a} \cdot \mathbf{b}$$

$$\mathbf{c}_K = \mathbf{a} \times \mathbf{b} \tag{9}$$

$$\mathbf{c}_\sigma = -a_0\mathbf{b} - \mathbf{a}b_0 \tag{10}$$

And in the quaternion notation we get

$$C = AB = (-a_0b_0 - \mathbf{a} \cdot \mathbf{b})\hat{\mathbf{1}} + (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{K} + (-a_0\mathbf{b} - \mathbf{a}b_0) \cdot \boldsymbol{\sigma} \tag{11}$$

This matrix multiplication, in which I used $\hat{\mathbf{T}}\hat{\mathbf{T}} = -\hat{\mathbf{1}}$ and $\hat{\mathbf{T}}\mathbf{K} = -\boldsymbol{\sigma}$, implies that the space-time basis $(\hat{\mathbf{T}}, \mathbf{K})$ is being duplicated by a spin-norm basis $(\hat{\mathbf{1}}, \boldsymbol{\sigma})$ by the multiplication operation.

The relativistically relevant multiplications of two four-vectors are all in the form $C = A^TB$. The difference between AB and A^TB is in the sign of a_0. This results in

$$C = A^TB = (a_0b_0 - \mathbf{a} \cdot \mathbf{b})\hat{\mathbf{1}} + (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{K} + (a_0\mathbf{b} - \mathbf{a}b_0) \cdot \boldsymbol{\sigma} \tag{12}$$

From this it follows that the physically relevant norm of a four-vector, from a relativistic perspective, is the product A^TA and not the product AA:

$$\begin{aligned} C = A^TA = (a_0a_0 - \mathbf{a} \cdot \mathbf{a})\hat{\mathbf{1}} + (\mathbf{a} \times \mathbf{a}) \cdot \mathbf{K} + (a_0\mathbf{a} - \mathbf{a}a_0) \cdot \boldsymbol{\sigma} = \\ (a_0a_0 - \mathbf{a} \cdot \mathbf{a})\hat{\mathbf{1}} = c^2a_\tau^2\hat{\mathbf{1}}. \end{aligned} \tag{13}$$

The main quadratic form of the metric is $dR^T dR = (c^2dt^2 - d\mathbf{r}^2)\hat{\mathbf{1}} = c^2d\tau^2\hat{\mathbf{1}} = ds^2\hat{\mathbf{1}}$ with $ds = cd\tau$. The quadratic giving the distance of a point R to the origin of its reference system is given by $R^T R = (c^2t^2 - \mathbf{r}^2)\hat{\mathbf{1}} = c^2\tau^2\hat{\mathbf{1}} = s^2\hat{\mathbf{1}}$ with $s = c\tau$.

The multiplication of two four vectors can also be arranged as the multiplication of two tensors, a coordinate tensor times a metric tensor using that

$$(A_\mu \mathbf{K}^\mu)^T B_\nu \mathbf{K}^\nu = A_\mu B^\nu (\mathbf{K}_\mu)^T \mathbf{K}^\nu = C_\mu{}^\nu \mathbf{K}_\mu{}^\nu \tag{14}$$

with the metric tensor as

$$\mathbf{K}_\mu{}^\nu = (\mathbf{K}_\mu)^T \mathbf{K}^\nu = \left[-\hat{\mathbf{T}}, \hat{\mathbf{I}}, \hat{\mathbf{J}}, \hat{\mathbf{K}}\right] \begin{bmatrix} \hat{\mathbf{T}} \\ \hat{\mathbf{I}} \\ \hat{\mathbf{J}} \\ \hat{\mathbf{K}} \end{bmatrix} = \tag{15}$$

$$\begin{bmatrix} -\hat{\mathbf{T}}\hat{\mathbf{T}} & \hat{\mathbf{I}}\hat{\mathbf{T}} & \hat{\mathbf{J}}\hat{\mathbf{T}} & \hat{\mathbf{K}}\hat{\mathbf{T}} \\ -\hat{\mathbf{T}}\hat{\mathbf{I}} & \hat{\mathbf{I}}\hat{\mathbf{I}} & \hat{\mathbf{J}}\hat{\mathbf{I}} & \hat{\mathbf{K}}\hat{\mathbf{I}} \\ -\hat{\mathbf{T}}\hat{\mathbf{J}} & \hat{\mathbf{I}}\hat{\mathbf{J}} & \hat{\mathbf{J}}\hat{\mathbf{J}} & \hat{\mathbf{K}}\hat{\mathbf{J}} \\ -\hat{\mathbf{T}}\hat{\mathbf{K}} & \hat{\mathbf{I}}\hat{\mathbf{K}} & \hat{\mathbf{J}}\hat{\mathbf{K}} & \hat{\mathbf{K}}\hat{\mathbf{K}} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{1}} & -\sigma_I & -\sigma_J & -\sigma_K \\ \sigma_I & -\hat{\mathbf{1}} & -\hat{\mathbf{K}} & \hat{\mathbf{J}} \\ \sigma_J & \hat{\mathbf{K}} & -\hat{\mathbf{1}} & -\hat{\mathbf{I}} \\ \sigma_K & -\hat{\mathbf{J}} & \hat{\mathbf{I}} & -\hat{\mathbf{1}} \end{bmatrix}. \tag{16}$$

This multiplication product has a norm $\hat{\mathbf{1}}$ part, a space $\mathbf{K}$ part and a spin $\boldsymbol{\sigma}$ part. So the multiplication of two four vectors $A^T B = C$ has this multiplication matrix. The multiplication combines the properties of symmetric and anti-symmetric in one product.

The inevitable appearance of the spin-norm basis in the multiplication of two Synge minquats or Silberstein physical quaternions is why the minquats do not form a closed algebra under multiplication [11]. In my approach, the space-time basis $(\hat{\mathbf{T}}, \mathbf{K})$ doesn't form a closed algebra under multiplications, it needs a spin-norm complex dual $(\hat{\mathbf{T}}, \mathbf{K}) = \mathbf{i}(\hat{\mathbf{1}}, \boldsymbol{\sigma})$ to cover all of biquaternion space, *while only allowing real coordinates for* R_μ *and* P_μ *in* $R_\mu \mathbf{K}^\mu$ *and* $P_\mu \boldsymbol{\sigma}^\mu$. The obligation, chosen freely in a Kantian way, to only use real coordinates produces the dual basis in a unique way.

The physical sphere, the cosmos so to speak, then obtains a dual space-time/spin-norm basis as it's natural geometry. This duality will prove to mirror real physics with electric charges or monopoles as part of space-time and hypothetical magnetic monopoles as spin-norm entities, if at all possible. Electric currents exist in real space-time $(\hat{\mathbf{T}}, \mathbf{K})$ and magnetic monopole currents can only, if at all, exist in the 'imaginary' spin-norm $(\hat{\mathbf{1}}, \boldsymbol{\sigma})$ sphere as will be shown further on in this paper. If Synge's minquats are $R_\mu \mathbf{K}^\mu$ biquaternions, then $P_\mu \boldsymbol{\sigma}^\mu$ are pauliquats. The sum of minquats and pauliquats cover the whole of biquaternion space. The multiplication of a minquat with a minquat produces a minquat and a pauliquat. Electric currents must be represented by minquats and magnetic current by pauliquats, if at all. Intrinsic spin is a pauliquat, its Lorentz dual intrinsic polarization is a minquat. The existence of minquats and pauliquats defies electromagnetic super-symmetry as is striven for by the magnetic monopole research community. The combined minquat and pauliquat environment is a-symmetric.

2.4 The Lorentz transformation

A normal Lorentz transformation between two reference frames connected by a relative velocity v in the $x-$ or $\hat{\mathbf{I}}$-direction, with the usual $\gamma = 1/\sqrt{1 - v^2/c^2}$, $\beta = v/c$ and $r_0 = ct$, can be expressed as

$$\begin{bmatrix} r'_0 \\ r'_1 \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{bmatrix} \begin{bmatrix} r_0 \\ r_1 \end{bmatrix} = \begin{bmatrix} \gamma r_0 - \beta\gamma r_1 \\ \gamma r_1 - \beta\gamma r_0 \end{bmatrix}. \tag{17}$$

We want to connect this to our matrix representation of R as in Eq.(5) which gives

$$R'_{00} = \mathbf{i}r'_0 + \mathbf{i}r'_1 = \mathbf{i}\gamma r_0 - \mathbf{i}\beta\gamma r_1 + \mathbf{i}\gamma r_1 - \mathbf{i}\beta\gamma r_0 \tag{18}$$

$$R'_{11} = \mathbf{i}r'_0 - \mathbf{i}r'_1 = \mathbf{i}\gamma r_0 - \mathbf{i}\beta\gamma r_1 - \mathbf{i}\gamma r_1 + \mathbf{i}\beta\gamma r_0. \tag{19}$$

Now we want to introduce rapidity or hyperbolic Special Relativity in order to integrate Lorentz transformations into our matrix metric. In [12] I gave a brief history of rapidity in its relation to the Thomas precession and the geodesic precession. For this paper we only need elementary rapidity definitions. If we use the rapidity ψ as $e^{\psi} = \cosh\psi + \sinh\psi = \gamma + \beta\gamma$, the previous transformations can be rewritten as

$$R'_{00} = \mathbf{i}r'_0 + \mathbf{i}r'_1 = (\gamma - \beta\gamma)(\mathbf{i}r_0 + \mathbf{i}r_1) = R_{00}e^{-\psi} \tag{20}$$

$$R'_{11} = \mathbf{i}r'_0 - \mathbf{i}r'_1 = (\gamma + \beta\gamma)(\mathbf{i}r_0 - \mathbf{i}r_1) = R_{11}e^{\psi}. \tag{21}$$

As a result we have

$$R^L = \begin{bmatrix} R'_{00} & R'_{01} \\ R'_{10} & R'_{11} \end{bmatrix} = \begin{bmatrix} R_{00}e^{-\psi} & R_{01} \\ R_{10} & R_{11}e^{\psi} \end{bmatrix} = U^{-1}RU^{-1}. \tag{22}$$

In the expression $R^L = U^{-1}RU^{-1}$ we used the matrix U as

$$U = \begin{bmatrix} e^{\frac{\psi}{2}} & 0 \\ 0 & e^{-\frac{\psi}{2}} \end{bmatrix}. \tag{23}$$

But this means that we can write the result of a Lorentz transformation on R with a Lorentz velocity in the $\hat{\mathbf{I}}$-direction between the two reference systems as

$$R^L = r_0 \begin{bmatrix} \mathbf{i}e^{-\psi} & 0 \\ 0 & \mathbf{i}e^{\psi} \end{bmatrix} + r_1 \begin{bmatrix} \mathbf{i}e^{-\psi} & 0 \\ 0 & -\mathbf{i}e^{\psi} \end{bmatrix} + r_2 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + r_3 \begin{bmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{bmatrix}. \tag{24}$$

This can be written as

$$R^L = r_0 U^{-1}\hat{\mathbf{T}}U^{-1} + r_1 U^{-1}\hat{\mathbf{I}}U^{-1} + r_2\hat{\mathbf{J}} + r_3\hat{\mathbf{K}} = r_0\hat{\mathbf{T}}^L + r_1\hat{\mathbf{I}}^L + r_2\hat{\mathbf{J}} + r_3\hat{\mathbf{K}}. \tag{25}$$

But because we started with Eq.(17), we now have two equivalent options to express the result of a Lorentz transformation

$$R^L = r_0'\hat{\mathbf{T}} + r_1'\hat{\mathbf{I}} + r_2\hat{\mathbf{J}} + r_3\hat{\mathbf{K}} = r_0\hat{\mathbf{T}}^L + r_1\hat{\mathbf{I}}^L + r_2\hat{\mathbf{J}} + r_3\hat{\mathbf{K}}, \tag{26}$$

either as a coordinate transformation or as a basis transformation.

This result only works for Lorentz transformation between v_x-, v_1- or $\hat{\mathbf{I}}$-aligned reference systems. Reference systems which do not have their relative Lorentz velocity aligned in the $\hat{\mathbf{I}}$-direction will have to be space rotated into such an alignment before the Lorentz transformation in the form $R^L = U^{-1}RU^{-1}$ is applied. In principle, such a rotation in order to achieve the $\hat{\mathbf{I}}$ alignment of the primary reference frame to a secondary reference frame is always possible as an operation prior to a Lorentz transformation. This unique alignment between two frames of reference S and S', needed to match the physics with the algebra, is analyzed by Synge in [11, p. 41-48] and focuses on the concept of a communal photon. The requirement of reference system alignment is also the reason for the appearance of the Thomas precession and the Thomas-Wigner rotation if the axis are not aligned; the notion that two Lorentz transformations in different directions in space can always be substituted by the subsequent application of one space rotation and one single Lorentz transformation, see [12]. The communal photon of Synge is the one for which the relativistic Doppler shift between S and S' results in $\nu' = \nu e^{\pm\psi}$. The minquat algebra requires inertial observers to align their principal axis along such a communal photon, in my notation the $\hat{\mathbf{I}}$ axis.

The Lorentz transformation of the coordinates $(R^\mu)^L$ can be written as

$$\begin{bmatrix} r_0' \\ r_1' \\ r_2' \\ r_3' \end{bmatrix} = \Lambda_\nu{}^\mu R^\nu = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} \gamma r_0 - \beta\gamma r_1 \\ \gamma r_1 - \beta\gamma r_0 \\ r_2 \\ r_3 \end{bmatrix}$$

So the Lorentz transformation of $R = R_\mu \mathbf{K}^\mu = \mathbf{K}_\mu R^\mu$ can be presented as

$$\begin{aligned} R^L &= \mathbf{K}_\mu (R^\mu)^L = \mathbf{K}_\mu \Lambda_\nu{}^\mu R^\nu = (\mathbf{K}_\mu \Lambda_\nu{}^\mu) R^\nu = (\mathbf{K}_\nu)^L R^\nu \\ &= U^{-1}\mathbf{K}_\nu U^{-1} R^\nu = U^{-1}\mathbf{K}_\nu R^\nu U^{-1} = U^{-1} R U^{-1} \end{aligned} \tag{27}$$

This implies the identity $\mathbf{K}_\mu \Lambda_\nu^{\ \mu} = U^{-1}\mathbf{K}_\nu U^{-1}$, an identity that isn't possible for the coordinates only. The matrix representation of the basis is key to this identity, because the relativistic Doppler factor $e^{\pm\psi}$ appears differently attached to the matrix elements. As is the $\hat{\mathbf{1}}$ alignment of the two involved reference frames during the Lorentz transformation. Given that $\mathbf{K}_\mu = \mathbf{i}\boldsymbol{\sigma}_\mu$, the identity $\mathbf{K}_\mu \Lambda_\nu^{\ \mu} = U^{-1}\mathbf{K}_\nu U^{-1}$ can also be seen as an instruction for the Lorentz transformation of the Pauli spin matrices as a norm-spin four set $\boldsymbol{\sigma}_\mu = (\hat{\mathbf{1}}, \boldsymbol{\sigma})$.

The Lorentz transformation of A^T is also interesting, due to the importance of the product $C = A^T B$ and therefore the Lorentz transformation C^L. Given the inverse Lorentz transformation as

$$A^{L^{-1}} \equiv UAU \tag{28}$$

one can prove

$$\left(A^T\right)^{L^{-1}} = U\left(A^T\right)U = \left(U^{-1}AU^{-1}\right)^T = \left(A^L\right)^T \tag{29}$$

and

$$\left(A^T\right)^L = U^{-1}\left(A^T\right)U^{-1} = (UAU)^T = \left(A^{L^{-1}}\right)^T. \tag{30}$$

The result $\left(A^L\right)^T = UA^TU$ will be used in several important derivations in this paper, when the Lorentz transformation of a product and the possible invariance or Lorentz covariance has to be investigated, as in the next example.

Start with two inertial reference systems S_1 and S_2 connected by a constant relative velocity v, a Lorentz gamma factor $\gamma(v)$ and a rapidity factor $\psi(v)$ defining the Lorentz transformation matrix U. Given A and B in S_1 and their product in S_1 as $C = A^T B$. Then in S_2 one has A^L and B^L and their product $C^L = \left(A^L\right)^T B^L$. We then have

$$\begin{aligned} C^L = \left(A^L\right)^T B^L = \left(A^T\right)^{L^{-1}} B^L &= U\left(A^T\right)UU^{-1}BU^{-1} \\ &= UA^TBU^{-1} = UCU^{-1}. \end{aligned} \tag{31}$$

As a result, it is easy to prove that the quadratic $A^T A = c^2 a_\tau^2 \hat{\mathbf{1}}$ is Lorentz invariant. We have

$$\begin{aligned} (A^L)^T A^L = (A^T)^{-L} A^L &= UA^TUU^{-1}AU^{-1} = UA^TAU^{-1} \\ = U(c^2a_\tau^2)\hat{\mathbf{1}}U^{-1} &= UU^{-1}(c^2a_\tau^2)\hat{\mathbf{1}} = c^2a_\tau^2\hat{\mathbf{1}} = A^TA. \end{aligned} \tag{32}$$

So both quadratics $R^T R$ and $dR^T dR$ are Lorentz invariant scalars, as has been shown for every quadratic of four-vectors. But they aren't perfect quadratics, the last defined through the requirement $AA = |A|^2\hat{\mathbf{1}}$.

2.5 Adding the dynamic vectors

If I want to apply the previous to relativistic electrodynamics and to quantum physics, I need to further develop the mathematical language, the notation system and the biquaternion elements. I don't claim originality regarding the biquaternion foundations of my notation system. As indicated before, there is a whole subculture around quaternions an biquaternions in physics, see [9], [10], and I have been studying many of those papers. The justification for my paper is to be found in what it adds to this rather large subculture, as part of the more general *plethora of different vector formalisms currently in use [13]*.

But let's return to my project of formulating a pragmatic biquaternion math-phys language through which relativity and quantum can be synthesized. The most relevant dynamic four vectors must be given a biquaternion representation. The basic definitions I use for that purpose are quite common in the formulations of relativistic dynamics, see for example [14]. I start with a particle with a given three vector velocity as $\mathbf{v}$, a rest mass as m_0 and an inertial mass $m_i = \gamma m_0$, with the usual $\gamma = (\sqrt{1 - v^2/c^2})^{-1}$. I use the Latin suffixes as abbreviations for words, not for numbers. So m_i stands for inertial mass and U_p for potential energy. The Greek suffixes are used as indicating a summation over the numbers 0, 1, 2 and 3. So P_μ stands for a momentum four-vector coordinate row with components ($p_0 = \frac{1}{c}U_i, p_1, p_2, p_3$). The momentum three-vector is written as $\mathbf{p}$ and has components (p_1, p_2, p_3).

I define the coordinate velocity four vector as

$$V = V_\mu \mathbf{K}^\mu = \frac{d}{dt} R_\mu \mathbf{K}^\mu = c\hat{\mathbf{T}} + \mathbf{v} \cdot \mathbf{K} = v_0 \hat{\mathbf{T}} + \mathbf{v} \cdot \mathbf{K}. \tag{33}$$

The proper velocity four vector on the other hand will be defined using the proper time $\tau = t_0$, with $t = \gamma t_0 = \gamma \tau$, as

$$U = U_\mu \mathbf{K}^\mu = \frac{d}{d\tau} R_\mu \mathbf{K}^\mu = \frac{d}{\frac{1}{\gamma} dt} R_\mu \mathbf{K}^\mu = \gamma V_\mu \mathbf{K}^\mu = u_0 \hat{\mathbf{T}} + \mathbf{u} \cdot \mathbf{K}. \tag{34}$$

The momentum four vector will be, at least when we have the symmetry condition $\mathbf{p} = m_i \mathbf{v}$,

$$P = P_\mu \mathbf{K}^\mu = m_i V_\mu \mathbf{K}^\mu = m_i V = m_0 U_\mu \mathbf{K}^\mu = m_0 U. \tag{35}$$

The four vector partial derivative $\partial = \partial_\mu \mathbf{K}^\mu$ will be defined using the coordinate four set

$$\partial_\mu = \left[-\frac{1}{c} \partial_t, \nabla_1, \nabla_2, \nabla_3 \right] = [\partial_0, \partial_1, \partial_2, \partial_3]. \tag{36}$$

The electrodynamic potential four vector $A = A_\mu \mathbf{K}^\mu$ will be defined by the coordinate four set

$$A_\mu = \left[\frac{1}{c} \phi, A_1, A_2, A_3 \right] = [A_0, A_1, A_2, A_3] \tag{37}$$

The electric four current density vector $J = J_\mu \mathbf{K}^\mu$ will be defined by the coordinate four set

$$J_\mu = [c\rho_e, J_1, J_2, J_3] = [J_0, J_1, J_2, J_3], \tag{38}$$

with ρ_e as the electric charge density. The electric four current with a charge q will be also be written as J_μ and the context will indicate which one is used.

Although we defined these fourvectors using the coordinate column notation, we will often use the matrix or summation notation, as for example with $P = P_\mu \mathbf{K}^\mu$, written as

$$\begin{aligned} P &= p_0 \hat{\mathbf{T}} + p_1 \hat{\mathbf{I}} + p_2 \hat{\mathbf{J}} + p_3 \hat{\mathbf{K}} = p_0 \hat{\mathbf{T}} + \mathbf{p} \cdot \mathbf{K} \\ &= \begin{bmatrix} \mathbf{i}p_0 + \mathbf{i}p_1 & p_2 + \mathbf{i}p_3 \\ -p_2 + \mathbf{i}p_3 & \mathbf{i}p_0 - \mathbf{i}p_1 \end{bmatrix} = \begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{bmatrix}. \end{aligned} \tag{39}$$

The flexibility to use either of these notations is a strength of the math-phys language as developed in this paper. There are cases where one needs to go all the way to the internal scalar matrix notation to solve issues as for example the product rule in calculating a derivative, after which one returns to the more compact notation to evaluate the outcome.

2.6 The EM field in our language

I we apply the matrix multiplication rules to the electromagnetic field with four derivative ∂ and four potential A, with $\partial_0 = -\frac{1}{c}\partial_t$ and $A_0 = \frac{1}{c}\phi$, we get $B = \partial^T A$ as

$$B = \partial^T A = (-\frac{1}{c^2}\partial_t \phi - \boldsymbol{\nabla} \cdot \boldsymbol{A})\hat{\mathbf{I}} + (\boldsymbol{\nabla} \times \boldsymbol{A}) \cdot \mathbf{K} + \frac{1}{c}(-\partial_t \boldsymbol{A} - \boldsymbol{\nabla}\phi) \cdot \boldsymbol{\sigma}. \tag{40}$$

If we apply the Lorenz gauge $\mathbb{B}_0 = -\frac{1}{c^2}\partial_t \phi - \boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$ and the usual EM definitions of the fields in terms of the potentials we get

$$B = \partial^T A = \boldsymbol{B} \cdot \mathbf{K} + \frac{1}{c}\boldsymbol{E} \cdot \boldsymbol{\sigma}. \tag{41}$$

Using $\boldsymbol{\sigma} = -\hat{\mathbf{T}}\mathbf{K} = -\mathbf{i}\mathbf{K}$, this can also be written as

$$B = \partial^T A = (\boldsymbol{B} - \mathbf{i}\frac{1}{c}\boldsymbol{E}) \cdot \mathbf{K} = \overrightarrow{\mathbb{B}} \cdot \mathbf{K}. \tag{42}$$

The use of $\mathbb{B} = \boldsymbol{B} - \mathbf{i}\frac{1}{c}\boldsymbol{E}$ dates back to Minkowski's 1908 treatment of the subject [15]. In my opinion, the flexibility of easy switching between the different modes of notations makes my biquaternion variant suited for unification purposes.

Using $\mathbb{B}$ we can write B as

$$B = \mathbb{B}_1\hat{\mathbf{I}} + \mathbb{B}_2\hat{\mathbf{J}} + \mathbb{B}_3\hat{\mathbf{K}} = \overrightarrow{\mathbb{B}} \cdot \mathbf{K} =$$
$$\begin{bmatrix} \mathbf{i}\mathbb{B}_1 & \mathbb{B}_2 + \mathbf{i}\mathbb{B}_3 \\ -\mathbb{B}_2 + \mathbf{i}\mathbb{B}_3 & -\mathbf{i}\mathbb{B}_1 \end{bmatrix} = \begin{bmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{bmatrix}. \tag{43}$$

For the Lorentz transformation of B we can apply the result of the previous section to get $B^L = (\partial^L)^T A^L = (\partial^T)^{-L} A^L = U(\partial^T)UU^{-1}AU^{-1} = U(\partial^T A)U^{-1} = UBU^{-1}$, so

$$B^L = \begin{bmatrix} e^{\frac{\psi}{2}} & 0 \\ 0 & e^{-\frac{\psi}{2}} \end{bmatrix} \begin{bmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{bmatrix} \begin{bmatrix} e^{-\frac{\psi}{2}} & 0 \\ 0 & e^{\frac{\psi}{2}} \end{bmatrix} = \begin{bmatrix} B_{00} & B_{01}e^{\psi} \\ B_{10}e^{-\psi} & B_{11} \end{bmatrix} \tag{44}$$

which, when written out with $\boldsymbol{E}$ and $\boldsymbol{B}$ leads to the usual result for the Lorentz transformation of the EM field with the Lorentz velocity in the x-direction. But it can also be written as a transformation of the basis, while leaving the coordinates invariant:

$$B^L = UBU^{-1} = \mathbb{B}_1 U\hat{\mathbf{I}}U^{-1} + \mathbb{B}_2 U\hat{\mathbf{J}}U^{-1} + \mathbb{B}_3 U\hat{\mathbf{K}}U^{-1} = \tag{45}$$
$$\mathbb{B}_1\hat{\mathbf{I}} + \mathbb{B}_2\hat{\mathbf{J}}^L + \mathbb{B}_3\hat{\mathbf{K}}^L = \mathbb{B}_1 \begin{bmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{bmatrix} + \mathbb{B}_2 \begin{bmatrix} 0 & e^{\psi} \\ -e^{-\psi} & 0 \end{bmatrix} + \mathbb{B}_3 \begin{bmatrix} 0 & \mathbf{i}e^{\psi} \\ \mathbf{i}e^{-\psi} & 0 \end{bmatrix}.$$

The Lorentz transformation of the EM field can be performed by internally twisting the $(\hat{\mathbf{J}}, \hat{\mathbf{K}})$-surface perpendicular to the Lorentz velocity and in the process leaving the EM-coordinates invariant.

That the above equals the usual Lorentz transformation of the EM field can be shown by going back to [15], where he wrote the transformation in a form equivalent to

$$\begin{bmatrix} \mathbb{B}'_1 \\ \mathbb{B}'_2 \\ \mathbb{B}'_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \gamma & \mathbf{i}\beta\gamma \\ 0 & -\mathbf{i}\beta\gamma & \gamma \end{bmatrix} \begin{bmatrix} \mathbb{B}_1 \\ \mathbb{B}_2 \\ \mathbb{B}_3 \end{bmatrix} = \begin{bmatrix} \mathbb{B}_1 \\ \gamma\mathbb{B}_2 + \mathbf{i}\beta\gamma\mathbb{B}_3 \\ \gamma\mathbb{B}_3 - \mathbf{i}\beta\gamma\mathbb{B}_2 \end{bmatrix} \tag{46}$$

So we have

$$B'_{01} = \mathbb{B}'_2 + \mathbf{i}\mathbb{B}'_3 = \gamma\mathbb{B}_2 + \mathbf{i}\beta\gamma\mathbb{B}_3 + \mathbf{i}\gamma\mathbb{B}_3 + \beta\gamma\mathbb{B}_2 \tag{47}$$

and

$$B'_{10} = -\mathbb{B}'_2 + \mathbf{i}\mathbb{B}'_3 = -\gamma\mathbb{B}_2 - \mathbf{i}\beta\gamma\mathbb{B}_3 + \mathbf{i}\gamma\mathbb{B}_3 + \beta\gamma\mathbb{B}_2. \tag{48}$$

If we use the rapidity ψ as $e^{\psi} = \cosh\psi + \sinh\psi = \gamma + \beta\gamma$, this can be rewritten as

$$B'_{01} = \mathbb{B}'_2 + \mathbf{i}\mathbb{B}'_3 = (\gamma + \beta\gamma)(\mathbb{B}_2 + \mathbf{i}\mathbb{B}_3) = B_{01}e^{\psi} \tag{49}$$

and

$$B'_{10} = -\mathbb{B}'_2 + \mathbf{i}\mathbb{B}'_3 = (\gamma - \beta\gamma)(-\mathbb{B}_2 + \mathbf{i}\mathbb{B}_3) = B_{10}e^{-\psi}, \tag{50}$$

which leads to Eqn. (44).

2.7 The Maxwell Equations and the Lorentz force law

The Maxwell equations in our language can be given as, using $J = \rho V$, $\partial B = \mu_0 J$ and the Lorentz force law, with a four force density $\mathcal{F}$, as $JB = \mathcal{F}$. Maxwell's inhomogeneous wave equations can be written as $(-\partial^T \partial)B = -\mu_0 \partial^T J$ and with the Lorentz invariant quadratic derivative,

$$-\partial^T \partial = (\boldsymbol{\nabla}^2 - \frac{1}{c^2}\partial_t^2)\hat{\mathbf{1}} \tag{51}$$

we get the homogeneous wave equations of the EM field in free space in the familiar form as

$$(-\partial^T \partial)B = \boldsymbol{\nabla}^2 B - \frac{1}{c^2}\partial_t^2 B = 0. \tag{52}$$

I will look at $\partial B = \mu_0 J$ first. The underlying structure then also applies to the Lorentz Force Law and the inhomogeneous part of the wave equation. I start with

$$B = \partial^T A = \boldsymbol{B} \cdot \mathbf{K} + \frac{1}{c}\boldsymbol{E} \cdot \boldsymbol{\sigma}. \tag{53}$$

Then ∂B is given by

$$\partial B = \left(-\frac{1}{c}\partial_t \hat{\mathbf{T}} + \boldsymbol{\nabla} \cdot \mathbf{K}\right)\left(\boldsymbol{B} \cdot \mathbf{K} + \frac{1}{c}\boldsymbol{E} \cdot \boldsymbol{\sigma}\right) =$$
$$-(\boldsymbol{\nabla} \cdot \boldsymbol{B})\hat{\mathbf{1}} + \frac{1}{c}(\boldsymbol{\nabla} \cdot \boldsymbol{E})\hat{\mathbf{T}} + (\boldsymbol{\nabla} \times \boldsymbol{B} - \frac{1}{c^2}\partial_t \boldsymbol{E}) \cdot \mathbf{K} + \frac{1}{c}(\boldsymbol{\nabla} \times \boldsymbol{E} + \partial_t \boldsymbol{B}) \cdot \boldsymbol{\sigma} \tag{54}$$

If we interpret this result using the knowledge regarding the inhomogeneous Maxwell equations, we get an interesting result. First of all, the part of the Maxwell Equation with the dimension of the norm $\hat{\mathbf{1}}$ is zero and so is the part with the dimension of spin $\boldsymbol{\sigma}$. The space-time parts $\mathbf{K}$ and $\hat{\mathbf{T}}$ equal the space-time parts of the four current $\mu_0 J$. So we get

$$\partial B = -(\boldsymbol{\nabla} \cdot \boldsymbol{B})\hat{\mathbf{1}} + \frac{1}{c}(\boldsymbol{\nabla} \cdot \boldsymbol{E})\hat{\mathbf{T}} + (\boldsymbol{\nabla} \times \boldsymbol{B} - \frac{1}{c^2}\partial_t \boldsymbol{E}) \cdot \mathbf{K} + \frac{1}{c}(\boldsymbol{\nabla} \times \boldsymbol{E} + \partial_t \boldsymbol{B}) \cdot \boldsymbol{\sigma} =$$
$$0\hat{\mathbf{1}} + \mu_0 c \rho \hat{\mathbf{T}} + \mu_0 \boldsymbol{J} \cdot \mathbf{K} + 0\boldsymbol{\sigma} = \mu_0 J. \tag{55}$$

So the spin-norm part of the Maxwell Equations equals zero and the space-time part equals the space-time four current density times μ_0. In the line of this interpretation, magnetic monopoles and the correlated magnetic monopole current should be searched in the pauliquat dimensions of spin-norm, not in the minquat dimensions of space-time.

As for the Lorentz covariance of the Maxwell Equations, this can be demonstrated quite easily. Given the four-vectors ∂, A and J in reference system S_1, with the Maxwell Equations as $\partial(\partial^T A) = \mu_0 J$, then in reference system S_2 we have the four-vectors ∂^L, A^L and J^L and the covariant Maxwell Equations given as $\partial^L(\partial^L)^T A^L = \mu_0 J^L$. In S_2 this can be proven through

$$\partial^L(\partial^L)^T A^L = \partial^L(\partial^T)^{L^{-1}} A^L = U^{-1}\partial U^{-1} U(\partial^T) U U^{-1} A U^{-1} = U^{-1}\partial(\partial^T) A U^{-1} = U^{-1}\partial B U^{-1} = U^{-1}\mu_0 J U^{-1} = \mu_0 J^L. \quad (56)$$

So if we have $\partial B = \mu_0 J$ in one frame of reference, this transforms as $\partial^L B^L = \mu_0 J^L$ in another frame of reference, which means that the equation maintains its form, it is Lorentz covariant. We have form-invariance of the equations.

I will look at $JB = F$ now, with $J = qV$. The underlying structure for the Lorentz Force Law is the same as for the Maxwell equations. So JB is given by

$$JB = \left(cq\hat{\mathbf{T}} + \boldsymbol{J}\cdot\mathbf{K}\right)\left(\boldsymbol{B}\cdot\mathbf{K} + \frac{1}{c}\boldsymbol{E}\cdot\boldsymbol{\sigma}\right) = -(\boldsymbol{J}\cdot\boldsymbol{B})\hat{\mathbf{1}} + \frac{1}{c}(\boldsymbol{J}\cdot\boldsymbol{E})\hat{\mathbf{T}} + (\boldsymbol{J}\times\boldsymbol{B} + q\boldsymbol{E})\cdot\mathbf{K} + (\frac{1}{c}\boldsymbol{J}\times\boldsymbol{E} - cq\boldsymbol{B})\cdot\boldsymbol{\sigma} \quad (57)$$

If we interpret this result using the knowledge regarding the Lorentz Force Law, we get an interesting result. First of all, the part of the Lorentz force law with the dimension of the norm $\hat{\mathbf{1}}$ is zero and so is the part with the dimension of spin $\boldsymbol{\sigma}$. The space-time parts $\mathbf{K}$ and $\hat{\mathbf{T}}$ equal the space-time parts of the four force F. Thus we get

$$JB = -(\boldsymbol{J}\cdot\boldsymbol{B})\hat{\mathbf{1}} + \frac{1}{c}(\boldsymbol{J}\cdot\boldsymbol{E})\hat{\mathbf{T}} + (\boldsymbol{J}\times\boldsymbol{B} + q\boldsymbol{E})\cdot\mathbf{K} + (\frac{1}{c}\boldsymbol{J}\times\boldsymbol{E} - cq\boldsymbol{B})\cdot\boldsymbol{\sigma} = 0\hat{\mathbf{1}} + \frac{1}{c}P\hat{\mathbf{T}} + \boldsymbol{F}\cdot\mathbf{K} + 0\boldsymbol{\sigma} = F. \quad (58)$$

So the spin-norm pauliquat part of the Lorentz Force Law equals zero and the space-time minquat part equals the space-time four force.

In both cases, ∂B and BJ, we get a dual spin-norm and space-time product, with the spin-norm equal zero and the non-zero space-time leading to the inhomogeneous four-vectors of current and force. Speculations about magnetic monopoles are connected to these spin-norm parts, the set spanned by pauliquats. In my analysis, if spin-norm is the twin dual of space-time and as such an integral aspect of the metric as foreseen in [16], then searches for magnetic monopoles should focus on this spin-norm aspect of the vacuum.

But from a purely geometric perspective, the product of three four-vectors like in $BJ = \partial^T AJ = F$, we can separate the coordinate four sets ∂_μ, A^ν, and J^μ from the metric basis, as in $BJ = ((\partial_\mu A^\nu)J^\mu)((\mathbf{K}_\mu^T\mathbf{K}^\nu)\mathbf{K}^\mu)$, and focus on the metric product alone. We then get

$$\mathbf{K}_{\mu}{}^{\nu}\mathbf{K}^{\mu} = (\mathbf{K}_{\mu}^{T}\mathbf{K}^{\nu})\mathbf{K}^{\mu} = \begin{bmatrix} -\hat{\mathbf{T}}\hat{\mathbf{T}} & \hat{\mathbf{I}}\hat{\mathbf{T}} & \hat{\mathbf{J}}\hat{\mathbf{T}} & \hat{\mathbf{K}}\hat{\mathbf{T}} \\ -\hat{\mathbf{T}}\hat{\mathbf{I}} & \hat{\mathbf{I}}\hat{\mathbf{I}} & \hat{\mathbf{J}}\hat{\mathbf{I}} & \hat{\mathbf{K}}\hat{\mathbf{I}} \\ -\hat{\mathbf{T}}\hat{\mathbf{J}} & \hat{\mathbf{I}}\hat{\mathbf{J}} & \hat{\mathbf{J}}\hat{\mathbf{J}} & \hat{\mathbf{K}}\hat{\mathbf{J}} \\ -\hat{\mathbf{T}}\hat{\mathbf{K}} & \hat{\mathbf{I}}\hat{\mathbf{K}} & \hat{\mathbf{J}}\hat{\mathbf{K}} & \hat{\mathbf{K}}\hat{\mathbf{K}} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{T}} \\ \hat{\mathbf{I}} \\ \hat{\mathbf{J}} \\ \hat{\mathbf{K}} \end{bmatrix} = \tag{59}$$

$$\begin{bmatrix} -\hat{\mathbf{T}}\hat{\mathbf{T}}\hat{\mathbf{T}} + \hat{\mathbf{I}}\hat{\mathbf{T}}\hat{\mathbf{I}} + \hat{\mathbf{J}}\hat{\mathbf{T}}\hat{\mathbf{J}} + \hat{\mathbf{K}}\hat{\mathbf{T}}\hat{\mathbf{K}} \\ -\hat{\mathbf{T}}\hat{\mathbf{I}}\hat{\mathbf{T}} + \hat{\mathbf{I}}\hat{\mathbf{I}}\hat{\mathbf{I}} + \hat{\mathbf{J}}\hat{\mathbf{I}}\hat{\mathbf{J}} + \hat{\mathbf{K}}\hat{\mathbf{I}}\hat{\mathbf{K}} \\ -\hat{\mathbf{T}}\hat{\mathbf{J}}\hat{\mathbf{T}} + \hat{\mathbf{I}}\hat{\mathbf{J}}\hat{\mathbf{I}} + \hat{\mathbf{J}}\hat{\mathbf{J}}\hat{\mathbf{J}} + \hat{\mathbf{K}}\hat{\mathbf{J}}\hat{\mathbf{K}} \\ -\hat{\mathbf{T}}\hat{\mathbf{K}}\hat{\mathbf{T}} + \hat{\mathbf{I}}\hat{\mathbf{K}}\hat{\mathbf{I}} + \hat{\mathbf{J}}\hat{\mathbf{K}}\hat{\mathbf{J}} + \hat{\mathbf{K}}\hat{\mathbf{K}}\hat{\mathbf{K}} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{T}} - \hat{\mathbf{T}} - \hat{\mathbf{T}} - \hat{\mathbf{T}} \\ \hat{\mathbf{I}} - \hat{\mathbf{I}} + \hat{\mathbf{I}} + \hat{\mathbf{I}} \\ \hat{\mathbf{J}} + \hat{\mathbf{J}} - \hat{\mathbf{J}} + \hat{\mathbf{J}} \\ \hat{\mathbf{K}} + \hat{\mathbf{K}} + \hat{\mathbf{K}} - \hat{\mathbf{K}} \end{bmatrix}, \tag{60}$$

with no norm-spin $(\hat{\mathbf{1}}, \boldsymbol{\sigma})$ product in the end result. The product of three four-vectors in this metric/geometry environment should produce a space-time four vector only, as is reflected in the Maxwell equations and the Lorentz Force Law. In other words, the multiplication of three minquats produces a pure minquat, not a pauliquat or a sum of a pauliquat and a minquat. Looking for magnetic monopoles as 'symmetric completion' of the Maxwell Equations and the Lorentz Force Law makes no sense in the metric/geometry developed in this paper because it implies looking for non-zero $(\hat{\mathbf{1}}, \boldsymbol{\sigma})$ results from $\mathbf{K}_{\mu}{}^{\nu}\mathbf{K}^{\mu}$. The metric $(\hat{\mathbf{T}}, \mathbf{K})$; $(\hat{\mathbf{1}}, \boldsymbol{\sigma})$ dimensionality analysis implies that only non-zero $(\hat{\mathbf{T}}, \mathbf{K})$ results are possible and that excludes magnetic monopole four forces and four currents.

3 The Dirac spin level

3.1 The Dirac-Weyl environment's disconnect to Minkowsky space-time

In the twenties of the previous century, the quadratic relativistic scalar Klein-Gordon wave equation could not be applied to the relativistic electron including (Pauli-) spin. In his search for a solution, Dirac linearized the Klein-Gordon equation by introducing four by four matrices as duplexes of the two by two Pauli matrices. In his two seminal 1928 papers he introduced the Clifford four set $(\beta, \boldsymbol{\alpha})$ and, using what were later called the gamma matrices, the covariant Clifford four set $(\beta, \boldsymbol{\gamma})$ [17], [18]. The Pauli matrices were incorporated in these matrices. Weyl later found another covariant Clifford four set, which relates to the Dirac covariant set as low velocity relativistic to high velocity relativistic gamma matrices Clifford four set [19]. Weyl explicitly discussed parity P and time-reversal T properties of these representations [20], [21, p. 109].

All these gamma or gamma-like matrices can be represented as two by two matrices of the biquaternion pauliquat basis $(\hat{\mathbf{1}}, \boldsymbol{\sigma})$. But using this biquaternion basis $(\hat{\mathbf{1}}, \boldsymbol{\sigma})$ as a basis for the gamma matrices makes it difficult to establish a connection between the gamma four-vector as a basis in an imaginary Dirac space-time and the Minkowsky space-time of special relativity. With the spinor wave objects as setting up an Hilbert space, the disconnect increases even more. The Born rule connecting the intensity of the waves absolute value to probabilities of outcomes in the real world in the laboratory successfully decreases the gap without entirely closing it.

Using result of the previous section, I can build gamma-equivalent matrices with a direct connection to Minkowskian space-time. It is my opinion that the $(\hat{\mathbf{T}}, \mathbf{K})$ biquaternion minquat basis will provide a solid foundation for connecting the Clifford four sets of Relativistic Quantum Mechanics to ordinary relativistic Minkowski space-time. This direct connection, through the construction of the Dirac environment from basic elements, will prove to be insightful and fruitful.

3.2 The Weyl matrices in dual minquat space-time mode as bèta matrices.

In my math-phys language and with a Möbius kind of doubling in mind I can define matrices through the application of parity or point reflection P and time reversal or present reflection T of the energy-momentum four vector P as

$$\begin{bmatrix} P & P \\ P^P & P^T \end{bmatrix} = \begin{bmatrix} P & P \\ -P^T & P^T \end{bmatrix} =$$
$$p_0 \begin{bmatrix} \hat{\mathbf{T}} & \hat{\mathbf{T}} \\ \hat{\mathbf{T}} & -\hat{\mathbf{T}} \end{bmatrix} + p_1 \begin{bmatrix} \hat{\mathbf{I}} & \hat{\mathbf{I}} \\ -\hat{\mathbf{I}} & \hat{\mathbf{I}} \end{bmatrix} + p_2 \begin{bmatrix} \hat{\mathbf{J}} & \hat{\mathbf{J}} \\ -\hat{\mathbf{J}} & \hat{\mathbf{J}} \end{bmatrix} + p_3 \begin{bmatrix} \hat{\mathbf{K}} & \hat{\mathbf{K}} \\ -\hat{\mathbf{K}} & \hat{\mathbf{K}} \end{bmatrix} =$$
$$p_0 \begin{bmatrix} \hat{\mathbf{T}} & \hat{\mathbf{T}} \\ \hat{\mathbf{T}} & -\hat{\mathbf{T}} \end{bmatrix} + \mathbf{p} \cdot \begin{bmatrix} \mathbf{K} & \mathbf{K} \\ -\mathbf{K} & \mathbf{K} \end{bmatrix} \tag{61}$$

The problem with this matrix is that it doesn't represent a Clifford four-set; it doesn't square to $2E^2 \mathbb{1}$.

I can split the quadruple of P into two duplexes $P_\mu \beta^\mu + P_\mu \xi^\mu$. The bèta's are defined through, constructed as, the parity duplex

$$\not{P} = P_\mu \beta^\mu = \begin{bmatrix} 0 & P \\ -P^T & 0 \end{bmatrix} = p_0 \begin{bmatrix} 0 & \hat{\mathbf{T}} \\ \hat{\mathbf{T}} & 0 \end{bmatrix} + \mathbf{p} \cdot \begin{bmatrix} 0 & \mathbf{K} \\ -\mathbf{K} & 0 \end{bmatrix} = p_0 \beta_0 + \mathbf{p} \cdot \boldsymbol{\beta} =$$
$$p_0 \begin{bmatrix} 0 & \hat{\mathbf{T}} \\ \hat{\mathbf{T}} & 0 \end{bmatrix} + p_1 \begin{bmatrix} 0 & \hat{\mathbf{I}} \\ -\hat{\mathbf{I}} & 0 \end{bmatrix} + p_2 \begin{bmatrix} 0 & \hat{\mathbf{J}} \\ -\hat{\mathbf{J}} & 0 \end{bmatrix} + p_3 \begin{bmatrix} 0 & \hat{\mathbf{K}} \\ -\hat{\mathbf{K}} & 0 \end{bmatrix} \tag{62}$$

with $\not{P} = P_\mu \beta^\mu = p_0 \beta_0 + p_1 \beta_1 + p_2 \beta_2 + p_3 \beta_3$.

The xi's are defined through, constructed as, the time reversed duplex

$$P_\mu \xi^\mu = \begin{bmatrix} P & 0 \\ 0 & P^T \end{bmatrix} = p_0 \begin{bmatrix} \hat{\mathbf{T}} & 0 \\ 0 & -\hat{\mathbf{T}} \end{bmatrix} + \mathbf{p} \cdot \begin{bmatrix} \mathbf{K} & 0 \\ 0 & \mathbf{K} \end{bmatrix} = p_0 \xi_0 + \mathbf{p} \cdot \boldsymbol{\xi} =$$
$$p_0 \begin{bmatrix} \hat{\mathbf{T}} & 0 \\ 0 & -\hat{\mathbf{T}} \end{bmatrix} + p_1 \begin{bmatrix} \hat{\mathbf{I}} & 0 \\ 0 & \hat{\mathbf{I}} \end{bmatrix} + p_2 \begin{bmatrix} \hat{\mathbf{J}} & 0 \\ 0 & \hat{\mathbf{J}} \end{bmatrix} + p_3 \begin{bmatrix} \hat{\mathbf{K}} & 0 \\ 0 & \hat{\mathbf{K}} \end{bmatrix} \tag{63}$$

with $P_\mu \xi^\mu = p_0\xi_0 + p_1\xi_1 + p_2\xi_2 + p_3\xi_3$.

The relation with the metric of the previous section is direct. In the β_μ space is mirrored, so the space-time double is obtained through the parity operation. In the ξ_μ time is reversed, so the space-time double is obtained through the T operation.

Of these two, only the β_μ matrices are a Clifford four set; only for them does the square of $\not{P}$ give the desired outcome as in $\not{P}\not{P} = -E\not{1}$. The β_μ matrices are the minquat equivalent of the Weyl-gamma matrices, the latter as based on doubling the pauliquat spin-norm set. If I use $\hat{\mathbf{T}} = \mathbf{i}\hat{\mathbf{1}}$ and $\mathbf{K} = \mathbf{i}\boldsymbol{\sigma}$, the result is

$$\beta_\mu = (\beta_0, \boldsymbol{\beta}) = \left(\begin{bmatrix} 0 & \mathbf{i}\hat{\mathbf{1}} \\ \mathbf{i}\hat{\mathbf{1}} & 0 \end{bmatrix}, \begin{bmatrix} 0 & \mathbf{i}\boldsymbol{\sigma} \\ -\mathbf{i}\boldsymbol{\sigma} & 0 \end{bmatrix} \right) = (\mathbf{i}\hat{\mathbf{1}}, \mathbf{i}\boldsymbol{\gamma}) = \mathbf{i}\gamma_\mu \tag{64}$$

which relates the parity dual β_μ to the Weyl gamma representation. The question then is how to represent the equivalent of the Dirac representation in the *beta* minquat environment.

3.3 From the Weyl and Dirac equations to the Dirac bèta matrices

The trick in finding Clifford four-sets is connected to the problem of the quadratics and to the problem of formulating equations in the Dirac environment. The quadratics of the energy-momentum four-vectors in the Clifford representation have to be reducible to the Klein Gordon energy condition $P^T P = E^2\hat{\mathbf{1}}$ with $E = \frac{U_0}{c} = m_0 c$. The Weyl bèta representation of $\not{P}$ matches this requirement. The ξ_μ representation doesn't.

In the Weyl and Dirac equations we can split $-E^2\not{1}$ using the ξ matrix, as $\not{E}^2 = (E\xi)^2 = -E^2\not{1}$, with the eigen time matrix ξ, defined as

$$\xi = \begin{bmatrix} \hat{\mathbf{T}} & 0 \\ 0 & \hat{\mathbf{T}} \end{bmatrix}. \tag{65}$$

The Weyl or chiral equation stems from the quadratic $\not{P}\not{P} = \not{E}\not{E}$ in the space-time Weyl representation.

$$\not{P}\not{P} = \begin{bmatrix} 0 & P \\ -P^T & 0 \end{bmatrix} \begin{bmatrix} 0 & P \\ -P^T & 0 \end{bmatrix} = \begin{bmatrix} -PP^T & 0 \\ 0 & -P^T P \end{bmatrix} =$$
$$\begin{bmatrix} -E^2\hat{\mathbf{1}} & 0 \\ 0 & -E^2\hat{\mathbf{1}} \end{bmatrix} = -E^2\not{1} = \not{E}\not{E} \tag{66}$$

So we have $\not{P}\not{P} - \not{E}\not{E} = 0$. This leads to $(\not{P} - \not{E})(\not{P} + \not{E}) = 0$. If we split this into two equations, $\not{P} - \not{E} = 0$ and $\not{P} + \not{E} = 0$, then only the trivial all zero solution is possible. As a consequence, the Weyl equations only apply to zero-restmass particles, like the neutrino's. If we add the Dirac spinors, then we get $\Psi^\dagger(\not{P} - \not{E})(\not{P} + \not{E})\Psi = 0$, which can be split into $\Psi^\dagger(\not{P} - \not{E}) = 0$ and $(\not{P} + \not{E})\Psi = 0$. By interpreting the spinors as waves or wave-like fields all the solutions of those equations can be interpreted as eigenvalue solutions of related operators and we get the Weyl wave equations as

$$\Psi^\dagger \hat{\not{P}} = \Psi^\dagger \not{E} \tag{67}$$

$$\hat{\not{P}}\Psi = -\not{E}\Psi \tag{68}$$

if we use $\hat{\not{P}} = -\mathbf{i}\hbar\not{\partial}$ and a four column dual spinor Ψ. When applied to zero restmass particles, this reduces to $\Psi^\dagger \hat{\not{P}} = 0$ and $\hat{\not{P}}\Psi = 0$.

The Dirac equation stems from the quadratic form $\not{P}\not{P} = (p_0\beta_0 + \mathbf{p}\cdot\boldsymbol{\beta})^2 = -E^2\not{1}$, as

$$\not{P}\not{P} = \begin{bmatrix} p_0\hat{\mathbf{T}} & \mathbf{p}\cdot\mathbf{K} \\ -\mathbf{p}\cdot\mathbf{K} & -p_0\hat{\mathbf{T}} \end{bmatrix} \begin{bmatrix} p_0\hat{\mathbf{T}} & \mathbf{p}\cdot\mathbf{K} \\ -\mathbf{p}\cdot\mathbf{K} & -p_0\hat{\mathbf{T}} \end{bmatrix} =$$

$$\begin{bmatrix} (-p_0^2 + \mathbf{p}^2)\hat{\mathbf{1}} & 0 \\ 0 & (-p_0^2 + \mathbf{p}^2)\hat{\mathbf{1}} \end{bmatrix} = -E^2\not{1}. \tag{69}$$

This leads to the two options for the Dirac equations

$$(\hat{p}_0\beta_0 + \hat{\mathbf{p}}\cdot\boldsymbol{\beta})\Psi = E\mathbf{i}\not{1}\Psi \tag{70}$$

$$\Psi^\dagger(\hat{p}_0\beta_0 + \hat{\mathbf{p}}\cdot\boldsymbol{\beta}) = -E\Psi^\dagger\mathbf{i}\not{1} \tag{71}$$

if we use $\hat{P} = -\mathbf{i}\hbar\partial$ and a four column spinor Ψ.

From this we can derive the Dirac bèta matrices, i.e. the bèta-matrices in the Dirac representation. The Dirac representation mixes the bèta and the xi representation and thus represents a PT dual. I nevertheless, using the gamma tradition, use the bèta and Feynman slash symbols for both representations in the time-space $(\hat{\mathbf{T}}, \mathbf{K})$ basis. This gives for the Dirac bèta representation

$$\not{P} = P_\mu\beta^\mu = p_0 \begin{bmatrix} \hat{\mathbf{T}} & 0 \\ 0 & -\hat{\mathbf{T}} \end{bmatrix} + \mathbf{p}\cdot \begin{bmatrix} 0 & \mathbf{K} \\ -\mathbf{K} & 0 \end{bmatrix} = p_0\beta_0 + \mathbf{p}\cdot\boldsymbol{\beta} =$$

$$p_0 \begin{bmatrix} \hat{\mathbf{T}} & 0 \\ 0 & -\hat{\mathbf{T}} \end{bmatrix} + p_1 \begin{bmatrix} 0 & \hat{\mathbf{I}} \\ -\hat{\mathbf{I}} & 0 \end{bmatrix} + p_2 \begin{bmatrix} 0 & \hat{\mathbf{J}} \\ -\hat{\mathbf{J}} & 0 \end{bmatrix} + p_3 \begin{bmatrix} 0 & \hat{\mathbf{K}} \\ -\hat{\mathbf{K}} & 0 \end{bmatrix}. \tag{72}$$

As with the Weyl representation, in the Dirac representation we have $\beta_\mu = \mathbf{i}\gamma_\mu$.

So in the space-time representation we have the Weyl $\not{P}$ as

$$\not{P}_w = \begin{bmatrix} 0 & P \\ -P^T & 0 \end{bmatrix} \tag{73}$$

and the Dirac $\not{P}$ as

$$\not{P}_d = \begin{bmatrix} p_0\hat{\mathbf{T}} & \mathbf{p}\cdot\mathbf{K} \\ -\mathbf{p}\cdot\mathbf{K} & -p_0\hat{\mathbf{T}} \end{bmatrix} \tag{74}$$

.

3.3.1 The transformation from the Dirac to the Weyl representation and vice versa

The transformation from the Weyl to the Dirac representation and vice versa is an operator that is usually written as S. Given the Weyl and Dirac bèta representations of Eqn.(73) and Eqn.(74), the transformation matrix can easily been found and equals one of the usual forms of S:

$$S = \frac{1}{\sqrt{2}}\begin{bmatrix} \hat{\mathbf{1}} & \hat{\mathbf{1}} \\ -\hat{\mathbf{1}} & \hat{\mathbf{1}} \end{bmatrix} \tag{75}$$

It has the property $\beta_0 S = S^{-1}\beta_0$ and the directly related $S\beta_0 = \beta_0 S^{-1}$.

The switch from the Weyl β_w^ν to the Dirac β_d^ν is then given by $\beta_d^\nu = S\beta_w^\nu S^{-1}$ and the switch from the Dirac to the Weyl representation by the inverse $\beta_w^\nu = S^{-1}\beta_d^\nu S$. We then also have the transformation $\not{P}_w = S^{-1}\not{P}_d S$ and $\not{P}_d = S\not{P}_w S^{-1}$.

3.4 Lorentz transformations of the vectors in the Dirac and Weyl representation environments

In the Pauli level part of this paper I developed the $(\hat{\mathbf{T}}, \mathbf{K})$ relativistic approach. This resulted in the Lorentz transformation of a four vector $P = (p_0\hat{\mathbf{T}}, \boldsymbol{p}\cdot\mathbf{K})$ as $P^L = U^{-1}PU^{-1}$ and the Lorentz transformation of its time reversal P^T as $(P^L)^T = (P^T)^{L^{-1}} = UP^TU$ with U as

$$U = \begin{bmatrix} e^{\frac{\psi}{2}} & 0 \\ 0 & e^{-\frac{\psi}{2}} \end{bmatrix} \tag{76}$$

and the rapidity ψ. The quadratic P^TP is a Lorentz invariant scalar $\frac{U_0^2}{c^2}\hat{\mathbf{1}} = E^2\hat{\mathbf{1}}$ with the dimension of the norm $\hat{\mathbf{1}}$. If in the space-time minquat β_μ representation we have the Weyl $\not{P}$ in a reference system S as

$$\not{P} = \begin{bmatrix} 0 & P \\ -P^T & 0 \end{bmatrix} \tag{77}$$

then in reference system S' we have P^L and so also the Weyl $\not{P}^L$ as

$$\not{P}^L = \begin{bmatrix} 0 & P^L \\ -(P^L)^T & 0 \end{bmatrix} = \begin{bmatrix} 0 & U^{-1}PU^{-1} \\ -UP^TU & 0 \end{bmatrix} \tag{78}$$

The question then is which matrix can generate this result. The obvious answer is

$$\not{P}_w^L = \Lambda \not{P}_w \Lambda^{-1} = \begin{bmatrix} U^{-1} & 0 \\ 0 & U \end{bmatrix} \begin{bmatrix} 0 & P \\ -P^T & 0 \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & U^{-1} \end{bmatrix} = \begin{bmatrix} 0 & U^{-1}PU^{-1} \\ -UP^TU & 0 \end{bmatrix} \quad (79)$$

with the Lorentz transformation matrix

$$\Lambda = \begin{bmatrix} U^{-1} & 0 \\ 0 & U \end{bmatrix} \quad (80)$$

and its inverse Λ^{-1}.

As for the generator of Λ, we have

$$\Lambda = \begin{bmatrix} U^{-1} & 0 \\ 0 & U \end{bmatrix} = \begin{bmatrix} e^{-\frac{\psi}{2}} & 0 & 0 & 0 \\ 0 & e^{\frac{\psi}{2}} & 0 & 0 \\ 0 & 0 & e^{\frac{\psi}{2}} & 0 \\ 0 & 0 & 0 & e^{-\frac{\psi}{2}} \end{bmatrix} =$$

$$\begin{bmatrix} \cosh\left(\frac{\psi}{2}\right) & 0 & 0 & 0 \\ 0 & \cosh\left(\frac{\psi}{2}\right) & 0 & 0 \\ 0 & 0 & \cosh\left(\frac{\psi}{2}\right) & 0 \\ 0 & 0 & 0 & \cosh\left(\frac{\psi}{2}\right) \end{bmatrix} +$$

$$\begin{bmatrix} -\sinh\left(\frac{\psi}{2}\right) & 0 & 0 & 0 \\ 0 & \sinh\left(\frac{\psi}{2}\right) & 0 & 0 \\ 0 & 0 & \sinh\left(\frac{\psi}{2}\right) & 0 \\ 0 & 0 & 0 & -\sinh\left(\frac{\psi}{2}\right) \end{bmatrix} =$$

$$\cosh\left(\frac{\psi}{2}\right) \begin{bmatrix} \hat{\mathbf{1}} & 0 \\ 0 & \hat{\mathbf{1}} \end{bmatrix} + \sinh\left(\frac{\psi}{2}\right) \begin{bmatrix} -\sigma_I & 0 \\ 0 & \sigma_I \end{bmatrix} =$$

$$\not{1} \cosh\left(\frac{\psi}{2}\right) + \alpha_I \sinh\left(\frac{\psi}{2}\right) = \not{1} e^{\alpha_I \left(\frac{\psi}{2}\right)} \quad (81)$$

with the α_I defined in the Weyl presentation as:

$$\alpha_I = \begin{bmatrix} -\sigma_I & 0 \\ 0 & \sigma_I \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \quad (82)$$

The inverse is then obviously given by $\Lambda^{-1} = \not{1} e^{-\alpha_I \left(\frac{\psi}{2}\right)}$.

The Klein Gordon energy-momentum condition's Lorentz invariance or covariance depends on the product $\not{p}^L\not{p}^L$. Using the previous result, we have for the Lorentz transformation of the product $\not{p}\not{p}$ in the Weyl representation

$$\begin{gathered}\not{p}^L\not{p}^L = \Lambda\not{p}\Lambda^{-1}\Lambda\not{p}\Lambda^{-1} = \Lambda\not{p}\not{p}\Lambda^{-1} = \\ \Lambda(-E^2\not{1})\Lambda^{-1} = -E^2\not{1}\Lambda\Lambda^{-1} = -E^2\not{1} = \not{p}\not{p},\end{gathered} \tag{83}$$

so a Lorentz invariant product. This ensures the Lorentz invariance of the Klein Gordon energy-momentum condition $\not{p}\not{p} = \not{E}\not{E}$ in the Weyl representation.

In the Dirac version, where $\not{p} = p_0\beta_0 + \mathbf{p}\cdot\boldsymbol{\beta}$, things get more complicated. We have to start with the Dirac $\not{p}_d$ in the primary reference system and we want to end up with $\not{p}_d^L$ in the secondary reference system. We know how to transform between the Dirac and the Weyl representations and we know how to Lorentz transform the Weyl $\not{p}_w$. This means we have to go from Dirac to Weyl in the primary reference system, then Lorentz transform the Weyl four vector to the secondary reference system and then transform back from the Weyl to the Dirac representation, three operations in total. The total result gives

$$\not{p}_d^L = S\Lambda S^{-1}\not{p}_d S\Lambda^{-1}S^{-1}. \tag{84}$$

For the Klein Gordon equation in the Dirac representation, we get the Lorentz invariance through

$$\begin{aligned}\not{p}_d^L\not{p}_d^L = S\Lambda S^{-1}\not{p}_d S\Lambda^{-1}S^{-1}S\Lambda S^{-1}\not{p}_d S\Lambda^{-1}S^{-1} &= \\ S\Lambda S^{-1}\not{p}_d S\Lambda^{-1}\Lambda S^{-1}\not{p}_d S\Lambda^{-1}S^{-1} &= \\ S\Lambda S^{-1}\not{p}_d S S^{-1}\not{p}_d S\Lambda^{-1}S^{-1} &= \\ S\Lambda S^{-1}\not{p}_d\not{p}_d S\Lambda^{-1}S^{-1} &= \\ S\Lambda S^{-1}(-E^2\not{1})S\Lambda^{-1}S^{-1} &= \\ -E^2\not{1}S\Lambda S^{-1}S\Lambda^{-1}S^{-1} &= \\ -E^2\not{1}S\Lambda\Lambda^{-1}S^{-1} &= \\ -E^2\not{1}SS^{-1} &= \\ -E^2\not{1} &= \\ \not{p}_d\not{p}_d&.\end{aligned} \tag{85}$$

In details, with rapidity ψ, the operator $S\Lambda S^{-1}$ is given as

$$S\Lambda S^{-1} = \begin{bmatrix}\cosh(\frac{\psi}{2})\hat{\mathbf{1}} & \sinh(\frac{\psi}{2})\sigma_I \\ \sinh(\frac{\psi}{2})\sigma_I & \cosh(\frac{\psi}{2}\hat{\mathbf{1}}\end{bmatrix} = \not{1}\cosh(\frac{\psi}{2}) + \alpha_I\sinh(\frac{\psi}{2}) = \not{1}e^{\left(\alpha_I\frac{\psi}{2}\right)}, \tag{86}$$

with $\not{1} e^{\left(\alpha_I \frac{\psi}{2}\right)}$ as the generator of the Lorentz boost. The α_I in the Dirac representation is defined as as:

$$\alpha_I = \begin{bmatrix} 0 & \sigma_I \\ \sigma_I & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}. \tag{87}$$

The operator $S\Lambda^{-1}S^{-1}$ is given as

$$S\Lambda^{-1}S^{-1} = \begin{bmatrix} \cosh(\frac{\psi}{2})\hat{\mathbf{1}} & -\sinh(\frac{\psi}{2})\sigma_I \\ -\sinh(\frac{\psi}{2})\sigma_I & \cosh(\frac{\psi}{2}\hat{\mathbf{1}} \end{bmatrix} = \\ \not{1}\cosh(\frac{\psi}{2}) - \alpha_I \sinh(\frac{\psi}{2}) = \not{1} e^{-\left(\alpha_I \frac{\psi}{2}\right)}. \tag{88}$$

In the transformation of the four vector we have $\not{P}_d = P_\mu \beta^\mu$. Because the operators only work on the matrix aspect of each of the elements of β^μ, the Lorentz transformation can also be written as

$$\not{P}^L = e^{\left(\alpha_I \frac{\psi}{2}\right)} \not{P} e^{-\left(\alpha_I \frac{\psi}{2}\right)} = S\Lambda S^{-1} P_\mu \beta^\mu S\Lambda^{-1}S^{-1} = P_\mu S\Lambda S^{-1} \beta^\mu S\Lambda^{-1}S^{-1} \tag{89}$$

and we can focus on

$$(\beta^\mu)^L = S\Lambda S^{-1} \beta^\mu S\Lambda^{-1}S^{-1} \tag{90}$$

thus interpreting the Lorentz transformation as a boost of the dual minquat metric.

Using the Lorentz transformation expression of the operator combinations $S\Lambda S^{-1}$ and $S\Lambda^{-1}S^{-1}$ in terms of the rapidity and the hyperbolic trigonometric expressions, we can calculate the result on the bèta matrices of the $S\Lambda S^{-1}$ and $S\Lambda^{-1}S^{-1}$ operators. After some calculations this results in

$$(\beta^\mu)^L = S\Lambda S^{-1} \beta^\mu S\Lambda^{-1}S^{-1} = \Lambda_\nu{}^\mu \beta^\nu = \beta^\mu \tag{91}$$

with, given the usual Lorentz boost $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ and $\beta = \frac{v}{c}$,

$$(\beta^\mu)^L = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}^L = \Lambda_\nu^\mu \beta^\nu = \\ \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} \gamma\beta_0 - \beta\gamma\beta_1 \\ \gamma\beta_1 - \beta\gamma\beta_0 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \beta^\mu. \tag{92}$$

Independently from the previous direct calculations based on known Λ, S and $\Lambda_\nu{}^\mu$, the Lorentz transformation of P can also be presented as a transformation of the coordinates P_μ with a fixed metric K^μ, see the results of Eqn.(27). For the bèta dual this applies as well. Thus one either transforms the coordinates P_μ or one transforms the metric in β^μ, but not both. The end result will be the same. For the first we get

$$\not{P}^L = (P_\mu \beta^\mu)^L = (P_\mu)^L \beta^\mu = (P_\nu \Lambda_\mu{}^\nu)\beta^\mu = P'_\mu \beta^\mu. \tag{93}$$

But this can also be written as

$$\not{P}^L = (P_\nu \Lambda_\mu{}^\nu)\beta^\mu = P_\nu(\Lambda_\mu{}^\nu \beta^\mu) = P_\nu \beta'^\nu. \tag{94}$$

So we have $(\beta^\nu)^L = \Lambda_\mu{}^\nu \beta^\mu$ and we have, in the Dirac representation, $(\beta^\nu)^L = S\Lambda S^{-1}\beta^\nu S\Lambda^{-1}S^{-1}$, leading to

$$\Lambda_\mu{}^\nu \beta^\mu = S\Lambda S^{-1}\beta^\nu S\Lambda^{-1}S^{-1}. \tag{95}$$

In the space-time Weyl representation the results are the same, giving

$$\Lambda\beta^\mu\Lambda^{-1} = \Lambda_\mu{}^\nu\beta^\mu. \tag{96}$$

A remark is necessary: one has to keep track of the representation one is in, Weyl or Dirac, because the same $\Lambda_\mu{}^\nu$ and β^μ symbols are used but they aren't equal in the respective representations.

The ease of the Lorentz transformation and the proving of Lorentz covariance or invariance in the developed math-phys environment can be contrasted with the usual approach as critically analyzed and alternatively presented in [8]. The relation $S\Lambda S^{-1}\beta^\mu S\Lambda^{-1}S^{-1} = \Lambda_\mu{}^\nu\beta^\mu$ for the Dirac matrices in this paper has been constructed using already known matrices. I do not use this relationship as a starting point in the process of finding the operator $S\Lambda S^{-1}$, as is done in the literature. As I mentioned in the introduction, in [5, p. 147, Eqn. 5.102], the " S" is a black box, whereas in my approach I opened the box and found " S" $= S\Lambda^{-1}S^{-1}$, a relation that I constructed and then used to prove $S\Lambda S^{-1}\beta^\mu S\Lambda^{-1}S^{-1} = \Lambda_\mu{}^\nu\beta^\mu$ instead of assuming it first and solving it later. I do not assume and solve, I construct and prove instead. This was possible because of its connection to the Lorentz transformation approach in the biquaternion representation of the Pauli level physics.

My approach confirms the claim that the bèta matrices can transform like a 'regular' four-vector, but it also confirms the approach that the bèta matrices remain fixed during a Lorentz transformation. One has to realizes that in the Feynman $\not{P} = P_\mu\beta^\mu$ notation, the Lorentz transformation is either performed on P_μ with fixed β^μ or on β^μ with fixed P_μ: one either transforms the coordinates or one transforms the metric, but not both. Either the metric aspect of $\not{P}$ is Doppler twisted or the dynamic variables are, but not both.

3.5 Lorentz transformations of the spinors in the Weyl representation environments

The Lorentz transformation of the spinors is known to be half the Lorentz transformation of a four vector. In case of the Weyl bèta representation we have $\not{p}^L = \Lambda\not{p}\Lambda^{-1}$, so we expect to have either $\Psi^L = \Lambda\Psi$ or $\Psi^L = \Lambda^{-1}\Psi$.

Now, in physics, the Lorentz transformation of EM-waves represents a relativistic Doppler boost, represented by the factor $e^{\psi} = \gamma + \gamma\beta$ shifting the frequency and wavelength to the red or to the blue. Matter waves and the associated phenomena are duly called so because they exhibit wave phenomena as refraction and interference. So matter waves should have wave-fronts and crests and troughs and as such undergo the equivalent of Doppler shifts when observed from $v\hat{\mathbf{I}}$-boosted reference systems. But measurements always involve intensities, never pure waves, so the intensities should exhibit quantum Doppler shifts.

We further know that if the spinor Ψ represents a matter wave, then $\Theta = \not{p}\Psi$ also represents a matter wave and both should Lorentz transform identically. From this we can infer that $\Psi^L = \Lambda\Psi$, because then

$$\Theta^L = (\not{p}\Psi)^L = \not{p}^L\Psi^L = \Lambda\not{p}\Lambda^{-1}\Psi^L = \Lambda\not{p}\Lambda^{-1}\Lambda\Psi = \Lambda\not{p}\Psi = \Lambda\Theta \tag{97}$$

From $\Psi^L = \Lambda\Psi$ we can derive the relation

$$(\Psi^L)^\dagger = (\Lambda\Psi)^\dagger = \Psi^\dagger\Lambda, \tag{98}$$

due to the fact that Λ is diagonal real and thus equal to its conjugate transpose.

We then get for Lorentz transformation of the intensity $\Psi^\dagger\Psi$ of the matter wave Ψ

$$\begin{aligned}(\Psi^\dagger\Psi)^L = (\Psi^L)^\dagger(\Psi^L) = (\Lambda\Psi)^\dagger(\Lambda\Psi) = \Psi^\dagger\Lambda\Lambda\Psi = \Psi^\dagger\Lambda^2\Psi = \Psi^\dagger e^{\alpha_I\psi}\Psi = \\ \Psi^\dagger\Psi\cosh(\psi) + \Psi^\dagger\alpha_I\Psi\sinh(\psi) = \Psi^\dagger\Psi\gamma + \Psi^\dagger\alpha_I\Psi\gamma\beta,\end{aligned} \tag{99}$$

with alpha matrix α_I, Lorentz boost $\gamma = \cosh(\psi)$, $\gamma\beta = \sinh(\psi)$. This is the quantum equivalent of a Doppler boost with rapidity ψ, as should be expected for a wave phenomenon when observed from a moving reference system.

In the Weyl representation, boosting the probability density doesn't mix the spinors because we have a diagonal matrix in the Lorentz boost operator, as

$$\begin{aligned}(\Psi^\dagger\Psi)^L = \begin{bmatrix}\Psi_1^* & \Psi_2^* & \Psi_3^* & \Psi_4^*\end{bmatrix}\begin{bmatrix}\gamma-\gamma\beta & 0 & 0 & 0 \\ 0 & \gamma+\gamma\beta & 0 & 0 \\ 0 & 0 & \gamma+\gamma\beta & 0 \\ 0 & 0 & 0 & \gamma-\gamma\beta\end{bmatrix}\begin{bmatrix}\Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4\end{bmatrix} \\ = \gamma\Psi_1^*\Psi_1 - \gamma\beta\Psi_1^*\Psi_1 + \gamma\Psi_2^*\Psi_2 + \gamma\beta\Psi_2^*\Psi_2 \\ +\gamma\Psi_3^*\Psi_3 + \gamma\beta\Psi_3^*\Psi_3 + \gamma\Psi_4^*\Psi_4 - \gamma\beta\Psi_4^*\Psi_4 = \\ \Psi_1^*\Psi_1 e^{-\psi} + \Psi_2^*\Psi_2 e^{\psi} + \Psi_3^*\Psi_3 e^{\psi} + \Psi_4^*\Psi_4 e^{-\psi}.\end{aligned} \tag{100}$$

The factor $\gamma \pm \gamma\beta = e^{\pm\psi}$ represents a relativistic wavelenght/frequency Doppler shift of the intensities.

In Wave Mechanics, the equations are wave equations and the Lagrangians are the intensities of those waves. The Klein-Gordon energy-momentum condition is Lorentz Invariant, the linearized Dirac equation transforms like a wave Ψ and the Lagrangian wave intensity derived from that equation transforms Doppler like with a factor $e^{\alpha_I \psi}$.

The conditon $\Psi^L = \Lambda\Psi$ gives

$$\Psi_w^L = \Lambda\Psi_w = \begin{bmatrix} U^{-1} & 0 \\ 0 & U \end{bmatrix} \begin{bmatrix} \Psi_w^1 \\ \Psi_w^2 \end{bmatrix} = \begin{bmatrix} U^{-1}\Psi_w^1 \\ U\Psi_w^2 \end{bmatrix}. \tag{101}$$

Important in this last equation is the result that the bispinors Ψ^1 and Ψ^2 do not mix in the Lorentz transformation in the space-time Weyl representation.

3.6 Lorentz transformations of the spinors in the Dirac representation environments

The same line of reasoning will give us the Lorentz transformation rules for the spinors in the space-time Dirac representation, respectively

$$\Psi_d^L = S\Lambda S^{-1}\Psi_d \tag{102}$$

and

$$(\Psi^L)_d^\dagger = (\Psi_d^\dagger)S\Lambda S^{-1}. \tag{103}$$

For the intensities we then get

$$(\Psi_d^\dagger \Psi_d)^L = (\Psi^L)_d^\dagger \Psi_d^L = (\Psi_d^\dagger)S\Lambda S^{-1}S\Lambda S^{-1}\Psi_d = (\Psi_d^\dagger)S\Lambda^2 S^{-1}\Psi_d. \tag{104}$$

In the Dirac representation, we have to calculate $S\Lambda^2 S^{-1}$ in order to be able to evaluate the result. In details, with rapidity ψ, the operator $S\Lambda^2 S^{-1}$ is given as

$$S\Lambda^2 S^{-1} = \begin{bmatrix} \cosh(\psi)\hat{\mathbf{1}} & \sinh(\psi)\sigma_I \\ \sinh(\psi)\sigma_I & \cosh(\psi)\hat{\mathbf{1}} \end{bmatrix} = \mathbb{1}\cosh(\psi) + \alpha_I \sinh(\psi) = \mathbb{1} e^{(\alpha_I \psi)}, \tag{105}$$

with $\mathbb{1} e^{(\alpha_I \psi)}$ as the generator of the Lorentz boost delivered Doppler shift of the probability/field density, as

$$(\Psi^\dagger \Psi)^L = \Psi^\dagger e^{(\alpha_I \psi)} \Psi = \Psi^\dagger \Psi \cosh(\psi) + \Psi^\dagger \alpha_I \Psi \sinh(\psi). \tag{106}$$

The operator $S\Lambda S^{-1}$ for the Lorentz transformation of the Dirac spinor Ψ exactly matches the one in [22].

Zooming in further and using $\cosh(\psi) = \gamma$ and $\sinh(\psi) = \gamma\beta$, we get for the Dirac representation

$$(\Psi^\dagger\Psi)^L = \begin{bmatrix} \Psi_1^* & \Psi_2^* & \Psi_3^* & \Psi_4^* \end{bmatrix} \begin{bmatrix} \gamma & 0 & \gamma\beta & 0 \\ 0 & \gamma & 0 & -\gamma\beta \\ \gamma\beta & 0 & \gamma & 0 \\ 0 & -\gamma\beta & 0 & \gamma \end{bmatrix} \begin{bmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \end{bmatrix}$$
$$= \gamma\Psi_1^*\Psi_1 + \gamma\beta\Psi_1^*\Psi_3 + \gamma\Psi_2^*\Psi_2 - \gamma\beta\Psi_2^*\Psi_4$$
$$+\gamma\Psi_3^*\Psi_3 + \gamma\beta\Psi_3^*\Psi_1 + \gamma\Psi_4^*\Psi_4 - \gamma\beta\Psi_4^*\Psi_2. \tag{107}$$

We see that in the Dirac representation, boosting the probability density mixes the spinors and thus the particles and the anti-particles, the electrons and the positrons.

The structure of these transformations look familiar. If we define $\gamma' = \cosh(\frac{\psi}{2})$ and $\gamma'\beta' = \sinh(\frac{\psi}{2})$, we get the Lorentz transformation of Ψ as

$$\Psi^L = \begin{bmatrix} \gamma'\hat{\mathbf{1}} & \gamma'\beta'\sigma_I \\ \gamma'\beta'\sigma_I & \gamma'\hat{\mathbf{1}} \end{bmatrix} \begin{bmatrix} \Psi^1 \\ \Psi^2 \end{bmatrix} = \begin{bmatrix} \gamma'\hat{\mathbf{1}}\Psi^1 + \gamma'\beta'\sigma_I\Psi^2 \\ \gamma'\hat{\mathbf{1}}\Psi^2 + \gamma'\beta'\sigma_I\Psi^1 \end{bmatrix}. \tag{108}$$

In the hyperbolic formulation, the details of the Lorentz transformation of Ψ gives

$$\Psi^L = \begin{bmatrix} (\Psi^1)^L \\ (\Psi^2)^L \end{bmatrix} = \begin{bmatrix} \cosh(\frac{\psi}{2})\hat{\mathbf{1}} & \sinh(\frac{\psi}{2})\sigma_I \\ \sinh(\frac{\psi}{2})\sigma_I & \cosh(\frac{\psi}{2}\hat{\mathbf{1}} \end{bmatrix} \begin{bmatrix} \Psi^1 \\ \Psi^2 \end{bmatrix} =$$
$$\begin{bmatrix} \cosh(\frac{\psi}{2})\hat{\mathbf{1}}\Psi^1 + \sinh(\frac{\psi}{2})\sigma_I\Psi^2 \\ \cosh(\frac{\psi}{2})\hat{\mathbf{1}}\Psi^2 + \sinh(\frac{\psi}{2})\sigma_I\Psi^1 \end{bmatrix}. \tag{109}$$

What we see here is that the Lorentz transformation of the Dirac spinor mixes the two twin Pauli spinors Ψ^1 and Ψ^2. As a consequence, one cannot Lorentz transform a single Pauli spinor in the Dirac representation, so a Lorentz transformation of the Pauli equation without the full Dirac twin is impossible. The Pauli equation on its own cannot possibly be relativistic, not because of the Pauli spin matrices, as is usually thought [23], but due to the spinors involved.

3.7 Connecting the beta-matrices to the gamma-matrices and to the Dirac alpha and spin matrices

My reversed order of the Pauli spin matrices, with $\sigma_I = \sigma_z$, $\sigma_J = \sigma_y$, $\sigma_K = \sigma_x$ and $\boldsymbol{\sigma} = (\sigma_I, \sigma_J, \sigma_K)$ implies that the usual (x, y, z) order of the gamma matrices are reversed correspondingly, with $\gamma_1 = \gamma_I = \gamma_z$, $\gamma_2 = \gamma_J = \gamma_y$, $\gamma_3 = \gamma_K = \gamma_x$ and $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3) = (\gamma_I, \gamma_J, \gamma_K))$.

The set of gamma matrices in the Dirac representation, $\gamma_\mu = (\beta, \boldsymbol{\gamma}) = (\gamma_0, \boldsymbol{\gamma})$, can then be defined as

$$\gamma_\mu = (\beta, \boldsymbol{\gamma}) = (\gamma_0, \boldsymbol{\gamma}) = \left(\begin{bmatrix} \hat{\mathbf{1}} & 0 \\ 0 & -\hat{\mathbf{1}} \end{bmatrix}, \begin{bmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{bmatrix} \right) \tag{110}$$

The set of gamma matrices in the Weyl representation, $\gamma_\mu = (\gamma_0, \boldsymbol{\gamma})$, can be defined as

$$\gamma_\mu = (\gamma_0, \boldsymbol{\gamma}) = \left(\begin{bmatrix} 0 & \hat{\mathbf{1}} \\ \hat{\mathbf{1}} & 0 \end{bmatrix}, \begin{bmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{bmatrix} \right) \tag{111}$$

In my $(\hat{\mathbf{1}}, \boldsymbol{\sigma})$ norm-spin basis the Dirac set $\alpha_\mu = (\not{1}, \boldsymbol{\alpha})$ can be represented as

$$\alpha_\mu = (\not{1}, \boldsymbol{\alpha}) = \left(\begin{bmatrix} \hat{\mathbf{1}} & 0 \\ 0 & \hat{\mathbf{1}} \end{bmatrix}, \begin{bmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{bmatrix} \right). \tag{112}$$

The most straightforward doubling of the Pauli level norm-spin set $(\hat{\mathbf{1}}, \boldsymbol{\sigma})$ is the Dirac level norm-spin set $\Sigma_\mu = (\not{1}, \boldsymbol{\Sigma})$ defined as

$$\Sigma_\mu = (\not{1}, \boldsymbol{\Sigma}) = \left(\begin{bmatrix} \hat{\mathbf{1}} & 0 \\ 0 & \hat{\mathbf{1}} \end{bmatrix}, \begin{bmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{bmatrix} \right). \tag{113}$$

The tensor $-\beta_\mu \beta^\nu = -\mathbf{i}\gamma_\mu \mathbf{i}\gamma^\nu = \gamma_\mu \gamma^\nu$ is given by

$$-\beta_\mu \beta^\nu = \gamma_\mu \gamma^\nu = \begin{bmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} \begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix} =$$

$$\begin{bmatrix} \gamma_0\gamma_0 & \gamma_1\gamma_0 & \gamma_2\gamma_0 & \gamma_3\gamma_0 \\ \gamma_0\gamma_1 & \gamma_1\gamma_1 & \gamma_2\gamma_1 & \gamma_3\gamma_1 \\ \gamma_0\gamma_2 & \gamma_1\gamma_2 & \gamma_2\gamma_2 & \gamma_3\gamma_2 \\ \gamma_0\gamma_3 & \gamma_1\gamma_3 & \gamma_2\gamma_3 & \gamma_3\gamma_3 \end{bmatrix} = \begin{bmatrix} \not{1} & -\alpha_1 & -\alpha_2 & -\alpha_3 \\ \alpha_1 & -\not{1} & -\mathbf{i}\Sigma_3 & \mathbf{i}\Sigma_2 \\ \alpha_2 & \mathbf{i}\Sigma_3 & -\not{1} & -\mathbf{i}\Sigma_1 \\ \alpha_3 & -\mathbf{i}\Sigma_2 & \mathbf{i}\Sigma_1 & -\not{1} \end{bmatrix}. \tag{114}$$

Thus, the product $-\beta_\mu \beta^\nu$ firmly connects the minquat domain to the pauliquat domain on the Dirac level. The product of two Dirac level duplex minquats produces a mixture of a duplex minquat and a a duplex pauliquat, as was the case on the Pauli level.

3.8 Lorentz transformation of the EM field in the Weyl-Dirac environment

We can apply this to the product $\not{B} = \not{\partial}\not{A}$, which then results in

$$\partial\!\!\!/ A\!\!\!/ = \begin{bmatrix} -\frac{1}{c}\partial_t \hat{\mathbf{T}} & \nabla \cdot \mathbf{K} \\ -\nabla \cdot \mathbf{K} & \frac{1}{c}\partial_t \hat{\mathbf{T}} \end{bmatrix} \begin{bmatrix} \frac{1}{c}\phi \hat{\mathbf{T}} & \boldsymbol{A} \cdot \mathbf{K} \\ -\boldsymbol{A} \cdot \mathbf{K} & -\frac{1}{c}\phi \hat{\mathbf{T}} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{21} \\ B_{21} & B_{22} \end{bmatrix} =$$
$$\begin{bmatrix} -\boldsymbol{B} \cdot \mathbf{K} & \frac{1}{c}\boldsymbol{E} \cdot \hat{\mathbf{T}}\mathbf{K} \\ \frac{1}{c}\boldsymbol{E} \cdot \hat{\mathbf{T}}\mathbf{K} & -\boldsymbol{B} \cdot \mathbf{K} \end{bmatrix} = -\mathbf{i}\boldsymbol{B} \cdot \begin{bmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{bmatrix} - \frac{1}{c}\boldsymbol{E} \cdot \begin{bmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{bmatrix} =$$
$$-\mathbf{i}\boldsymbol{B} \cdot \boldsymbol{\Sigma} - \frac{1}{c}\boldsymbol{E} \cdot \boldsymbol{\alpha} = -\frac{1}{c}(\mathbf{E} \cdot \boldsymbol{\alpha} + \mathbf{i}c\mathbf{B} \cdot \boldsymbol{\Sigma}). \tag{115}$$

with

$$B_{11} = B_{22} = (\frac{1}{c^2}\partial_t \phi + \nabla \cdot \boldsymbol{A})\hat{\mathbf{I}} - (\nabla \times \boldsymbol{A}) \cdot \mathbf{K} \tag{116}$$

and

$$B_{12} = B_{21} = -\frac{1}{c}\partial_t \boldsymbol{A} \cdot \hat{\mathbf{T}}\mathbf{K} - \frac{1}{c}\nabla\phi \cdot \hat{\mathbf{T}}\mathbf{K}. \tag{117}$$

The end result $B\!\!\!/ = -\frac{1}{c}(\mathbf{E} \cdot \boldsymbol{\alpha} + \mathbf{i}c\mathbf{B} \cdot \boldsymbol{\Sigma})$ applies for both the Weyl and the Dirac presentations and includes the use of the Lorenz gauge. In the perspective of the approach of this paper, this can be interpreted as a photon field – hypercomplex metric interaction product. The product is located in the pauliquat domain on the level of the Dirac-Weyl duplex of the Pauli space-time duplex. The product of two bèta matrices isn't a bèta matric but a metric-intrinsic 'polarization'-'spin' dual 'six-vector' like entity. As on the Pauli-level, the set of the PT-duplex minquat bèta matrices isn't a closed set for multiplications. Multiplication transports us from the space-time domain to the spin-norm domain in a double duplex way.

The Lorentz transformation of this product is straightforward. In the Weyl representation, we get

$$B\!\!\!/^L = \partial\!\!\!/^L A\!\!\!/^L = \Lambda\partial\!\!\!/\Lambda^{-1}\Lambda A\!\!\!/\Lambda^{-1} = \Lambda\partial\!\!\!/ A\!\!\!/\Lambda^{-1} = \Lambda B\!\!\!/\Lambda^{-1} =$$
$$-\frac{1}{c}(\mathbf{E} \cdot \Lambda\boldsymbol{\alpha}\Lambda^{-1} + \mathbf{i}c\mathbf{B} \cdot \Lambda\boldsymbol{\Sigma}\Lambda^{-1}). \tag{118}$$

In the Dirac representation the result will be

$$B\!\!\!/^L = S\Lambda S^{-1} B\!\!\!/_d S\Lambda^{-1} S^{-1} =$$
$$-\frac{1}{c}(\mathbf{E} \cdot S\Lambda S^{-1}\boldsymbol{\alpha} S\Lambda^{-1} S^{-1} + \mathbf{i}c\mathbf{B} \cdot S\Lambda S^{-1}\boldsymbol{\Sigma} S\Lambda^{-1} S^{-1}. \tag{119}$$

It is of course also possible to perform the Lorentz transformation on the coordinates of **E** and **B** and leave the Weyl-Dirac alpha (the intrinsic 'polarization' when enhanced by the Bohr magneton) and Weyl-Dirac Sigma (the intrinsic 'magnetization' when enhanced by the Bohr magneton) unaltered. The result will be that a Lorentz transformation mixes the alpha and the Sigma, or, alternatively, that it mixes the electric and magnetic fields.

3.9 Lorentz transformation, Dirac adjoint and Dirac probability current

The Dirac adjoint is defined as

$$\overline{\Psi} = \Psi^{\dagger}\gamma_0 = -\mathbf{i}\Psi^{\dagger}\beta_0. \tag{120}$$

The Lorentz transformation properties of the Dirac adjoint are problematic, to say the least. In the Weyl representation, $\Psi^{\dagger}$ Lorentz transforms as $(\Psi^L)^{\dagger} = \Psi^{\dagger}\Lambda$. The Lorentz transformation of the Dirac adjoint, $\overline{\Psi}^L$ should then be

$$\overline{\Psi}^L = \left(\Psi^{\dagger}\gamma_0\right)^L = \Psi^{\dagger}\Lambda\gamma_0^L. \tag{121}$$

But the transformation properties of $\gamma_0 = -\mathbf{i}\beta_0$ depend on many contextual circumstances. As part of a Feynman slash vector $\not{A}$, one can treat the matrices as fixed during a Lorentz transformation under the condition that the coordinates undergo the Lorentz transformation. So it is allowed to treat $\gamma_0 = -\mathbf{i}\beta_0$ as a Lorentz 'scalar' and keep it fixed, but not at all times and unconditionally. In my approach, γ_0 is not unconditionally a reference frame independent fixed matrix, it's reference frame independence is contextual.

The Dirac adjoint is used to get the products $\overline{\Psi}\gamma_0\Psi$ and $J^{\nu} = \overline{\Psi}\gamma^{\nu}\Psi = \Psi^{\dagger}\not{1}\Psi + \Psi^{\dagger}\boldsymbol{\alpha}\Psi$. If we look at the probability density tensor $\Phi_{\mu}{}^{\nu} = \Psi^{\dagger}\gamma_{\mu}\gamma^{\nu}\Psi =$, we recognize the elements of the Dirac probability current J^{ν} in this tensor:

$$\Phi_{\mu}{}^{\nu} = \Psi^{\dagger}\gamma_{\mu}\gamma^{\nu}\Psi = \begin{bmatrix} \Psi^{\dagger}\not{1}\Psi & -\Psi^{\dagger}\alpha_1\Psi & -\Psi^{\dagger}\alpha_2\Psi & -\Psi^{\dagger}\alpha_3\Psi \\ \Psi^{\dagger}\alpha_1\Psi & -\Psi^{\dagger}\not{1}\Psi & -\Psi^{\dagger}\mathbf{i}\Sigma_3\Psi & \Psi^{\dagger}\mathbf{i}\Sigma_2\Psi \\ \Psi^{\dagger}\alpha_2\Psi & \Psi^{\dagger}\mathbf{i}\Sigma_3\Psi & -\Psi^{\dagger}\not{1}\Psi & -\Psi^{\dagger}\mathbf{i}\Sigma_1\Psi \\ \Psi^{\dagger}\alpha_3\Psi & -\Psi^{\dagger}\mathbf{i}\Sigma_2\Psi & \Psi^{\dagger}\mathbf{i}\Sigma_1\Psi & -\Psi^{\dagger}\not{1}\Psi \end{bmatrix}. \tag{122}$$

This tensor has transparent Lorentz transformation properties. If the coordinates, eventually to be attached to this tensor, are kept fixed, then the gamma-matrices can be Lorentz transformed and we get for the tensor, in the Dirac representation and using the bèta matrices:

$$\begin{aligned}\left(\Phi_{\mu}{}^{\nu}\right)^L &= \left(\Psi^L\right)^{\dagger}(\beta_{\mu})^L(\beta^{\nu})^L(\Psi)^L = \left(\Psi^{\dagger}S\Lambda S^{-1}\right)\left(S\Lambda S^{-1}\beta_{\mu}S\Lambda^{-1}S^{-1}\right) \\ &\left(S\Lambda S^{-1}\beta^{\nu}S\Lambda^{-1}S^{-1}\right)\left(S\Lambda S^{-1}\Psi\right) = \Psi^{\dagger}S\Lambda\Lambda S^{-1}\beta_{\mu}{}^{\nu}\Psi = \Psi^{\dagger}e^{\alpha_I\psi}\beta_{\mu}{}^{\nu}\Psi = \\ &\Phi_{\mu}{}^{\nu}\cosh(\psi) + \Psi^{\dagger}\alpha_I\beta_{\mu}{}^{\nu}\Psi\sinh(\psi) = \gamma\Phi_{\mu}{}^{\nu} + \gamma\beta\Psi^{\dagger}\alpha_I\beta_{\mu}{}^{\nu}\Psi.\end{aligned} \tag{123}$$

In $\left(\Phi_{\mu}{}^{\nu}\right)^L = \gamma\left(\Phi_{\mu}{}^{\nu} + \beta\Psi^{\dagger}\alpha_I\beta_{\mu}{}^{\nu}\Psi\right)$, the γ represents the scalar velocity clock-effect and the β represents the geometric $\hat{\mathbf{I}}$-direction dependence of the relativistic Doppler shift. The result represents a quantum-relativistic Doppler shift with Doppler shift factor $e^{\alpha_I \psi}$. From the tensor $\Phi_\mu{}^\nu$ the Dirac probability density current and probability density scalar can be derived.

The Dirac current can be arrived at by using the coordinate velocity's rest system coordinates as V^ν to get

$$J^\nu = \Phi_\mu{}^\nu V^\mu =$$

$$\begin{bmatrix} \Psi^\dagger \mathbb{1}\Psi & -\Psi^\dagger\alpha_1\Psi & -\Psi^\dagger\alpha_2\Psi & -\Psi^\dagger\alpha_3\Psi \\ \Psi^\dagger\alpha_1\Psi & -\Psi^\dagger\mathbb{1}\Psi & -\Psi^\dagger\mathbf{i}\Sigma_3\Psi & \Psi^\dagger\mathbf{i}\Sigma_2\Psi \\ \Psi^\dagger\alpha_2\Psi & \Psi^\dagger\mathbf{i}\Sigma_3\Psi & -\Psi^\dagger\mathbb{1}\Psi & -\Psi^\dagger\mathbf{i}\Sigma_1\Psi \\ \Psi^\dagger\alpha_3\Psi & -\Psi^\dagger\mathbf{i}\Sigma_2\Psi & \Psi^\dagger\mathbf{i}\Sigma_1\Psi & -\Psi^\dagger\mathbb{1}\Psi \end{bmatrix} \begin{bmatrix} c \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} c\Psi^\dagger\mathbb{1}\Psi \\ c\Psi^\dagger\alpha_1\Psi \\ c\Psi^\dagger\alpha_2\Psi \\ c\Psi^\dagger\alpha_3\Psi \end{bmatrix}. \tag{124}$$

But this object isn't complete because it involves two β_μ's and only one set of coordinates, in this case V^μ. But it does indicate that it is problematic to interpret the Dirac probability density current as a real current, moving through space-time, see [2, p. 24].

The point to make here is to show that the Dirac adjoint and the Dirac probability current can be seen as part of the more general tensor product with at its core the product $\beta_\mu\beta^\nu$. In my opinion, the Dirac probability current should be interpreted as part of a tensor, not as a stand alone four vector. If $\beta_\mu\beta^\nu$ would be at the origin of the adjoint $\overline{\Psi} = \Psi^\dagger\gamma_0$, then the probability density tensor $\Phi_\mu{}^\nu = \Psi^\dagger\beta_\mu\beta^\nu\Psi$ would be thé quantum-relativistic object to study in this context. This tensor has a transparent Lorentz transformation. The continuity equation for the Dirac probability density current would then be derived from the more general closed system condition for this probability density tensor

$$\partial_\nu\Phi_\mu{}^\nu = \partial_\nu\Psi^\dagger\gamma_\mu\gamma^\nu\Psi = 0. \tag{125}$$

The closed system condition is going back to von Laue [24]. It might be interesting to further explore the relevance of this condition for the probability density of relativistic quantum mechanics.

4 Conclusion

In the Pauli-level part of this paper, I constructed a biquaternion space-time metric that matches the Minkowski space-time requirements and at the same time contains the Pauli spin environment. The space-time metric K_μ and its spin-norm dual σ_μ are not inert like the original Minkowski metric η_μ but can be internally Doppler twisted. Dynamic vectors in this space-time are $P_\mu K^\mu$ products. The connection between the Minkowski space-time metric η_μ and the biquaternion space-time metric K_μ can be made but isn't straightforward. It is unclear for example in what dimension one should imagine K_μ's spin-norm dual σ_μ.

Nevertheless was it possible to construct the Weyl-Dirac gamma-matrices environment from the biquaternion Pauli-level building blocks. To arrive at the the Dirac environment I added the PT dual to both the space-time metric K_μ and its spin-norm dual σ_μ. This construction from basic elements with known space-time connection allowed me to construct the Lorentz transformation operators on the Weyl-Dirac level.

This connection also allowed me to critically assess the confusing situation regarding the Lorentz transformation properties of the Dirac matrices. In my opinion, the Feynman slash objects are the Weyl-Dirac analogues of the Pauli-level $P_\mu K^\mu$ dynamic vectors, combinations of four-sets of dynamic variables and a space-time metric. In Lorentz transformations of these dynamic vectors, one either Lorentz transforms the metric object K^μ or the dynamic variables P_μ but not both.

In my opinion, the Weyl-Dirac matrices contain the space-time Minkowski metric, but in a complicated way. The gamma matrices γ_μ are not to be interpreted as ordinary dynamic four-vectors related to real particles or fields, but as objects with a complicated relationship with the space-time metric of Special Relativity. According to my analysis, the Dirac adjoint contains the time element present in $\mathbf{i}\gamma_0$, but also highly indirect, making its Lorentz transformation properties extremely complicated. Treating the Dirac adjoint's importance for physics as originating in the probability density tensor seems advisable.

The results of this paper might be useful for those teaching Relativistic Quantum Mechanics. Not for direct use in college teaching, but as background knowledge regarding the confusing situation around the roots of RQM. RQM is a logical positivist, Copenhagen Interpretation product, bringing order to the experimental spectroscopic data of the first half of the twentieth century. In this bringing order in the available spectroscopic data, connecting the mathematical-experimental order to physical reality beyond the Born stochastic rule was not within the reach of the theorists of the Copenhagen School. This disconnect still haunt us, especially when teaching RQM to students who expect to learn more about nature itself. Reducing this disconnect might be a positive contribution of this paper.

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