Finitist Results Concerning Physics

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This paper uses transfinite ordinals to prove the distance between any two given points and the interval of time between any two given instants can only be finite, and that, under certain conditions, the number of events between any two events is always finite. It also proves a contradiction involving the actual infinity hypothesis on which the spacetime continuum is grounded. The alternative of a discrete spacetime is then considered, and the consideration leads, via Pythagoras digital theorem, to the conclusion that the factor for converting between continuous and digital geometries is the relativistic Lorentz factor if length is replaced with the product of speed and time in a isotropic space. These finitist results suggest the convenience to consider the possibility of a digital interpretation of special relativity.

1. Actual and Potential infinity

According to the hypothesis of the actual infinity the elements of an infinite collection exist all at once, *in the act*, as a complete totality. Subsumed into the Axiom of Infinity, this hypothesis is one of the pillars supporting the mainstream of contemporary mathematics. According to G. Cantor, Platonism (even theo-Platonism [22]) is behind the concept of the actual infinity:

> . . . in my opinion the absolute reality and legality of the natural numbers is much higher than that of the sensory world. This is so because of a unique and very simple reason, namely, that natural numbers exist in the highest degree of reality, both separately and collectively in their actual infinitude, in the form of eternal ideas in Intellectus Divinus. ([15]; reference and (Spanish) text in [9])

> . . . I am only an instrument of a higher power, which will continue to work after me in the same way as it manifested itself thousands of years ago in Euclid and Archimedes . . . ([5, pp 104-105])

With such convictions, "as firm as a rock" [6, p. 298], Cantor did not need additional hypotheses to found his theory of transfinite numbers. He simply took it for granted that all finite cardinals exist as a complete totality: [4, pp 103-104]:

> The first example of a transfinite aggregate is given by the totality of finite cardinal numbers v; we call its cardinal number Aleph-zero and denote it by $\aleph_o$; thus we define $\aleph_o = \{\bar{\bar{v}}\}$.

where $\{\bar{\bar{v}}\}$ is Cantor's notation for the cardinal of the set $\{v\}$ of all finite cardinals ($|\mathbb{N}|$ in modern notation). According to Cantor, the list of natural numbers exists as a complete totality despite the fact that no last natural number completes the list. To emphasize this sense of completeness, consider the task of counting the successive natural numbers 1, 2, 3,. . . In agreement with the hypothesis of the actual infinity we could count *all* natural numbers in any finite interval of time by performing the following supertask (an infinite sequence of actions carried out in a finite interval of time [12]):

> Count each of the successive natural numbers $1, 2, 3...$ at each of the successive instants t_1, t_2, t_3... of a strictly increasing sequence of instants $\langle t_i \rangle$ within the finite real interval (t_a, t_b), being t_b the limit of the sequence.

In these conditions, at t_b *all* natural numbers would have been counted. *All*. But the fact of pairing the elements of two infinite sequences (of natural numbers and of instants) does not prove both sequences exist as complete totalities. Both paired sequences could also be potentially infinite. Indeed, there is an alternative to the actual infinity hypothesis: the hypothesis of the potential infinity (more pragmatic than platonic), which rejects the existence of *complete* infinite totalities, and then the possibility of counting *all* natural numbers. From this perspective, the natural numbers result from the endless process of counting: it is always possible to count numbers greater than any given number. But it is impossible to complete the process of counting all of them, just because it is an endless process. So, the complete list of all natural numbers makes no sense. For this and other reasons, Bolzano, Dedekind and Cantor tried to prove the existence of actual infinities. Bolzano's proof goes as follow ([16, p 112]):

> One truth is the proposition that Plato was Greek. Call this p_1. But then there is another truth p_2, namely the proposition that p_1 is true [But then there is another truth p_3, namely the proposition that p_2 is true]. And so *ad infinitum*. Thus the set of truths is infinite.

But the existence of an endless sequence of inferences (p_1 is true, then p_2 is true, then p_3 is true, then . . .) does not prove the existence of a complete infinite totality of inferences. It only proves the existence of an endless (potentially infinite) sequence of inferences. Dedekind's proof is similar (taken from [16, p 113]):

> Given some arbitrary thought s_1, there is a separate thought s_2, namely that s_1 can be object of thought [there is a separate thought s_3, namely that s_2 can be object of thought]. And so ad infinitum. Thus the set of thoughts is infinite.

The above comment on Bolzano proof also applies to Dedekind's. Dedekind gave another proof a little more detailed, albeit with the same formal defect, based on his definition of infinite set [7, p. 112]. And finally, Cantor's proof: ([11, p 25], [16, p. 117]):

> Each potential infinite presupposes an actual infinity.

or ([3, p. 404] English translation [20, p. 3]):

> ... in truth the potential infinity has only a borrowed reality, insofar as a potentially infinite concept always points towards a logically prior actually infinite concept whose existence it depends on.

It is then clear why the existence of an (actual) infinite set had to be finally established by means of an axiom, the Axiom of Infinity. The following two sections make use of the subsequent infinitist mathematics in two opposite directions. Their respective conclusions could benefit physics.

2. On infinite distances, times and sequences of events

Transfinite ordinals are used here to prove a result relevant to physics: that distances, times and sequences of events can only be finite (the proof makes use of some basics of Euclidean geometry not referred to in the proof).

Proposition-*The length of a straight line with two endpoints is always finite. And the distance between any two given points is always finite.*

Proof.-Let A and B be any two points; AB the unique straight line joining them; and P_1 a point of AB at any finite distance AP_1 (length of AP_1) from A. Let $\mathbf{P}$ be the sequence of all successive points $P_1, P_2, P_3 \ldots$ of AB defined according to: $\forall P_{i \geq 1}$: iff $P_iB \geq AP_1$, take a point P_{i+1} in P_iB separated from P_i by a distance AP_1. Consider the closed segment $[Q, B]$ whose length is also AP_1. It holds: $\forall P_\alpha \in \mathbf{P}$ and $P_\alpha \in [A, Q]$, there will be a point $Q' \in [Q, B]$ such that $P_\alpha Q' \geq AP_1$ because $P_\alpha B \geq AP_1$. In consequence, there must be in $[Q, B]$ one point (and only one because $QB = AP_1$) P_ϕ of $\mathbf{P}$, otherwise $\mathbf{P}$ would not contain all points P_i of AB such that $P_{i-1}P_i = AP_1$, which is not the case. So, the sequence $\mathbf{P}$ has a last element P_ϕ. The endpoints A and B and the sequence $\mathbf{P}$ define in AB a sequence $\mathbf{S}$ of successive adjacent segments: $[A, P_1]$, $(P_1, P_2]$, $(P_2, P_3] \ldots (P_\phi B]$ of the same length AP_1 , except at most the last one $P_\phi B \leq AP_1$. In the ordering $\mathbf{O}$ of $\mathbf{S}$, there is a first element $[A, P_1]$; a last element $(P_\phi, B]$; each element $(P_i, P_{i+1}]$ has an immediate predecessor $(P_{i-1}, P_i]$ (or $[A, P_1]$), except $[A, P_1]$, and an immediate successor $(P_{i+1}, P_{i+2}]$ (or $(P_\phi, B]$), except $(P_\phi, B]$; no element exists between any two of its successive elements; and any non empty subsequence **S'** of $\mathbf{S}$, containing, for instance, $(P_v, P_{v+1}]$, will also contain an element that precedes in the ordering $\mathbf{O}$ of $\mathbf{S}$ all elements of **S'** except itself: one of the elements $[A, P_1]$, $(P_1, P_2]$, $(P_2, P_3] \ldots (P_v, P_{v+1}]$. Therefore, $\mathbf{S}$ is a well ordered sequence, to which an ordinal number can be assigned [4, p. 152]. In addition, $\mathbf{S}$ cannot be non-denumerable [2]. The ordinal of $\mathbf{S}$ cannot be the least transfinite ordinal ω because the sequences whose ordinal is ω (as the ω-ordered sequence of all finite ordinals 1, 2, 3,...) have not a last element, which is not the case of $\mathbf{S}$. So, if the ordinal of $\mathbf{S}$ were infinite, it would be greater than ω, in which case there would be a first element succeeding all elements $[A, P_1]$, $(P_1, P_2]$, $(P_2, P_3] \ldots$ indexed by the sequence of all finite ordinals 1, 2, 3,... which can only be the limit of all them $(P_\omega P_{\omega+1}]$ [4, Theorem I, p. 158] (P_{w+1} could be B). Take in AB a point R at any given distance from P_ω less than AP_1, and in the direction from P_ω to A. The point R could only belong to a segment $(P_v, P_\omega]$ immediately preceding $(P_\omega, P_{\omega+1}]$ (or $(P_\omega, B]$). But $(P_v, P_\omega]$ is impossible because there is not a last finite ordinal v whose immediate successor $v + 1$ is ω. Hence, the ordinal of $\mathbf{S}$ cannot be infinite but finite. $\mathbf{S}$ can only have a finite number of elements. And being finite the sum of any finite number of finite lengths, AB has a finite length. And the distance between any two given points (the length of the straight line joining them) is always finite. □

The same above argument proves the time elapsed between any two given instants can only be finite. And that the number of events between any two events is always finite if the interval of time between any two of them is equal or greater than any given finite interval of time.

3. A proof of inconsistency

The historical controversy between the potential and the actual infinity came to a practical end when set theory was formally established, subsuming the hypothesis of the actual infinity into the Axiom of Infinity. Since then, the actual infinity has been absolutely hegemonic in contemporary mathematics (although some relevant authors as Kronecker, Poincaré, Brouwer, Wittgenstein, Kleene, among others, rejected it). But set theory, and other related theories as supertask theory, also contain the instruments to develop arguments questioning the consistency the hypothesis of the actual infinity. Over the last 25 years, more than 30 of such arguments have been completed (pending its publication, a summary is available in [14]). It would be good news for physicists if at least one of those proofs were correct, as they would be freed from the tedious calculations needed to remove the sterile infinities from their equations. What follows is the shortest of those arguments (half a page, including comments). It has been selected as a tribute to J.J. Thomson (1921-1984) and P. Benacerraf (1931-) for their seminal debate on supertasks [21, 1].

Supertasks are performed by supermachines: theoretical devices intended to facilitate the discussions on the actual infinity, although their physical possibilities have also been analyzed (see for instance [19], [8], [13], [18]). So, let SM be a supermachine that counts natural numbers in such a way that it counts each of the successive natural numbers 1, 2, 3... at each of the successive instants t_1, t_2, t_3... of a strictly increasing sequence of instants $\langle t_n \rangle$ in the real interval (t_a, t_b), being t_b the limit of $\langle t_n \rangle$. In addition, SM has a red LED $\mathbf{L}$ that turns *on* if, and only if, AM counts an even number; and turns *off* if, and only if, SM counts an odd number, and so that the counting of the number and the change of state of $\mathbf{L}$ are simultaneous and instantaneous events. The one to one correspondence f between $\mathbb{N}$ and $\langle t_n \rangle$ defined by $f(n) = t_n, \forall n \in \mathbb{N}$ proves that at t_b *all* natural numbers have been counted by SM. The conclusions on the state of $\mathbf{L}$ at t_b will not be deduced from its successive states while performing the supertask of counting all natural numbers, as Thomson did with his lamp [21], otherwise Benacerraf's criticism would be inevitable [1]. They will deduced from the fact of being a LED with two, and only two, states, *on* and *off*, so that no other alternative exist. Thus, if after performing the supertask, SM continues to be the same counting machine it was before beginning the supertask, i.e. if performing a supertask does not arbitrarily violate a legitimate formal definition, as that of SM, then its LED $\mathbf{L}$ can only be either *on* or *off*, simply because, according to its legitimate definition, $\mathbf{L}$ can only be either *on* or *off*, and it will always be either *on* or *off*, independently of the number of times it has been turned *on* and *off*. Assume, then, that at t_b, the LED $\mathbf{L}$ is *on* (the same argument applies if it is *off*). One of the following two exhaustive and mutually exclusive alternatives must be true:

1. At t_b, $\mathbf{L}$ is *on* because SM counted a last even number that left it *on*.
2. At t_b, $\mathbf{L}$ is *on* because of any other reason.

The first alternative is impossible if *all* natural numbers have been counted: each even number has an immediate odd successor and there is not a last natural number, neither even nor odd. The second alternative implies the formal definition of SM has been arbitrarily violated: **L** turns *on* if, and only if, SM counts an even number, which excludes the possibility of being turned *on* by any other reason. Since the same argument applies if **L** is *off* at t_b, we must conclude that if the ω-ordered list of natural numbers exists as a complete infinite totality, then, once completed the supertask of counting all of them, **L** can be neither *on* nor *off*; though, by definition, it will be either *on* or *off*. The alternative to this contradiction is the arbitrary violation of a legitimate definition with the only purpose to justify that **L** can change its state by reasons different from the reason defined as the unique reason by which **L** can change its state: if, and only if, SM counts a natural number, being both events simultaneous and instantaneous. But assuming the arbitrary violation of a definition when convenient means any thing can be proved. So this alternative is formally unacceptable. Notice again the above contradiction on the state of **L** at t_b has not been drawn from its successive states while performing the supertask, but from the fact of being a LED with two definite, precise and unique states: *on* and *off*, and so that it turns *on* if, and only if, SM counts an even number, and it turns *off* if, and only if, SM counts an odd number. Thus, SM definition forces the actual infinity to leave a track of its existence through the state of **L** at t_b, and what it leaves is an inconsistency. By contrast, from the hypothesis of the potential infinity, only finite totalities of numbers can be counted, as large as wished but always finite, and depending of the parity of the last counted number, **L** will be either *on* or *off*, in agreement with the definition of SM.

4. Digital relativity

Most of decimal expansions of the real numbers are ω–ordered sequences of decimals that, according to the hypothesis of the actual infinity, exist all at once, as complete totalities of $\aleph_o$ decimals each. Consider the firsts 1.76×10^{79} decimals of any of them. In standard text of 5mm per numeral, this *minuscule* number (compared with $\aleph_o$) of decimals would be a lineal string of numerals longer than the diameter of the visible universe. It is not hard to imagine Ockham opinion on a physical constant (and on the corresponding universe) whose decimal expansion is a string of numerals longer than the diameter of the universe. And $\aleph_o$ is minuscule compared with $2^{\aleph_o}$, the power of the continuum, the cardinal of the set of real numbers; or the number of points of any segment of the real line; or the number of points of the whole tridimensional universe. Recall that a lineal interval trillions of times less than Planck length has the same number of points ($2^{\aleph_o}$) as the whole tridimensional universe. Formal physics is made of this infinitist mathematics. Particularly one of its most successfully theories: the theory of special relativity, a theory on the spacetime continuum. But space and time could also be discontinuous, digital, even if the hypothesis of the actual infinity is consistent. Let alone if it is not, as the above argument of the counting supermachine suggests. If it is not, space and time *could only be* of a discrete nature. Surprisingly, if that were the case, the theoretical and experimental success of special relativity could be explained in terms of a coincidence: Lorentz factor has the same algebraic form as the factor for converting continuous in discrete geometries. This section proves that is the case in a isotropic spacetime.

At least since Heisenberg [10, pp. 68-72], there has been a growing interest in discrete spacetimes (DST), even in experimental terms [17], although all attempts to approach this digital vision of space and time have been made within the framework of infinitist mathematics, the hegemonic, and practically unique, stream of mathematics from the beginning of XX century. Though for the reasons given in the precedent section, that could not be the best framework. In any case, at the moment we must accept we know nothing on the actual geometry of DST. Notwithstanding, some elementary conclusions can be logically drawn from the own concept of discreteness. For instance, DST should be made of indivisible units of space (geons) and of time (chronons); the distance between two geons should be an integer number of geons; the interval between two chronons an integer number of chronons; the number of geons of the hypotenuse of a right triangle should be equal to the number of geons of its greater leg (Pythagoras digital theorem); nothing can move a distance less than one geon; nothing can last less than one chronon; integer numbers should play in DST the same role as the real numbers play in the continuum; speed should be defined as the ratio of the integer number of geons an object traverses to the integer number of elapsed chronons; there would be a maximum speed of one geon per chronon. In addition, if DST is isotropic, as physical space seems to be, its geons should be anyway isometric. Under this last assumption it is possible to convert between continuous and discrete hypotenuses. Indeed, let h, x and y be the respective number of geons of the hypotenuse and legs of a right triangle in DST, and let λ be the length of a geon in the continuous geometry. Assume $x < y$. In the discrete geometry of DST we will have: $h = y$. In classical Euclidean geometry the length of the hypotenuse will no longer be $h\lambda$ but $h'\lambda$, being $h' > h$, because it is greater than the length $y\lambda$ of the greatest leg (note that while h, x and y are natural numbers, λ and h' are real numbers). According to classical Pythagoras theorem, it can be written:

$$(h'\lambda)^2 = (x\lambda)^2 + (y\lambda)^2; \quad y = \sqrt{h'^2 - x^2} \tag{1}$$

The ratio between the continuous and the discrete hypotenuse can be written:

$$\frac{h'\lambda}{h\lambda} = \frac{h'}{h} = \frac{h'}{y} = \frac{h'}{\sqrt{h'^2 - x^2}} = \frac{1}{\sqrt{1 - \left(\frac{x}{h'}\right)^2}} \tag{2}$$

where the last term on the right side of (2) as the algebraic form of the relativistic Lorentz factor γ. It can ve rewritten as:

$$\frac{h'\lambda}{h\lambda} = \frac{1}{\sqrt{1 - \left(\frac{x\lambda}{h'\lambda}\right)^2}} \tag{3}$$

Let ϕ be a photon that moves through a vertical distance $y\lambda$ in the rest frame RF_o of its source. Assume ϕ moves the same vertical distance $y\lambda$ from the perspective of another inertial frame RF_v while RF_o moves with respect to RF_v the horizontal distance $x\lambda$ at a uniform velocity v parallel to X_v for a time t_v. So, ϕ moves with respect to RF_v along the hypotenuse of a right triangle whose legs are $y\lambda$ and $x\lambda = vt_v$, i.e. along $h'\lambda$. And it will hold $h'\lambda = ct_v$. Therefore, (3) can be rewritten:

$$\frac{h'\lambda}{h\lambda} = \frac{1}{\sqrt{1 - \left(\frac{vt_v}{ct_v}\right)^2}} = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} = \gamma \tag{4}$$

which proves the ratio between the continuous hypotenuse and its corresponding discrete alternative is the relativistic Lorentz factor γ.

In conclusion, the above finitist conclusions suggest the convenience to consider the possibility of a digital interpretation of spacetime and of its main physical theory, though the new interpretation should be developed within a new finitist mathematics framework, of which everything is to be done (current discrete mathematics are infinitist). In these new conditions, discreteness could surely account for the weirdness of relativity related to the universal character of the speed of light.

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