Proving Unproved Euclidean Propositions
on a New Foundational Base

Updated version (2021/02/26) of the published version

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Abstract.-This paper introduces a new foundational basis for Euclidean geometry that includes productive definitions of concepts so far primitive, or formally unproductive, allowing to prove a significant number of axiomatic statements, unproved propositions and hidden postulates, among them the strong form of Euclid’s First Postulate, Euclid’s Second Postulate, Hilbert’s Axioms I.5, II.1, II.2, II.3, II.4 and IV.6, Euclid’s Postulate 4, Posidonius-Geminus’ Axiom, Proclus’ Axiom, Cataldi’s Axiom, Tacquet’s Axiom 11, Khayyam’s Axiom, Playfair’s Axiom, Euclid’s Postulate 5. The proposed foundation is more formally detailed and productive than other classical and modern alternatives, and at least as accessible as any of them.

1. Introduction

After more than two millennia of discussions on Euclid’s original geometry and at a time in which such discussions have been practically abandoned, this article introduces a new foundational basis for Euclidean geometry that includes productive definitions of concepts so far formally unproductive, as sidedness, betweenness, straight line, straightness, angle, or plane, among others (all of them properly legitimized by axioms or by formal proofs). The result is an enriched Euclidean geometry in which it is possible to prove some propositions that were proved to be unprovable on other Euclidean geometry bases. It will be introduced in the next sections. Conventions and general fundamentals are the objectives of Section 2. Section 3 introduces the new foundational basis: 28 definitions, 10 axioms and 40 corollaries (8 of the 10 axioms and most of the 40 corollaries are implicit (hidden) postulates in other Euclidean geometries). Section 5 introduces plane Euclidean geometry through 43 basic propositions and corollaries on triangles, perpendicularity, parallelism, and convergence.

2. Conventions and General Fundamentals

The nth axiom, corollary, definition, postulate and proposition will be referred to, respectively, as [Ax. n], [Cr. n], [Df. n], [Ps. n] and [Pr. n]. The same letters, for instance AB or BA, will be used to denote a line of endpoints A and B [Df. 1], as well as its length [Df. 8], and the distance between A and B [Df. 14] if AB is a straight line [Df. 10]. Unless otherwise indicated, different letters will denote different points, including endpoints. When convenient, lines will also be denoted by lower case Latin letters, whether or not indexed. Symbols as 0, +, −, =, ≠, ≤, etc. will be used conventionally. The expressions ‘point in a plane,’ and ‘point of a line’ will be used as synonyms. The same goes for ‘line in a plane’ and ‘line of a plane’. Closed lines [Df. 2] will be referred to as such closed lines, or by specific names, as circle [Df. 18]. As in classical Euclidean geometry [5, p. 8]; [3, p. 153], in Euclidean geometry a straight line is a particular type of line. So, and in contrast with modern English, in Euclidean geometry ‘line’ and ‘straight line’ are not synonyms. Asterisked expressions as ‘for instance∗’, ‘for example∗’, ‘assume∗’ etc., will always indicate that only one of the possible alternatives in a proof will be considered and proved, because the other alternatives can be proved in the same way. Proofs begin with the symbol ⊢ and end with the symbol □. The biconditional logical connective will be shortened by the term ‘iff’. And, unless otherwise indicated, the word ‘number’ will always mean natural number.

The following definitions and postulates are not exclusive to geometry, they have a general use in all sciences. For that reason they have been separated from the very fundamentals of geometry.

Definition 1 A quantity to which a real number can be assigned is said a numerical quantity. Numerical quantities that can be symbolically represented and operated with one another according to the procedures and laws of algebra, are said operable values.

Definition 2 An operable value is said to vary in a continuous way iff for any two different operable values of the corresponding variation, the variation contains any operable value greater than the less and less than the greater of those two operable values.

Definition 3 Metric properties and metric transformations: properties (transformations) to which operable values that vary in a continuous way are univocally assigned: to each quantity of the property (transformation) a unique and exclusive operable value, even zero, is assigned.

Definition 4 To define an object is to give the properties that unequivocally identifies the object. Objects with the same definition are said of the same class. To draw objects is to make a descriptive representation of them by means of graphics or texts, or by both of them, without the drawing modifies neither their properties nor their relative relations with other objects, if any.

Postulate A Of any two operable values, either they are equal to each other, or one of them is greater than the
other, and the other is less than the one. Symbolic representations of equal operable values, or of equal objects, are interchangeable in any expression where they appear.

**Postulate B** To be less than, equal to, or greater than, are transitive relations of operable values that are preserved when adding to, subtracting from, multiplying or dividing by the same operable value, the operable values so related. Metric properties (transformations) are algebraically operable through their corresponding operable values.

**Postulate C** Belonging to, and not belonging to, are mutually exclusive relations. Belonging to is a reflexive and transitive relation.

Contrarily to, for instance, fuzzy set theory or non-Boolean logics, this Euclidean geometry assumes [Ps. C], according to which it is not possible for an object to partially belong and partially not to belong to another object.

### 3. Foundational basis of Euclidean geometry

**Fundamentals on lines**

**Definition 1** Endedness.- A point at which a line ends is said endpoint. If such a point belongs to the line, the line is said closed at that end; if not, the line is said open at that end. Two endpoints, whether or not in the line, define two opposite directions in the corresponding line, each from an endpoint, said initial, to the other, said final.

**Definition 2** Collinearity.-A line is said through one or more given points iff such given points are points of the line. A line whose points belong, all of them, to a given line is said a segment of the given line. Points, or segments, or points and segments, are said collinear if they belong to the same line, and non-collinear if they do not belong to the same line.

**Definition 3** Commonness.-Two lines (surfaces) are said different from each other iff one of them has at least one point that is not in the other. Points and segments belonging to different lines are said common to them, otherwise they are said non-common to them. Non-collinear lines with at least one common segment are said locally collinear. Lines without common segments but with at least one common point are said intersecting lines, and their common points are also said intersection points. Intersecting lines are said to cut or to intersect one another at their intersection points.

**Definition 4** Adjacency.-Lines whose only common point is a common endpoint are said adjacent at that common endpoint iff no point of any of them is a non-common endpoint of any of the others. Lines containing all points of a given line, and only them, are said to make the given line. If any two points of a line are the common endpoints of only two segments of the line, the line is said closed or figure. Otherwise it is said non-closed. Closed lines that are not self-intersecting are said simply closed.

**Definition 5** Sidedness.-Adjacent lines containing all points of a given line, and only them, whose common endpoint is a given point of the given line, and whose non-common endpoints are the endpoints of the given line, if any, are said sides of the given point in the given line.

**Definition 6** Betweenness.-A point is said to be between two given points of a line, iff it is a point of the line and each of the given points is in a different side of the point in the line.

**Definition 7** Uniformity.-Lines whose segments have the same definition as the whole line are said uniform. Two or more uniform lines are said mutually uniform iff any segment of any of them has the same definition as any segment of any of the others.

**Definition 8** Metricity.-Length (area) is a metric property of lines (figures) of which arbitrary units can be defined. Lengths (areas) are said equal iff their corresponding operable values are equal. Lines (figures) with a
finite length (area) are said finite. If the sides of a point of a line have the same length, the point is said to bisect the line.

**Axiom 1** Any number of points, lines, surfaces and solids can be considered and drawn, being, or not, the points in lines, the lines in surfaces and the surfaces in solids.

**Axiom 2** A line has at least two points, at least one point between any two of its points, and at most two endpoints, whether or not in the line.

**Axiom 3** Two adjacent lines make a line, and a point of a line can be common to any number of any other different lines, either collinear, or non-collinear, or locally collinear.

**Axiom 4** Except endpoints, each point of a non-closed line has two, and only two, sides in the line, whose lengths are greater than zero and sum the length of the whole line.

Unless otherwise indicated, only non-self-intersecting and non closed lines whose endpoints, if any, belong to the line will be considered.

**Corollary 1** The number of points of a line is greater than any given number.

▷ It is an immediate consequence of [Axs. 1, 2]. □

**Corollary 2** Each side of a point, except endpoints, of a line is a segment of the line and both sides make the line.

▷ Except endpoints, a point \( P \) of a line \( l \) [Ax.1, Cr. 1]
has two, and only two, sides in \( l \) [Ax. 4],
which are two lines adjacent at \( P \) [Df. 5]
containing all points of \( l \), and only them [Df. 5].
So, each side is a segment of the line [Dfs. 5, 2],
and both sides make the line \( l \) [Ax. 3, Df. 4]. □

**Corollary 3** Any point of a line is in one, and only in one, of the two sides of any other point, except endpoints, of the line.

▷ Except endpoints, a point \( P \) of a line \( l \) [Ax.1]
has two, and only two, sides in \( l \) [Ax. 4].
Any other point of \( l \) [Cr. 1]
will be in one of such sides [Cr. 2],
and only in one of them, otherwise both sides would not be adjacent at \( P \) [Df. 4],
which is impossible [Dfs. 5, 4]. □

**Corollary 4** A point is in a line with two endpoints iff, being not an endpoint of the line, it is between the endpoints of the line.

▷ If a point \( P \) is between the two endpoints of a line \( AB \) [Axs. 1, 2, 4, Df. 6],
it is in \( AB \) [Df. 6].
If a point \( P \) is in a line \( AB \) [Cr. 1],
it has just two sides in \( AB \) [Ax. 4],
whose respective non-common endpoints are the endpoints \( A \) and \( B \) of \( AB \) [Dfs. 5, 4].
So, \( P \) is between both endpoints \( A \) and \( B \) [Df. 6]. □

**Corollary 5** Any two points of a line are the endpoints of a segment of the line. And the line has a number of segments and a number of points between any two of its points greater than any given number.

▷ Let \( P \) and \( Q \) be any two points of a line \( l \) different from its endpoints, if any [Ax.1, Cr. 1].
\( Q \) has two sides in \( l \) [Ax. 4],
which are two lines adjacent at \( Q \) [Df. 5]
that contains all points of \( l \) and only them [Cr. 2].
So, in one, and only in one, of such lines will be $P$ [Cr. 3].
In turn, $P$ has two sides in that side of $Q$ [Df. 5, Ax. 4],
the side $PQ$ in which it is $Q$ and the side in which it is not $Q$ [Cr. 3].

$PQ$ is a line [Df. 5]
al of whose points belong to $l$ [Df. 5].
Hence, $PQ$ is a segment of $l$ [Df. 2].
Being $P$ and $Q$ any two of its points, $l$ has a number of segments and a number of points between any two of
its points greater than any given number [Crs. 1, 4] □

**Corollary 6** A segment of a segment of a line, it is also a segment of the line.

Let $RS$ be a segment of a segment $PQ$ of a line $l$ [Ax.1, Cr. 5].

$PQ$ is a line whose points belong to $l$ [Df. 2].
$RS$ is a line whose points belong to $PQ$ [Df. 2],
and then to $l$ [Ps. C].
So, $RS$ is a segment of $l$ [Df. 2]. □

**Corollary 7** If a point is between two given points of a given line, it is also between the given points in any
line of which the given line is a segment.

Let $R$ be a point of a segment $PQ$ of a line $l'$ [Ax.1, Cr. 5],
which is a segment of another line $l'$ [Cr. 5].

Since $PQ$ is a segment of $l'$, it is also a segment of $l$ [Cr. 6].

So, $R$ is a point of a segment $PQ$ of $l$ [Df. 2],
and then a point of $l$ [Df. 2]
between $P$ and $Q$ [Cr. 4]. □

**Corollary 8** (A variant of Hilbert’s Axiom II.2) At least one of any three points of a line is between the other
two.

Let $P$, $Q$ and $R$ be any three points of any line $l$ [Ax.1, Cr. 1].

At least one of them, for example* $Q$, will not be an endpoint of $l$ [Ax. 2].

$P$ can only be in one of the two sides of $Q$ in $l$ [Cr. 3];
$R$ can only be in one of the two sides of $Q$ in $l$ [Cr. 3].

So, either $P$ and $R$ are in different sides of $Q$ in $l$, or they are in the same side of $Q$ in $l$. If $P$ and $R$ are in
different sides of $Q$ in $l$, then $Q$ is between $P$ and $R$ in $l$ [Df. 6].

If not, $P$ and $R$ are in the same side of $Q$ in $l$, which is a segment $l'$ of $l$ [Cr. 2],
one of whose endpoints is $Q$ [Df. 5].

If $R$ is an endpoint of $l'$, $P$ can only be between the endpoints $Q$ and $R$ of $l'$ [Cr. 4],
and then between $Q$ and $R$ in $l$ [Cr. 7].
If $R$ is not an endpoint of $l'$, it has two sides in $l'$ [Ax. 4]:
the side $RQ$ in which it is $Q$, and the side in which it is not $Q$ [Cr. 3].
If $P$ is in $RQ$, it is between $R$ and $Q$ in $l'$ [Cr. 4],
and then between $R$ and $Q$ in $l$ [Cr. 7].
If $P$ is in the side of $R$ in $l'$ in which it is not $Q$, then $P$ and $Q$ are in different sides of $R$ in $l'$, and $R$ is between $P$ and $Q$ in $l'$ [Df. 6]
and then between $P$ and $Q$ in $l$ [Cr. 7].
So, in all possible cases [Ax. 4, Cr. 3]
at least one of the three points is between the other two in $l$. □
Corollary 9 (Hilbert’s Axioms II.3, II.1) One, and only one, of any three points of a line is between the other two.

Let \( P, Q \) and \( R \) be any three points of any line \( l \) [Ax.1, Cr. 1].

At least one of them, for example \( Q \), will be between the other two, \( P \) and \( R \), in \( l \) [Cr. 8], in which case \( Q \) is a point of \( PR \) [Cr. 4].

So, \( Q \) has two sides in \( PR \) [Ax. 4], which are two lines, \(QP\) and \( QR\), adjacent at \( Q \) [Df. 5].

\( P \) cannot be between \( Q \) and \( R \), otherwise it would be in \( QR \) [Cr. 4], \( QP \) would be a segment of \( QR \) [Cr. 5], all points \( QP \) [Cr. 1] would be points of \( QR \) [Df. 2], and \( QP \) and \( QR \) would not be adjacent at \( Q \) [Df. 4], which is impossible [Df. 5].

For the same reasons \( R \) cannot be between \( P \) and \( Q \) either. Therefore, one [Cr. 8], and only one, of any three points of a line is between the other two. □

Corollary 10 (a variant of Hilbert’s Axiom II.4) Of any four points of a line, two of them are between the other two.

Let \( P, Q, R \) and \( S \) be any four points of a line \( l \) [Ax.1, Cr. 1].

Consider any three of them, for instance \( P, Q \) and \( R \). One, and only one, of them, for instance \( Q \), will be between the other two, \( P \) and \( R \) [Cr. 9], and \( Q \) will be in \( PR \) [Cr. 4].

Of the other three points \( P, R \) and \( S \), one, and only one, of them will be between the other two [Cr. 9]: if \( P \) is between \( R \) and \( S \), it is in \( SR \) [Cr. 4], so that \( PR \) is a segment of \( SR \) [Cr. 5].

Therefore \( Q \), which is in \( PR \), is also in \( SR \) [Cr. 6].

So, \( Q \) and \( P \) are between \( R \) and \( S \) [Cr. 4].

For the same reasons, if \( R \) is between \( P \) and \( S \) then \( Q \) and \( R \) are between \( P \) and \( S \); and if \( S \) is between \( P \) and \( R \), then \( Q \) and \( S \) are between \( P \) and \( R \). So, in all possible cases [Ax. 4, Cr. 3] two of the four points are between the other two. □

Corollary 11 Two segments can only be either collinear or non-collinear. And if a segment of a given line is non-collinear with another segment of another given line, then both given lines are also non-collinear.

Since belonging to is a reflexive relation [Ps. C] and segments are lines [Df. 2], any two segments \( l_1 \) and \( l_2 \) [Ax. 1] belong to a line, even if the line is the own segment itself [Df. 2].

So, \( l_1 \) and \( l_2 \) will be either collinear, or non-collinear, or collinear and non-collinear. If they were collinear and non-collinear they would be segments that belong to the same line \( l \) [Df. 2], and segments that do not belong to the same line \( l \) [Df. 2], which is impossible [Ps. C].

So, \( l_1 \) and \( l_2 \) can only be either collinear or non-collinear. Let now \( l'_1 \) be a segment of a line \( l_1 \) and \( l'_2 \) another segment of a line \( l_2 \) [Cr. 5], such that \( l'_1 \) and \( l'_2 \) are non-collinear [Df. 2].

If \( l_1 \) and \( l_2 \) were collinear, they would be segments of the same line \( l \) [Df. 2], and being their respective segments \( l'_1 \) and \( l'_2 \) also segments of \( l \) [Cr. 6],
Corollary 12 If two points of a line have a given property, and all points between any two points with the given property have also the given property, then the line has a unique segment whose points are all points of the line with the given property.

Let $A$ and $B$ be two points $[\text{Ax. 1, Cr. 5}]$

with a given property (gp-points for short) of a line $l$ such that all points of $l$ between any two of its gp-points are also gp-points. So, $l$ has a number of gp-points greater than any given number $[\text{Cr. 5}]$.

Let a segment whose points are gp-points, except at most its endpoints, be referred to as gp-segment. Any gp-point $C$ of $l$ is at least in the gp-segment $AC$ of $l$ $[\text{Crs. 5, 4}]$.

It is then impossible for $P$ not to be a gp-point, and for $DG$ not to be a gp-segment. $\square$

Corollary 13 The length of a finite line is greater than the length of each of the sides of any of its points, except endpoints, and it is greater than zero. The length of each side is equal to the length of the whole line minus the length of the other side. And the length of a segment of the line is less than the length of the whole line if at least one endpoint of the segment is not an endpoint of the line.

Let $P$ be a point of a finite line $AB$ $[\text{Df. 8, Axs. 1, 2}]$.

Assume the length $AP$ is not less than the length $AB$. It will be $AP \geq AB$ $[\text{Ps. A}]$.

And being $AB = AP + PB$ $[\text{Ax. 4}]$,

it would hold $AP \geq AP + PB$ $[\text{Ps. A}]$.

Hence, $0 \geq PB$ $[\text{Ps. B}]$,

which is impossible $[\text{Ax. 4}]$.

So, it must be $AP < AB$ $[\text{Ps. A}]$.

And for the same reasons $PB < AB$. Therefore, and being $0 < PB$ $[\text{Ax. 4}]$,

it holds $0 < AB$ $[\text{Ps. B}]$.

So, the length of any line is greater than zero. And from $AP + PB = AB$ $[\text{Ax. 4}]$,

it follows immediately $AP = AB - PB$; $PB = AB - AP$ $[\text{Ps. B}]$.

Let now $Q$ be any point of $AB$ different from $P$ $[\text{Crs. 1}]$.

It will be in one, and only in one, of the sides of $P$ in $AB$ $[\text{Cr. 3}]$.

for instance* in $AP$. It has just been proved that $AP < AB$. If $Q$ were the endpoint $A$ of $AP$ we would have $QP = AP$ $[\text{Ps. A}]$.

If not, and for the same reasons above, it will be $QP < AP$. So that we can write $QP \leq AP$, and then $QP < AB$ $[\text{Ps. B, A}]$.

Therefore, the length of a segment of $AB$ is less than $AB$ if at least one if its endpoints $P$ is not an endpoint of $AB$. $\square$
Fundamentals on straight lines

**Definition 9** Extensible lines.-To produce (extend) a given line by a given length is to define a line, said production (extension) of the given line, so that the production has the given length, is adjacent to the given line and the production and the produced lines are of the same class as the given line. Lines that can be produced from any of its endpoints by any given length are said extensible.

**Definition 10** Straight lines: extensible and mutually uniform lines that cannot be locally collinear nor have non-common points between common points.

**Definition 11** Straightness.-Three or more points are said to be in straight line with one another iff they are in the same straight line, whether or not produced. A point is said in straight line with a given straight line iff it is in straight line with any two points of the given straight line, whether or not produced. Collinear straight lines whose common line is a straight line, and only them, are said to be in straight line with one another.

**Axiom 5** Any two points can be the endpoints of a straight line, and only both points are necessary to draw the straight line.

**Corollary 14** A segment of a straight line is also a straight line.

- It is an immediate consequence of [Ax. 5, Dfs. 10, 7]. □

**Corollary 15** (Strong form of Euclid’s First Postulate) Any two points can be the endpoints of one, and only of one, straight line.

- Assume two different straight lines $l_1$ and $l_2$ have the same endpoints $A$ and $B$. At least one of them will have a point which is not in the other [Df. 3].

  And they would have at least one non-common point between the two common points $A$ and $B$, which is impossible [Dfs. 10].

  So, any two points can be the endpoints of one [Ax. 5], and only of one, straight line. □

Hereafter, to join two points will always mean to consider and draw the unique straight line whose endpoints are both points.

**Corollary 16** (Strong form of Euclid’s Second Postulate) There is one, and only one, way of producing a given straight line by any given length and from any of its endpoints, being the produced line a straight line; and the given straight line and its production, adjacent straight lines in straight line with each other.

- Let $AB$ be any straight line [Ax. 1, Cr. 15].

  It can be produced from any of its endpoints, for example* from $B$, by any given length [Dfs. 10, 9] to a point $C$, so that $BC$ and $AC$ are straight lines [Dfs. 10, 9, 4], and $AB$ and $BC$ are adjacent segments [Dfs. 10, 9].

  Assume $AB$ can be produced from $B$ by the same given length to another point $C'$. The straight lines $AC$, $AC'$ [Dfs. 10, 9] would have a common segment $AB$ [Cr. 5]; they would be collinear since they cannot be locally collinear [Dfs. 10, 3, Cr. 11]; and $BC$ and $BC'$ would be two segments of the same line $l$ [Cr. 5], both adjacent at $B$ to $AB$ [Ax. 5, Df. 9].

  Being $C$ and $C'$ different points of the same line $l$, one of them, for example* $C'$, would be between $B$ and the other in $l$ [Cr. 9], and we would have $BC' < BC$ [Cr. 13], which is not the case. So, $C'$ can only be the point $C$. And being $BC$ a straight line [Dfs. 10, 9, 4], it is the unique straight line joining $B$ and $C$ [Cr. 15].

  So, there is a unique way of producing a straight line by a given length from any of its endpoints. And being segments of the produced straight line $AC$, the straight lines $AB$ and $BC$ are in straight line with each other [Df. 11] □

**Corollary 17** Through any two points, any number of collinear straight lines of different lengths can be drawn.
It is an immediate consequence of [Df. 2, Crs. 15, 16]. □

Corollary 18 Two straight lines with two common points belong to the same straight line.

All points between any two common points of two straight lines are also common points [Df. 10].

So, if two straight lines \(AB\) and \(CD\) have two common points, they have a unique segment \(PQ\) with all of their common points [Cr. 12].

\(PQ\) is a straight line [Cr. 14].

\(AB\) and \(PQ\) belongs to the same straight line \(l\) [Df. 2]; and \(PQ\) and \(CD\) belong to the same straight line \(l\) [Df. 2].

Therefore, \(AB\) and \(CD\) belong to the same straight line \(l\) [Ps. C]. □

Corollary 19 Any point between the endpoints of a given straight line can be common to any number of intersecting straight lines not in straight line with the given straight line, and that point is the only common point of those straight lines and the given straight line.

Any point \(P\) between the endpoints of a straight line \(l\) [Ax. 1, Cr. 15]
can be common to any number \(n\) of non-collinear straight lines [Ax. 3],
which being non-collinear cannot be in straight line with the given straight line [Df. 11].

Assume there is a second common point \(Q\) of \(l\) and of any one of those \(n\) intersecting straight lines \(l'\), whether or not producing \(l\) and \(l'\) [Cr. 16].

Both straight lines would have a common segment \(PQ\) [Df. 10, Cr. 4]
and they would be locally collinear [Df. 3],
which is impossible [Df. 10].

Therefore, \(P\) is the only intersection points of \(l\) and each of those \(n\) intersecting straight lines, even arbitrarily producing \(l\) and any of the \(n\) intersecting straight lines. □

Corollary 20 There is a number of points greater than any given number that are not in straight line with any two given points, or with a given straight line.

Let \(A\) and \(B\) be any two points [Ax. 1].

Join \(A\) and \(B\) [Cr. 15],

and let \(PC\) be a straight line non-collinear with \(AB\) that intersects \(AB\) at \(P\) [Cr. 19].

\(P\) is the only common point of both straight lines even arbitrarily produced [Cr. 19].

So, \(PC\) has a number of points greater than any given number [Cr. 1] none of which, except \(P\), is in straight line with \(A\) and \(B\) [Df. 11].

On the other hand, if \(AB\) is any straight line, it has just been proved there is a number greater than any given number of points that are not in straight line with \(A\) and \(B\), and then with \(AB\) [Df. 11]. □

Corollary 21 Each endpoint of a given straight line can be the common endpoint of any number of adjacent straight lines not in straight line with the given straight line.

Let \(AB\) be any straight line [Ax. 1, Cr. 15].

There is a number greater than any given number of points not in straight line with \(AB\) [Cr. 20].

Join each of them with, for instance*, the endpoint \(A\) of \(AB\) [Cr. 15].

Each of these straight lines are adjacent at \(A\) to \(AB\) [Df. 4].

If any of them, for instance* \(AP\), were in straight line with \(AB\), they would be segments of the same straight line [Df. 11],
and \(P\) would be in straight line with \(AB\) [Df. 11],
which is not the case. □
Corollary 22 If two adjacent straight lines are not in straight line, then no point of any of them, except their common endpoint, is in straight line with the other.

Let $AB$ and $AC$ be two straight lines adjacent at $A$ not in straight line with each other [Cr. 21].

Let $P$ be a point of, for instance*, $AB$ [Cr. 1].

$A$, $P$ and $B$ belong to $AB$. So, if $P$ were in straight line with $AC$, it would be in straight line with $A$ and $C$ [Df. 11].

$P$, $A$ and $C$ would belong to the same straight line [Df. 11], and then $A$, $P$, $B$ and $C$ would belong to the same straight line [Ps. C], which is not the case [Df. 11]. □

Fundamentals on planes

Definition 12 Plane: a surface that contains at least three points not in straight line and any straight line through any two of its points. A line is said in a plane iff all of its points are points of the plane. Lines in a plane are said plane lines. Points, or lines, or points and lines in the same plane are said coplanar.

Definition 13 Sides of a given straight line in a plane: parts of the plane that contain all points of the plane, and only them, each part with at least two common points and at least two non-common points, where a point is said common, or common to all parts, if it is in straight line with the given straight line; and non-common if it is not, being said non-common of a part iff it is in that part. Any other straight line is said to be in one of those parts iff all of its points between its endpoints are non-common points of that part.

Axiom 6 Any three points lie in a plane, in which any straight line has just two sides. Any other straight line is in one of such sides iff its endpoints are in that side.

Corollary 23 (A variant of Hilbert’s Axiom I.5) A plane has a number of points greater than any given number, any two of which can be joined by a unique straight line in that plane. And any given straight line is at least in a plane, in which it can be produced by any given length.

Let $P$, $Q$ and $R$ be any three points not in straight line [Cr. 20], and $Pl$ a plane in which they lie [Ax. 6].

$Pl$ has at least the points $P$, $Q$ and $R$ and all points of any straight line [Cr. 1] through any two of its points [Dfs. 12, Ax. 5, Cr. 17].

So, $Pl$ has a number of points greater than any given number [Cr. 1].

Let, then, $A$ and $B$ be any two points of $Pl$. Join $A$ and $B$ [Cr. 15], and produce $AB$ from $A$ and from $B$ by any given length to the respective points $A'$ and $B'$ [Cr. 16].

Since $A'B'$ is a straight line [Cr. 16] through two points $A$ and $B$ [Df. 2, Cr. 17] of $Pl$, $A'B'$ is in $Pl$ [Df. 12], so that all points of $A'B'$ are in $Pl$ [Df. 12], and then all points of its segment $AB$ are in $Pl$ [Df. 2, Cr. 5].

Hence, $Pl$ contains the unique straight line joining any two of its points $A$ and $B$ [Crs. 14, 15].

Let now $AB$ be any straight line [Ax. 1, Cr. 15], and $P$ and $Q$ any two of its points [Cr. 1].

There is a plane $Pl$ containing $A$, $P$ and $Q$ [Ax. 6], and the straight line $AB$ through $P$ and $Q$ is in $Pl$ [Df. 12].

Produce $AB$ from $A$ and from $B$ by any given length to the points $A'$ and $B'$ respectively [Cr. 16].

Since the produced straight line $A'B'$ is a straight line [Cr. 16] through two points $A$ and $B$ [Cr. 17] of $Pl$, it is a straight line of $Pl$ [Df. 12]. □

Corollary 24 A point of a plane can only be either common to both sides of a straight line in that plane, or
Let $A$ and $B$ be any two points of a plane $P$. [Ax. 6]

Join $A$ and $B$. [Cr. 15]

$AB$ is in $P$. [Cr. 23]

Let $P$ be any point of $P$. Either $P$ belongs to $AB$, whether or not produced [Cr. 16], or it does not [Ps. C].

If $P$ belongs to $AB$, whether or not produced [Cr. 16], $P$ is a point common to both sides of $AB$ [Ax. 6, Df. 13].

If $P$ does not belong to $AB$, whether or not produced [Cr. 16], $P$ cannot be in both sides of $AB$ [Df. 13], and being a point of $P$, it can only be in one, and only in one, of the two sides of $AB$ [Df. 13, Ax. 6].

So, it is a non-common point of that side, and only of it [Df. 13]. □

**Corollary 25** There is a plane containing any two adjacent straight lines not in straight line, being each of them in the same side of the other. And there is a plane containing any two intersecting and non-adjacent straight lines.

Let $AB$ and $AC$ be two straight lines adjacent at $A$ and not in straight line with each other [Cr. 21].

$A$, $B$ and $C$ are not in straight line, otherwise they would be in the same straight line [Df. 11], which is not the case [Df. 11].

So, there is a plane in which lie $A$, $B$ and $C$ [Ax. 6] and the adjacent straight lines $AB$ and $AC$ [Cr. 23].

$B$ is not in straight line with $AC$ [Cr. 22], so it is a non-common point of one of the sides of $AC$ [Df. 13].

Therefore $AB$ is in that side [Ax. 6].

For the same reasons $AC$ is in one of the sides of $AB$. Let now $l_1$ and $l_2$ be any two non-adjacent straight lines that intersect at a unique point $P$ [Cr. 19], $Q$ a point of $l_1$, and $R$ a point of $l_2$ [Cr. 1].

There is a plane containing $P$, $Q$ and $R$ [Ax. 6], the straight line $l_1$ through $Q$ and $P$ [Cr. 17, Df. 12], and the straight line $l_2$ though $R$ and $P$ [Cr. 17, Df. 12]. □

**Corollary 26** All points between two points of a straight line in the same side of a given straight line lie in that side of the given straight line, and that side has a number of non-common points greater than any given number.

Let $P$ and $Q$ be any two non-common points in the same side $Pl_1$ [Df. 13] of a straight line $l$ in a plane $P$ [Cr. 23].

Join $P$ and $Q$. [Cr. 15]

$PQ$ is in $Pl_1$ [Ax. 6].

All points between $P$ and $Q$ are non-common points of $Pl_1$ [Df. 13].

So, $Pl_1$ has a number of non-common points greater than any given number [Cr. 1]. □

**Corollary 27** In a plane and in each side of a straight line in that plane, it is possible the existence of a number greater than any given number of straight lines, whether or not adjacent, none of which is in straight line with any of the others.

(Fig. 2, left) Let $A$, $B$ and $C$ be any three points not in straight line [Cr. 20], and $P$ a plane in which they lie [Ax. 6].

Join $A$ and $B$. [Cr. 15]
and let $P_1$ and $P_2$ be the two sides of $AB$ in $Pl$ [Ax. 6].

$C$ will be a non-common point [Df. 13] of, for example*, $P_1$ [Cr. 24].

Join $C$ with $A$ and with $B$ [Cr. 15].

$CA$ and $CB$ are not in straight line, otherwise $A$, $C$ and $B$ would be in straight line [Df. 11], which is not the case. Join each of any number $n$ of points of $CA$ between $C$ and $A$ with a different point of $CB$ between $C$ and $B$ [Crs. 5, 15], and let $DE$ and $FG$ be any two of such straight lines, $D$ and $F$ in $CA$, and $E$ and $G$ in $CB$. The straight lines $DE$ and $FG$ cannot be in straight line with each other, otherwise they would be segments of the same straight line [Df. 11], and $D$, $E$, $F$ and $G$ would be in that straight line [Df. 2], so that $D$ would be in straight line with $E$ and $G$, and then with $CB$ [Df. 11], which is impossible [Cr. 22].

The same argument applies to the $n$ straight lines joining the same point $H$ of $CA$ between $A$ and $C$ (Fig. 2, right) with $n$ different points of $CB$ between $C$ and $B$ [Crs. 5, 15], being all of these straight lines adjacent at $H$ [Df. 4].

And being $CA$ and $CB$ in $P_1$ [Ax. 6], all of these straight lines in $Pl$, whether or not adjacent, have their respective endpoints on $P_1$ [Df. 13], so that all of them are in $P_1$ [Ax. 6]. □

**Corollary 28** The intersection point of two intersecting straight lines has its two sides in each of the intersecting straight lines in different sides of the other intersecting straight line in the plane that contains both straight lines.

(Fig. 3) Let $P$ be the unique intersection point of two straight lines [Cr. 19]

$AB$ and $CD$ in a plane $Pl$ [Cr. 25].

Since the only points of $Pl$ common to both sides of $CD$ in $Pl$ are the points in straight line with $CD$ [Df. 13], and $P$ is the only common point of $AB$ and $CD$, even arbitrarily produced [Crs. 16, 19], $P$ is the only point of $AB$ in straight line with $CD$ [Df. 11], and then the unique point common to both sides of $CD$ in $Pl$ [Df. 13].

Therefore, the endpoints $A$ and $B$ can only be non-common points of the sides of $CD$ in $Pl$ [Df. 13, Cr. 24].

So, if $PA$ and $PB$ were in the same side of $CD$ in $Pl$, the endpoints $A$ and $B$ would be non-common points of that side [Ax. 6], and being $P$ between them [Cr. 4],

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**Figure 2** – Corollary 27

**Figure 3** – Corollary 28
would also be a non-common point of that side [Cr. 26],
which is impossible because it is a common point of both sides [Cr. 24]. $\text{\textpi}$
Therefore, $PA$ and $PB$ cannot be in the same side of $CD$ in $Pl$. So, they must be in different sides of $CD$ in $Pl$. The same argument proves $PC$ and $PD$ can only be in different sides of $AB$ in $Pl$. □

**Corollary 29** The straight line joining any two non-common points, each in a different side of another given coplanar straight line, intersects the given straight line, or a production of it, at a unique point.

**Figure 4 – Corollary 29**

▷ (Fig. 4) Join any two non-common points $A$ and $B$ [Cr. 15] respectively in the sides $Pl_1$ and $Pl_2$ of a straight line $l$ in a plane $Pl$ [Cr. 23].
$AB$ is in $Pl$ [Cr. 23].
Except $A$ and $B$, all points of $AB$ are between $A$ and $B$ [Cr. 4].
If all points of $AB$ between $A$ and $B$ were non-common points of $Pl_1$, $AB$, including $B$, would be in $Pl_1$ [Df. 11, Ax. 6],
which is not the case. So, $AB$ contains points of $Pl_2$ other than $B$ and, for the same reason, points of $Pl_1$ other than $A$ [Cr. 1]
So, $AB$ has at least two points in each side of $l$. Since all points between two points of a straight line in the same side of another coplanar straight line are also in that side [Crs. 26],
$AB$ has a segment $AC$ whose points are all points of $AB$ in $Pl_1$ [Cr. 12].
And for the same reasons it also has a segment $BD$ whose points are all points of $AB$ in $Pl_2$ [Cr. 12].
If $C$ and $D$ were different points, all points of $AB$ between them [Cr. 5] would be in no side of $l$ in $Pl$, which is impossible because all points of $AB$ are points of $Pl$ [Df. 12], and all points of $Pl$ are points either of $Pl_1$, or of $Pl_2$, or of both of them [Df. 13].
So, $C$ and $D$ are the same point. Since all points between $A$ and $C$ are in $Pl_1$, $AC$ is in $Pl_1$ [Df. 13], and $C$ is also in $Pl_1$ [Ax. 6].
For the same reasons $D$ is in $Pl_2$. Since $C$ and $D$ are the same point, and this point belongs to $Pl_1$ and to $Pl_2$, it is a point of $l$, whether or not produced [Cr. 16, Df. 13].
So, it is an intersection point of $AB$ and $l$ [Df. 3] whether or not produced [Cr. 16].
And it is the unique intersection point of $AB$ and $l$, otherwise the non-common point $A$ of $Pl_1$ would be in straight line with at least two points of $l$ and it would be a common point of $Pl_1$ and $Pl_2$ [Dfs. 13, 11], which is impossible [Cr. 24]. □

**Corollary 30** A plane contains at least two non-intersecting straight lines, which can be intersected by any number of different coplanar straight lines.

▷ Let $l$ be a straight line in a plane $Pl$ [Cr. 23]; $Pl_1$ and $Pl_2$ the two sides of $l$ in $Pl$ [Ax. 6];
$A$, $B$ any two non-common points of $Pl_1$; and $C$, $D$ any two non-common points of $Pl_2$ [Cr. 26].
Joint $A$ with $B$; and $C$ with $D$ [Cr. 15].
$AB$ is in $Pl_1$, and $CD$ in $Pl_2$ [Ax. 6].
$AB$ and $CD$ cannot intersect with each other because the intersection point would be a common point of $Pl_1$ and $Pl_2$ [Df. 13]
while all of their points, even endpoints, are non-common points of Pl1 and of Pl2 respectively [Df. 13, Ax. 6].

On the other hand, AB and CD can be intersected by any number n of straight lines in Pl, each joining each of any n points of AB with a point of CD [Crs. 1, 15, 23]. □

**Fundamentals on distances**

**Definition 14** Distance between two points: length of the straight line joining both points. If both points are the same point, whether or not of a common point of different lines, the distance between them is said zero.

**Definition 15** Distance from a point not in a given line to the given line: the shortest distance between the point and a point of the given line, or of a production of the given line if the given line is a straight line and the point is not in straight line with it.

**Definition 16** Distancing direction and relative distancing.-Two non-common points in the same side of a given coplanar straight line and at different distances from the given straight line define a distancing direction in the straight line joining both points: from the nearest to the farthest of them. The difference between the distances to the given straight line from the endpoints of a segment of another straight in the same side of the given straight line is called relative distancing of the segment with respect to the given straight line.

**Definition 17** Parallel straight lines.-A straight line is said parallel to another coplanar straight line, iff all of its points are at the same distance, said equidistance, from the other straight line.

[Pr. 32] proves the existence of parallel straight lines. According to [Df. 14], the length of a straight line AB and the distance from A to B will be used as synonyms.

**Axiom 7** The distances from the points of a line to a fixed point or to another line vary in a continuous way.

**Corollary 31** The distance between any two given points is unique.

> It is an immediate consequence of [Cr. 15, Df. 14]. □

**Fundamentals on circles**

**Definition 18** Circle: a simply closed and plane line whose points are all points of the plane, and only them, at the same given finite distance, said radius, from a fixed point of that plane, said centre of the circle. A straight line joining any point of the circle with its centre is also said a radius of the circle. A segment of a circle is called arc, and the straight line joining its endpoints is a chord, or straight line subtending the arc. If the center of the circle is a point of a chord, the chord is said a diameter, and the corresponding arc a semicircle. Coplanar circles, and their corresponding segments, with the same centre are said concentric. The centre and any coplanar point at a distance from the centre less than its radius are said interior to the circle; if that distance is greater than the radius of the circle, the coplanar point is said exterior to the circle.

**Axiom 8** Any point of a plane can be the centre of a circle of any finite radius, being all points of any of its arcs in the same side of its chord.

**Corollary 32** A circle has interior points, other than its centre, and exterior points. And any point coplanar with a circle is either in the circle, or it is interior or exterior to the circle.

> Let O be the centre of a circle c in a plane Pl [Ax. 8],

and A any point of c [Df. 18].

Joint A with O [Cr. 15].

Produce OA from A by any given finite length to a point A′ [Cr. 16].

OA′ is in Pl [Cr. 23].

Let P be any point of OA [Cr. 5].

Since OP < OA and OA < OA′ [Cr. 13],

P is interior and A′ is exterior to c [Dfs. 14, 18].

Join now any point R of Pl [Cr. 23] with O [Cr. 15].

It holds RO ⊊ OA [Ps. A],

and R will be either in c (RO = OA), or it will be interior (RO < OA) or exterior (RO > OA) to c [Dfs. 14, 18]. □

**Corollary 33** A plane line intersects a coplanar circle at a point between its endpoints iff it has points interior and exterior to the circle.
Let $O$ be the centre and $AO$ the finite radius of a circle $c$ [Ax. 8] in a plane $Pl$; $BC$ a plane line in $Pl$ [Ax. 8], and $P$ and $Q$ two points of $BC$ [Cr. 1] such that $P$ is interior and $Q$ exterior to $c$ [Cr. 32].

Being $P$ interior to $c$, its distance to $O$ is less than $AO$ [Df. 18].

Being $Q$ exterior to $c$, its distance to $O$ is greater than $AO$ [Df. 18].

Therefore, there will be at least one point $R$ in $PQ$, and then in $BC$ [Cr. 1, 2], whose distance to $O$ is just $AO$ [Ax. 7, Df. 2].

And $R$ will also be in $c$ [Df. 18].

So, $R$ is an intersection point of $BC$ and $c$ [Df. 3].

On the other hand, if all points of a plane line $BC$ are interior (exterior) to $c$, none of its points is at a distance $AO$ from $O$ [Df. 18], and then no point of $BC$ is in $c$ [Df. 18].

Therefore $c$ and $BC$ have no point in common, and they do not intersect with each other [Df. 3].

**Corollary 34** Any point of a circle defines a unique diameter and two unique semicircles, each on a different side of the diameter.

Let $O$ be the centre and $AO$ the finite radius of a circle $c$ [Ax. 8], and let $P$ be any of its points [Cr. 1].

Join $P$ with the centre $O$ of $c$ [Cr. 15], and produce $PO$ from $O$ by any given length greater than $OA$ to a point $P'$ [Cr. 16].

Since $OP' > OA$, $PP'$ is a straight line with points interior, as $O$, and exterior, as $P'$, to $c$ [Cr. 16, Df. 18], $PP'$ intersects $c$ at a point $Q$ [Cr. 33].

Since $O$ is a point of $PQ$ and $PQ$ is unique [Cr. 15], $PQ$ is the unique diameter defined by $P$ [Df. 18].

And being $c$ a simply closed line, $P$ and $Q$ are the common endpoints of two semicircles of $c$ [Dfs. 4, 18, Ax. 8], each on a different side of its diameter $PQ$ [Ax. 8].

**Fundamentals on angles**

**Definition 19** Rigid transformations of lines: metric and reversible displacements of lines that preserve the definition and the metric properties of the displaced lines, each of whose points moves from an initial to a final position along a fixed finite line called trajectory, in any of the two opposite directions defined by the endpoints of the trajectory. If all points of the displaced line, except at most one, move around a fixed point and their trajectories are segments of concentric and coplanar circles whose centre is the fixed point, the rigid transformation is called rotation.

**Definition 20** Superpose two adjacent lines: to place them with at least two common points by means of rotations around their common endpoint. Lines with at least two common points are said superposed.

**Definition 21** Angle.-Two straight lines are said to make an angle greater than zero iff they are adjacent, one of them can be superposed on the other by two opposite rotations around their common endpoint, and the other can be superposed on the one by the same two rotations, though in opposite directions. The least of the rotations, of both if they are equal, is said (convex) angle, the greater one is said concave angle. The angle is said zero iff both straight lines are superposed. The angle is said to be in the side of one of the adjacent straight lines where the other adjacent straight lines lies. The straight lines and their common endpoint are said respectively sides and vertex of the angle. A side is said to make an angle with the other at their common vertex. A line joining a point on each side of the angle is said to subtend the angle, its points are called interior. The non-interior points are called exterior to the angle.

**Definition 22** Adjacent angles and union angle.-Two angles are said adjacent iff they have the same vertex, a common side, the first angle superposes its non-common side on the common side, and the second angle superposes the common side on its non-common side, both angles in the same directions of rotation. The angle
that superposes the non-common sides of both angles in the same direction of rotation of both angles is their union angle, which can be concave. If two adjacent angles are equal to each other, they are said to bisect their union angle.

**Definition 23** Straight angle.-The angle that make the two sides of a point, except endpoints, in a straight line is said straight angle.

**Definition 24** Acute, obtuse and right angles.-If a straight line cuts another given straight line and makes with it at the intersection point two adjacent angles that are equal to each other, both angles are said right angles, in which case, and only in it, the two sides of each angle are said perpendicular to each other, and the first straight line is also said perpendicular to the given one. Angles less (greater) than a right angle are said acute (obtuse).

**Definition 25** Interior and exterior points and angles.-If two given coplanar straight lines are intersected by another coplanar straight line, said common transversal, a point of this transversal, different from the intersection points, is said interior to the given straight lines if it is between the intersection points of the transversal with both given straight lines; otherwise it is said exterior to them. Of the angles that the common transversal makes with the two given coplanar straight lines at their intersection points, those whose sides in the transversal have only exterior points are said exterior angles; and those whose sides in the transversal have interior points are said interior angles.

**Definition 26** Alternate, corresponding and vertical angles.-Of the angles that a common transversal makes with two coplanar straight lines, the angles of a couple of non-adjacent angles are said alternate if they are both interior, or both exterior, and they are in different sides of the transversal; and corresponding if they are in the same side of the transversal, being the one interior and the other exterior. Of the angles that two intersecting straight lines make with each other at their intersection point, the couples of angles with no common side are said vertical angles.

**Axiom 9** It is possible for two adjacent straight lines to make any angle at their common endpoint.

**Corollary 35** Two straight lines make an angle greater than zero iff they are adjacent, being equal and unique the angle that each of the straight lines make with the other at their common endpoint, though both rotations are in opposite directions. And the adjacency point is their only common point, even arbitrarily produced

- Each of two coplanar adjacent straight lines [Cr. 25],
  and only them, makes with the other the same angle greater than zero at their common endpoint, though in opposite directions [Df. 21, Ax. 9].

  And that angle is unique [Dfs. 19, 3].

  And the only common point of both sides, even arbitrarily produced, is the vertex of the angle, otherwise both sides would be superposed [Df. 20] and they would make an angle zero [Df. 21]. □

**Corollary 36** The superposition by rotation of two adjacent straight lines around their common endpoint is a unique straight line.

- It is an immediate consequence of [Df. 20, Cr. 18]. □

**Corollary 37** An angle does not change by producing arbitrarily its sides from their non-common endpoints.

- Let $AB$ and $AC$ be two adjacent straight lines [Cr. 25] that make an angle $\alpha > 0$ at $A$ [Cr. 35].

  Produce $AB$ from $B$ and $AC$ from $C$ by any given length respectively to the points $B'$ and $C'$ [Cr. 16].

  The rotation $\alpha$ superposes at least two points of $AB'$ and $AC'$, because it is the least rotation that superposes at least two points of $AB$ and $AC$ [Df. 21] and all points of $AB$ and $AC$ are points respectively of $AB'$ and $AC'$ [Cr. 16, Df. 2].

  On the other hand, if a rotation $\alpha'$ less than $\alpha$ superposes $AB'$ on $AC'$ but not $AB$ on $AC$, the non-superposed (non-common) points of $AB$ and $CD$ would be between then common point $A$ of $AB'$ and $AC'$ and any other superposed (common) point of $AB'$ and $AC'$ [Df. 20], which is impossible [Df. 10].

  So, $AB'$ and $AC'$ also make at $A$ an angle $\alpha$. □
Corollary 38  Three adjacent straight lines define three angles at their common endpoint. And two intersecting straight lines define with each other at most four angles at their intersection point.

Three coplanar straight lines $AB$, $AC$ and $AD$ adjacent at the same point $A$ [Cr. 27] define three couples of coplanar straight lines adjacent at that point: $AB, AC$; $AB, AD$; and $AC, AD$ [Df. 4].

So, $AB, AC$ and $AD$ define three angles at that point $A$ [Cr. 35].

For the same reason, two intersecting straight lines define at most four angles whose two sides are not in the same straight line. □

Corollary 39 (Fig. 5) Three straight lines adjacent at the same point define a couple of adjacent angles at that point.

(Fig. 5) Three straight lines $r_1$, $r_2$, $r_3$ adjacent at $V$ define three angles $\alpha$, $\beta$ and $\gamma$ at $V$ [Cr. 38], and then three couples of angles: $\alpha$ and $\beta$; $\alpha$ and $\gamma$; and $\beta$ and $\gamma$. Being only three sides, the two angles of each of such couples must have a common side [Df. 21].

The angles of such couples that superpose their common side on their respective non-common sides can only be rotations either in the same or in opposite directions [Dfs. 1, 19].

In the second case, both angles are adjacent because any of them superposes its non common side on the common side in the same direction the other superposes the common side on its non-common side [Dfs. 21, 22]. In the first case both rotations, for instance* $\alpha$ and $\beta$, will be different [Dfs. 21, 19, 3]; and one of them, for instance* $\alpha$, will be less than the other [Ps. A].

Assuming* $\alpha$ superposes $r_1$ on $r_2$, and $\beta$ superposes $r_1$ on $r_3$ in the same direction of rotation, $\gamma$ can only be the angle that superposes $r_2$ on $r_3$. Hence, $\alpha$ and $\gamma$ are adjacent [Df. 22]. □

Corollary 40  Two adjacent straight lines make a straight line iff they make a straight angle at their common endpoint.

(Fig. 5) Three straight lines $r_1$, $r_2$, $r_3$ adjacent at $V$ make a straight angle $\alpha > 0$ [Ax. 9], and then three couples of angles: $\alpha$ and $\beta$; $\alpha$ and $\gamma$; and $\beta$ and $\gamma$. Being only three sides, the two angles of each of such couples must have a common side [Df. 21].

The angles of such couples that superpose their common side on their respective non-common sides can only be rotations either in the same or in opposite directions [Dfs. 1, 19].

In the second case, both angles are adjacent because any of them superposes its non common side on the common side in the same direction the other superposes the common side on its non-common side [Dfs. 21, 22]. In the first case both rotations, for instance* $\alpha$ and $\beta$, will be different [Dfs. 21, 19, 3]; and one of them, for instance* $\alpha$, will be less than the other [Ps. A].

Assuming* $\alpha$ superposes $r_1$ on $r_2$, and $\beta$ superposes $r_1$ on $r_3$ in the same direction of rotation, $\gamma$ can only be the angle that superposes $r_2$ on $r_3$. Hence, $\alpha$ and $\gamma$ are adjacent [Df. 22]. □

Corollary 41  A non-straight angle has a number of interior points and exterior points greater than any given number, and none of them is in a straight line with the sides of the angle.

Let $AB$ and $AC$ be two straight lines in a plane $Pl$, which are adjacent at $A$, are not in a straight line with each other [Cr. 25], and make any non-straight angle $\alpha > 0$ [Ax. 9].

Let $P$ be any point of $AB$ between the endpoints $A$ and $B$. [Ax. 2] and let $Q$ be any point of $AC$ between the endpoints $A$ and $C$. [Ax. 2]. Join $P$ and $Q$ [Cr. 15].

Produce $PQ$, for example* from $P$, by any length to a point $P'$ [Cr. 16].
$PQ$ has a number of points between $P$ and $Q$ greater than any given number [Cr. 5]
and all of them are interior to the angle $\alpha$ [Df. 21]
and none of them is in a straight line with $AB$, otherwise $AB$ and $P'Q$ would be in the same straight line. [Cr. 18],
$Q$ would be in straight line with $AB$, and $AC$ would be in straight line with $AB$ [Df. 11]
which is not the case. For the same reason, no interior point of $PQ$ is in a straight line with $AC$ [Cr. 18].
$PP'$ has a number of points between $P$ and $P'$ greater than any given number [Cr. 5],
all of which are exterior to the angle $\alpha$ Df. 21],
and for the same reason as with the interior points, none of them is in a straight line with $AB$ nor with $AC$ [Cr. 18]. □

FUNDAMENTALS ON POLYGONS

Definition 27 Polygon: Three or more finite coplanar straight lines, called sides, each of which is adjacent at
each of its two endpoints, called vertexes, to just one of the others, being not in straight line with each other,
and being their common endpoints their only intersection points, are said to make a polygon. Two sides of the
same or of different polygons are said equal iff they have the same length. Two polygons are said adjacent iff
they have a common side; opposite iff they have two opposite angles at a common vertex; similar iff the angles
of the one are equal to the angles of the other; and equal if they are similar and the sides of each angle of the
one are equal to the sides of the corresponding equal angle of the other. Polygons with at least one concave angle
are said concave. The angle each side makes with the production of another adjacent side is said exterior. A
straight line joining two points not in the same side of a polygon is a diagonal of the polygon. A diagonal bisects
a polygon iff it is the common side of two adjacent polygons with the same area.

Note the classical definition of diagonal is a particular case of the above general definition of diagonal.

Definition 28 Triangles and quadrilaterals. A polygon of three (four) sides is a triangle (quadrilateral). A
triangle (quadrilateral) is said equilateral if its three (four) sides are equal to one another. A triangle is said
isosceles if it has two equal sides; and scalene if the three of them are unequal. If one of its angles is a right
angle, it is said a right-angled triangle. A rectangle is a quadrilateral all of whose angles are right angles. An
equilateral rectangle is a square. And a parallelogram is a quadrilateral with two couples of equal and parallel
sides. Polygons with more than four sides are named pentagons, hexagons, heptagons etc. A polygon is said to
tie between two given lines iff its vertexes are in the given straight lines or in straight lines whose endpoints are
point of the given straight lines.

Axiom 10 The area of a polygon is greater than zero, and is the sum of the areas of the two adjacent polygons
defined by any of its diagonals. Equal polygons have equal areas.

Corollary 42 Any two adjacent sides of a polygon make an angle greater than zero at their common endpoint,
and the polygon has as many angles as sides.

> Being coplanar all sides of a polygon [Df. 27],
each couple of its adjacent sides makes a unique angle greater than zero at their common endpoint [Cr. 35].
So, the polygon has as many angles as couples of adjacent sides. Since each couple of adjacent sides is defined
by two adjacent sides, and each side defines two of such couples, one at each of its two endpoints [Df. 27],
the polygon has as many angles as sides. □

ON ANGLES AND TRIANGLES

In this section, all points and lines will be coplanar and, unless otherwise indicated, all angles will be greater
than zero. For the sake of brevity, three of the first 16 propositions of Euclid’s Elements (Book I) will be
used in this section, they will be referred to as [EBI, n], where $n$ is the proposition number [3]. The proofs of
these Euclid’s propositions make use of other two Euclid’s propositions not proved here either. The definitions,
common notions and postulates (first, second and third) used in the proofs of these 5 Euclid’s propositions are
also formal elements of the foundational basis of Euclidean geometry (Section 4). So, they can also be proved
within it, even in different and more formally detailed forms. They will be assumed to follow [Cr. 45] and to
precede [Pr. 10].

Proposition 1 (Euclid’s Proposition 3 extended) To take a point in a given finite straight line, produced if
necessary, at any given finite distance from a given point of the given straight line, and in any given direction
of the two opposite directions of the given straight line.

Figure 6 – Proposition 1

Let $AB$ be a finite straight line [Cr. 23];
$P$ the given point of $AB$ [Cr. 5];
$CD$ the given finite distance [Df. 14];
and the given direction in $AB$, for example, from $B$ to $A$ [Ax. 2, Df. 1].
Produce $AB$ from $A$ by the given distance $CD$ to a point $A'$ [Cr. 16].
With centre $P$ and radius $CD$ draw the circle $c$ [Ax. 8].
$P$ is interior to $c$ [Df. 18],
and being $PA' > AA'$ [Cr. 13]
and $AA' = CD$, it holds $PA' > CD$ [Ps. B].
Therefore, $A'$ is exterior to $c$ [Df. 18].
Hence, there is an intersection point $Q$ of $c$ and $BA'$ [Cr. 33].
$Q$ is in $AB$ [Df. 3],
whether or not produced, at the given finite distance $CD$ from the point $P$ of $AB$ [Df. 18];
and in the given direction from $B$ to $A$. □

From now on, to take a point in a straight line at a given finite distance from one of its points will always mean

to take the point in that straight line produced if necessary [Pr. 1]. And the distance between two points will
always be finite, which, in addition, is founded on the (non-strictly geometric) proposition [Pr. 13] that will be
proved in the next chapter.

**Proposition 2** All straight angles are equal to one another.

Figure 7 – Proposition 2

Let $P$ and $Q$ be any two points respectively of any two straight lines $AB$ and $CD$ [Crss. 1, 27],
and $\sigma$ and $\sigma'$ the respective straight angles that $PA$ makes at $P$ with $PB$, and $QC$ makes at $Q$ with $QD$ [Cr. 40].
Assume $\sigma' \neq \sigma$. In such a case a straight line $PE$ adjacent at $P$ to $PA$ and making an angle $\sigma'$ at $P$ with $PA$
is possible [Ax. 9].
Being each angle unique [Cr. 35]
and \( \sigma \neq \sigma' \), \( PE \) and \( PB \) will not be superposed [Df. 21]
and they will be adjacent at \( P \) [Dfs. 20, 4].

\( PA \) and \( PB \) are the two sides of \( \sigma \); and \( PA \) and \( PE \) the two sides of \( \sigma' \). Being \( \sigma \) and \( \sigma' \) straight angles, \( AB \) and \( AE \) are straight lines [Cr. 40];
\( AP \) is a common segment of them [Cr. 5]
and \( PE \) and \( PB \) are non-common segments of them [Dfs. 4, 3]
Consequently, \( AB \) and \( AE \) are locally collinear [Dfs. 2, 3],
which is impossible [Df. 10]
So, it is impossible for \( PB \) and \( PE \) to be adjacent at \( P \). The assumption \( \sigma' \neq \sigma \) is, then, impossible. And it can be concluded that all straight angles are equal to one another. \( \square \)

**Proposition 3** The union angle of two adjacent angles is the sum of both adjacent angles and is greater than each of them.

\[
\begin{align*}
\alpha + \beta &> \beta + \alpha \\
&> \alpha + \beta
\end{align*}
\]

**Proposition 4** (A variant of Euclid’s Proposition 13) If a straight line makes with another straight line two adjacent angles, these angles can be either equal or unequal to each other, and they always sum a straight angle.

\[
\begin{align*}
DA \text{, } DC \text{ and } DB \text{ are straight lines } & [Cr. 14] \\
\text{adjacent at } D \text{ } & [Df. 4].
\end{align*}
\]

So, \( DA \) makes at \( D \) with \( DC \), and \( DC \) at \( D \) with \( DB \) two adjacent angles \( \alpha \) and \( \beta \) [Cr. 39]
of which \( D \) is the common vertex and \( DC \) the common side [Df. 22];
\( \alpha \) and \( \beta \) can be either equal or unequal to each other [Ax. 9, Ps. A].
On angles and triangles

and their union angle $\sigma$ is the rotation $\alpha + \beta$ around $D$ [Pr. 3]

that in the same direction of rotation of $\alpha$ and $\beta$ superposes the non-common sides $DA$ and $DB$ respectively of $\alpha$ and $\beta$ [Df. 22],

and being $DA$ and $DB$ the two sides of $D$ in the straight line $AB$ [Df. 5, Ax. 4],

$\sigma$ is a straight angle [Cr. 40, Df. 23].

Therefore $\alpha$ and $\beta$ sum a straight angle [Pr. 3]. □

**Proposition 5** (Euclid’s Proposition 15) *The two angles of any couple of vertical angles are equal to each other.*

Proposition 6 Three given points define a triangle iff they are not in straight line, being the vertexes of the triangle the given points and its sides the straight lines joining them. A point defines a triangle with any two points iff it is a non-common points of on of the sides of the straight line joining the points.

Let $A$, $B$ and $C$ be any three points not in straight line [Cr. 20].

There is a plane $Pl$ that contains them [Ax. 6].

Join $A$ with $B$; $B$ with $C$; and $C$ with $A$ [Cr. 15].

$AB$, $BC$ and $AC$ are in $Pl$ [Cr. 23].
And none of them is in straight line with any of the others, otherwise \(A, B\) and \(C\) would be in straight line [Df. 11], which is not the case. \(B\) is the only common point of \(AB\) and \(BC\), otherwise they would be in straight line [Cr. 18] and \(A, B\) and \(C\) would be in straight line [Df. 11], which is not the case. So, \(AB\) and \(BC\) are adjacent at \(B\) [Df. 4]. For the same reason \(BC\) is adjacent at \(C\) to \(AC\), and \(AC\) adjacent at \(A\) to \(AB\). So, each of them is adjacent at each of its two endpoints to one, and only to one, of the others [Df. 4]. Therefore, \(A, B\) and \(C\) define a triangle \(ABC\) whose vertexes are \(A, B\) and \(C\) and whose sides are \(AB, BC\) and \(CA\) [Dfs. 28, 27]. Alternatively, if \(ABC\) is a triangle, its vertexes cannot be in straight line, otherwise they would be in the same straight line and \(ABC\) would not be a triangle [Df. 27, 28]. On the other hand, if \(P\) is any non-common point of one of the sides of a straight line \(l\) in \(Pl\), it cannot be in straight line with any couple of points \(Q\) and \(R\) of \(l\), even arbitrarily produced [Df. 13], so \(P, Q\) and \(R\) define a triangle, as it has just been proved. And if a point defines a triangle with any two points, it cannot be in straight line with these two points [Df. 27, 28] so, it cannot be a common point of the sides of the straight line through those points [Df. 11], and it must be a non-common point of one of such sides [Cr. 24]. □

**Corollary 43** The intersection point of two intersecting straight lines defines a triangle with any couple of points, each of one the intersecting straight lines.

▷ It is an immediate consequence of [Cr. 19, Df. 11, Pr. 6] □

**Corollary 44** A point of a side between two vertexes of a triangle is not in straight line with any of the other sides of the triangle.

▷ If a point \(P\) of a side* \(AB\) of a triangle \(ABC\) were in straight line with the side* \(BC\), these two sides would have two common points, \(B\) and \(P\), and they would belong to the same straight line [Cr. 18], which is impossible [Pr. 6]. □

**Proposition 7** If the length of a given line joining the centers of two circles is less than the sum of their radii, and each radius is less than the sum of the other radius and the length of the given line, then the circles intersect at two points, each on a different side of the given line and not in a straight line with it.

▷ Let \(AB\) be any straight line [Cr. 23], \(P\) any point of \(AB\) different from \(A\), and \(Q\) any point between \(AP\) different from \(P\) [Cr. 5]. Consider first the case \(P \neq B \neq Q \neq P\). In this case, \(A, B, P\) and \(Q\) satisfy \(AP < AB, BQ < AB\); and also \(AB = AQ + QB < AQ + QP + QB = AP + QB\) [Cr. 13, Pss. B, A]. Draw the circle \(c_1\) with centre \(A\) and radius \(AP\); and the circle \(c_2\) with centre \(B\) and radius \(BQ\) [Ax. 8]. Produce \(AB\) from \(A\) to a point \(P'\) such that \(AP' = AP\) [Pr. 1].

Figure 11 – Proposition 7
$P'P$ is a diameter of $c_1$ [Df. 18]
that define two semicircles $l$ and $l'$ of $c_1$ each on a different side of $AP$ [Cr. 34].
$P$ and $P'$ are the only points of $PP'$, even arbitrarily produced, at a distance $PA$ from $A$ [Cr. 13].
So, except $P$ and $P'$, no point of $l$ and $l'$ is in straight line with $AB$ [Dfs. 18, 11].
On the other hand, $P$ is interior to $c_2$ because $BP < BQ$ [Cr. 13, Df. 18],
and $P'$ is exterior to $c_2$ because $BP' > BQ$ [Cr. 13, Df. 18].
In consequence, $l$ and $l'$ intersect $c_2$ at two points $R$ and $S$ [Cr. 33],
none of which is in straight line with $AB$, otherwise $P$ and $Q$ would be the same point of $AB$ and $AP + QB = AB$ [Cr. 13],
which is not the case. The same above argument applies to the cases $P = B$ and/or $Q = A$, i.e the cases in which two, or three, of the straight lines $AB$, $AP$ and $BQ$ are equal. □

**Proposition 8** If two triangles have equal one of its sides and the two angles whose respective vertexes are the endpoints of that side, then the other two sides of each triangle are also equal to the corresponding two sides of the other.

\[\begin{align*}
\text{Figure 12 – Pr. 8}
\end{align*}\]

▷ Let $ABC$ be a triangle [Pr. 6]
with an angle $\alpha$ at $A$ and an angle $\beta$ at $B$ [Cr. 42].
Assume it is possible a triangle $ABC'$ with a side $AB$, an angle $\alpha$ at $A$, and an angle $\beta$ at $B$, and the side $BC'$ of $\beta$ different from $BC$, for instance $BC' < BC$ [Ps. A].
Being $ABC$ a triangle, $C$ is the only intersection point of $BC$ and $AC$ [Dfs. 27, 28],
even arbitrarily produced [Cr. 19]
So, $A$, $C'$ and $C$ are not in straight line [Df. 11]
and $AC'C$ is a triangle [Pr. 6].
Being $ABC$, $ABC'$ and $AC'C$ triangles, $AB$, $AC'$ and $AC$ are adjacent at $A$ [Df. 28]
where they make the adjacent angles $\alpha_1$ and $\alpha_2$ [Df. 22, Cr. 39]
whose union angle is the angle $\alpha$ that $AB$ makes at $A$ with $AC$ [Pr. 3],
and $\alpha_1 < \alpha$ [Pr. 3].
So, it is impossible a triangle with a side $AB$, an angle $\alpha$ at $A$, an angle $\beta$ at $B$ and a side $BC' \neq BC$. The same argument applies to the side $AC$. □

**Proposition 9** (Hilbert’s Axiom IV.6) *If two triangles have equal one of their angles and the two sides of that angle, then they have also equal their corresponding other two angles.*

▷ Let $ABC$ be a triangle [Pr. 6],
and $\alpha$, $\beta$ and $\gamma$ its corresponding angles respectively at $A$, $B$ and $C$ [Cr. 42].
A triangle $ABC'$ with an angle $\alpha$ at $A$, a side $AB$, a side $AC$ and an angle $\beta'$ at $B$ different from $\beta$ is impossible because, being $\beta$ unique [Cr. 35],
$\beta' \neq \beta$ implies that $\beta'$ will not superpose $BA$ on $BC$ but on $BC''$, where $C''$ is a point of $AC$, whether or not produced, different from $C$ [Df. 20],
otherwise $BC$ and $BC'$ would be the same straight line [Cr. 15].

So, $AC' \neq AC$ [Cr. 13].

For the same reasons it is impossible a triangle with a side $AB$, a side $AC$ an angle $\alpha$ at $A$ and an angle $\gamma'$ at $C$ such that $\gamma' \neq \gamma$. □

**Corollary 45** (Euclid’s Proposition 4) If two triangles have equal one of their corresponding angles and the two sides of that angle, then they have also equal their corresponding other two angles and their corresponding third side.

▷ It is an immediate consequence of [Prs. 9, 8]. □

**Proposition 10** (Euclid’s Proposition 8) If the three sides of a triangle are equal to the three sides of another triangle, then the three angles of the one are also equal to the corresponding three angles of the other.

▷ Let $ABC$ and $ABC'$ be two triangles [Pr. 6]

with a common side $AB$ and such that $BC = BC'$, $AC = AC'$. Assume $\beta' \neq \beta$. The angle $\beta'$ will not superpose $BA$ on $BC$ but on $BC'$ [Cr. 35],

where $C'$ can only be different from $C$ [Cr. 35].

Join $C$ and $C'$ [Cr. 15].

$C'$ cannot be in straight line with $A$ and $C$, otherwise it would be a point of $AC$, whether or not produced [Df. 11, Cr. 16],

different from $C$, and then $AC \neq AC'$ [Cr. 13],

which is not the case. So, $AC'C$ is an isosceles triangle [Cr. 6, Df. 28].

For the same reasons, $BC'C$ is an isosceles triangle. Since $ABC'$, $AC'C$ and $BC'C$ are triangles [Pr. 6], $C'B$, $C'A$ and $C'C$ are adjacent at $C'$ [Dfs. 28, 27].

Since $ABC$, $AC'C$ and $BC'C$ are triangles [Pr. 6],

$CA$, $CB$ and $CC'$ are adjacent at $C$ [Dfs. 28, 27].

So, $\gamma' + \delta' > \delta'$ and $\gamma + \delta > \delta$ [Cr. 39, Pr. 3].

On the other hand, in $BC'C$ it holds $\gamma' + \delta' = \delta$ [EBI, 5], and in $AC'C$: $\gamma + \delta = \delta'$ [EBI, 5].

In consequence, $\delta' > \delta$ and $\delta > \delta'$ [Ps. A],
which is impossible [Ps. A].

Therefore, the initial assumption is false, and the same rotation $\beta$ that superposes $AB$ on $BC$ superposes $AB$ on $BC'$ [Df. 20],

so that $\beta = \beta'$ [Df. 21].

And the other two angles of $ABC'$ are equal to the angles $\alpha$ and $\gamma$ of $ABC$ [Cr. 45]. □

**Proposition 11** (Euclid’s Propositions 18 and 19) *A side is the greatest of a triangle iff it subtends its greatest angle.*

(Fig. 15, left) Consider any two sides $AB$ and $AC$ of a triangle $ABC$ [Pr. 6], and assume* $AB < AC$ [Ps. A].

On $AC$ take a point $D$ such that $AD = AB$ [Pr. 1] and join $D$ with $B$ [Cr. 15].

$D$ is not in straight line with $AB$ nor with $BC$ [Cr. 44].

So, $ABD$ and $DBC$ are triangles [Pr. 6].

$ABD$ is isosceles [Df. 28], and then $\delta = \delta'$ [EBI, 5].

It also holds $\delta > \gamma$ [EBI, 16].

Since $ABC$, $ABD$ and $DBC$ are triangles, $BA$, $BD$ and $BC$ are adjacent at $B$ [Dfs. 28, 27], where they make the adjacent angles $\delta'$ and $\epsilon$ [Cr. 39] whose union angle is $\beta$ [Df. 22, Pr. 3].

Therefore, $\beta > \delta'$ [Pr. 3].

From $\beta > \delta'$, $\delta' = \delta$ and $\delta > \gamma$, it follows $\beta > \gamma$ [Ps. B].

Being $AB$ and $AC$ any two sides of $ABC$, it can be concluded that in a triangle the greatest side subtends the greatest angle [Df. 21].

Let now $\phi$ and $\varphi$ (Fig. 15, right) be any two angles of a triangle $EFC$ [Pr. 6, Cr. 42] and assume $\phi > \varphi$ [Ps. A].

$EG$ cannot be equal to $EF$, otherwise $\phi = \varphi$ [EBI, 5], which is not the case. Nor can it be less than $EF$, because in such a case the least side would subtend the greatest angle, which is impossible, as just proved. Therefore, $EG$ must be greater than $EF$ [Ps. A].

Being $\phi$ and $\varphi$ any two angles of a triangle, we conclude that the greatest angle is subtended by the greatest side. □
Proposition 12 (Euclid’s Proposition 20) In a triangle the sum of the lengths of any two of its sides is greater than the length of the remaining one.

Let $ABC$ be a triangle [Pr. 6].

Produce $AB$ from $A$ [Cr. 16]

to a point $D$ such that $AD = AC$ [Pr. 1].

Join $D$ and $C$ [Cr. 15].

Since $A$, $B$ and $C$ are not in straight line [Pr. 6],
the point $D$ of the production of $AB$ from $B$ cannot be in straight line with $B$ and $C$ [Df. 11],
and $DAC$ and $DBC$ are triangles [Pr. 6].

And being $DAC$ isosceles [Df. 28],

it holds $\delta = \delta'$ [EBI, 5].

Since $ABC$, $DBC$ and $DAC$ are triangles,
$CB$, $CA$ and $CD$ are straight lines adjacent at $C$ [Dfs. 28, 27];

where they make the adjacent angles $\gamma$ and $\delta'$ [Cr. 39]

whose union angle is $\phi$ [Df. 22, Pr. 3].

Therefore, $\phi > \delta'$ [Pr. 3].

Hence, $\phi > \delta$ [Ps. A].

In consequence $DB > BC$ [Pr. 11],
and then $AB + AD > BC; AB + AC > BC$ [Ps. A].

The same argument proves the sum of the lengths any other couple of sides of $ABC$ is greater than the length of the remaining one. □

On distances and perpendiculars
The next set theoretical proposition is not strictly necessary under the hypothesis that the length of a straight line $AB$ is finite as long as it has two definite endpoints $A$ and $B$. Here that hypothesis is not assumed but formally demonstrated. Although not developed here, the proposition has important consequences about finitism in geometry [4].

Proposition 13 In the Euclidean space $\mathbb{R}^3$, the length of a line with two endpoints is always finite. And the distance between any two given points is always finite and unique.

Let $AB$ be any line in the Euclidean space $\mathbb{R}^3$, and $\lambda > 0$ any finite length. Assume it is possible to define in $AB$ an infinite partition $P = AP_1, P_1P_2, P_2P_3, \ldots$ whose parts have, all of them, the same length $\lambda$. A point $x$ such that $xB < \lambda$ can only belong to a last part $P_{\omega}B$ of $P$. A point $y$ of $AP_{1<\omega}$ and a point $z$ of $P_{1<\omega}B$ can only belong respectively to $P_{1<\omega}P_{1<\omega}$ and $P_{1<\omega}P_{\omega+1}$. So, $P$ has a first element $AP_1$, a last element $P_\omega B$, and each element has an immediate predecessor (except $AP_1$), and an immediate successor (except $P_\omega B$). In addition, any subset $P'$ of $P$ containing for instance the element $P_\omega P_{\omega+1}$ will also contain a first element: one of the elements $AP_1, P_1P_2, \ldots, P_{\omega}P_{\omega+1}$. So, $P$ is well ordered and has an infinite ordinal $\alpha$ [2, p. 152]. Since $P$ has a last element $P_\omega B$ and $\omega$-ordered sets do not have last element, if $\alpha$ is infinite it must be greater than $\omega$, in whose case $P$ would have an $\omega$th element $P_\omega P_{\omega+1}$ (Theorem of the $\omega$th Term). But any point $u$ such that $uP_\omega < \lambda$ can only belong to the impossible immediate predecessor of $P_\omega P_{\omega+1}$. In consequence, $AB$ cannot be
On distances and perpendiculars

partitioned in a infinite number of parts of the same finite length, whatsoever it be. Therefore, it can only be partitioned in a finite number of parts of the same finite length. And being finite the sum of any finite number of finite lengths, $AB$ has a finite length. □ □

Note.-From now on, to produce a straight line will always mean to produce a straight line by a finite length. And the distances between points and between points and lines will always be finite.

**Proposition 14**  
*The length of straight line joining any two points interior to a circle is less than the sum of the lengths of two radii of the circle.*

![Figure 17 – Pr. 14](image)

Let $c$ be a circle with a centre $O$ and a finite radius $OA$ [Ax. 8, Pr. 13], and $P$ and $Q$ any two points interior to $c$ [Cr. 32].

It must be $OA < OA + OA$, otherwise $OA \geq OA + OA$ [Ps. A], and $0 \geq OA$ [Ps. B], which is impossible [Cr. 13].

Join $O$ with $P$ and with $Q$; and join $P$ with $Q$. [Cr. 15].

If $O, P$ and $Q$ are in straight line (Fig. 17, left), one of them will be between the other two [Cr. 9].

If $O$ is between $P$ and $Q$ then $PQ = OP + OQ$ [Cr. 13].

And being $OP < OA$, $OQ < OA$, it holds: $OP + OQ < OA + OQ$; $OQ + OA < OA + OA$ [Ps. B].

Therefore, $OP + OQ < OA + OA$, $PQ < OA + OA$ [Ps. B, A].

If $P$ is between $O$ and $Q$ it holds $PQ < OQ$ [Cr. 13]

and being $OQ < OA$ [Df. 18]

and $OA < OA + OA$, it must be $PQ < OA + OA$ [Ps. B].

The same argument applies if $Q$ is between $O$ and $P$. If $O$, $P$ and $Q$ are not in straight line (Fig. 17 right) they define a triangle $OPQ$ [Pr. 6]

in which it holds $PQ < OP + OQ$ [Pr. 12].

Being $OP < AO$; $OQ < AO$ [Df. 18],

and for the same reasons above, $PQ < OA + OA$ [Ps. B].

In consequence, the length $PQ$ is always less than the sum of the lengths of two of its radii. □

**Proposition 15**  
*(Euclid’s Proposition 11)* *Through a given point of a given straight line to draw a perpendicular to the given straight line.*

Let $AB$ be any given straight line [Cr. 23] and $P$ any given point of $AB$ [Cr. 1].

Assume* $PB < PA$. Take a point $C$ in $PA$ such that $PC = PB$ [Pr. 1].

With centers $C$ and $B$ and the same radius $CB$ draw the respective circles $c_1$ and $c_2$ [Ax. 8], which intersect at two points $Q$ and $R$ not in straight line with $CP$ [Pr. 7].

Join $Q$ with $C$, with $P$ and with $B$ [Cr. 15].

$QCP$ and $QPB$ are triangles because $Q$ is not in straight line with $C$, $P$, $B$ [Pr. 6],
and the three sides of $QCP$ are equal to the three sides of $QPB$.
Therefore $\rho = \rho'$ [Pr. 10].
And being $PC$, $PQ$ and $PB$ adjacent at $P$ [Dfs. 27, 28],
$P$ is the common vertex and $PQ$ the common side of $\rho y \rho'$. So, $\rho y \rho'$ are adjacent angles [Cr. 39, Df. 22],
and since they are equal to each other, they are right angles [Df. 24].
And $PQ$ is perpendicular to $AB$ through the given point $P$ [Df. 24]. □

**Proposition 16** (A variant of Euclid’s Proposition 12) *From a given point not in straight line with a given straight line, to draw a perpendicular to the given straight line, produced if necessary.*

![Figure 18](image18.png)

Let $AB$ be a straight line [Cr. 23]
and $P$ a point not in straight line with $AB$ [Cr. 20].
Take any point $Q$ in $AB$ [Cr. 1].
Join $P$ and $Q$ [Cr. 15]
and produce $PQ$ from $Q$ by any given length to a point $R$ [Cr. 16].
With centre $P$ and radius $PR$ draw the circle $c$ [Ax. 8].
Since $PQ$ is less than $PR$ [Cr. 13],
$Q$ is interior to $c$ [Df. 18].
In $AB$ and in the direction from $B$ to $A$, take a point $A'$ at a distance $PR + PR$ from $Q$ [Pr. 1].
Being $QA' = PR + PR$ and $Q$ interior to $c$, $A'$ cannot be interior to $c$ [Pr. 14].
So, it will be either a point of $c$ or exterior to $c$ [Cr. 32],
and in both cases there will be an intersection point $D$ of $c$ and $QA'$ [Cr. 33, Df. 3].
The same argument applied to the direction from $A$ to $B$ proves the existence of another intersection point $E$ of $AB$, produced from $B$ if necessary, and $c$. Join $P$ with $D$ and with $E$ [Cr. 15].

Bisect $DE$ at $S$ [EBI, 10],

where $S$ could also be the point $Q$, and join $S$ with $P$ [Cr. 15].

Being not $P$ in straight line with $AB$, it is not in straight line with any two points of $AB$ [Df. 11], whether or not produced [Cr. 16].

So, $PDS$ and $PSE$ are triangles [Pr. 6] with a common side $PS$, being also $PD = PE$ [Df. 18], and $SD = SE$ [Df. 8].

So, $\rho = \rho'$ [Pr. 10].

$SD$, $SP$ and $SE$ are adjacent at $S$ because $PDS$ and $PSE$ are triangles [Dfs. 28, 27].

So, $SD$, $SP$ and $SE$ make two adjacent angles $\rho$ and $\rho'$ at their common point $S$ [Cr. 39], which being equal, are right angles [Df. 24], and $SP$ is the perpendicular from $P$ to $AB$ [Df. 24], produced if necessary. \(\square\)

**Proposition 17** (Euclid’s Postulate 4) All right angles are equal to one another.

\(\triangleright\) Let $DC$ be a straight line perpendicular to another straight line $AB$ at any point $D$ of $AB$ [Pr. 15], and let $\rho_1$ and $\rho'_1$ be the respective adjacent right angles [Df. 24] that $DC$ makes at $D$ with $DA$, and $DC$ at $D$ with $DB$ [Cr. 39].

Since $DA$, $DC$ and $DB$ are adjacent at $D$ [Cr. 35], the union angle of $\rho_1$ and $\rho'_1$ is the angle that, in the same direction of rotation of $\rho$ and $\rho'_1$, superposes the non-common sides $DA$ and $DB$ respectively of $\rho_1$ and $\rho'_1$ [Df. 22, Pr. 3], which are the sides of the straight angle $\sigma_1$ that $DA$ makes at $D$ with $DB$ [Df. 23], and then $\sigma_1 = \rho_1 + \rho'_1$ [Pr. 3].

So then, any two adjacent right angles sum a straight angle [Pr. 2].

Let $\rho_2$, $\rho'_2$ be any other couple of adjacent right angles. As just proved, they sum a straight angle $\sigma_2$. Since $\sigma_1 = \sigma_2$ [Pr. 2], it holds $\rho_1 + \rho'_1 = \rho_2 + \rho'_2$ [Ps. B].

Assume $\rho_1 < \rho_2$. We would have $\rho_1 + \rho'_1 < \rho_2 + \rho'_2$ [Ps. B], and being $\rho_1 + \rho'_1 = \rho_2 + \rho'_2$ [Pr. 2], we can write $\rho_2 + \rho'_2 < \rho_2 + \rho'_1$ [Ps. A], and then $\rho'_2 < \rho'_1$ [Ps. B].

And being $\rho'_1 = \rho_1$ and $\rho'_2 = \rho_2$ [Df. 24], we get $\rho_2 < \rho_1$ [Ps. A], which contradicts our assumption. So $\rho_1$ cannot be less than $\rho_2$. The same argument applied to the assumption $\rho_1 > \rho_2$ proves $\rho_1$ cannot be greater than $\rho_2$ either. So it must be equal to $\rho_2$ [Ps. A].

Hence, all right angles, whether or not adjacent, are equal to one another. \(\square\)

**Corollary 46** A right angle is greater than zero. And two right angles sum a straight angle.

\(\triangleright\) It is an immediate consequence of [Cr. 35, Prs. 4, 17] \(\square\)

**Corollary 47** If one of the four angles that two intersecting straight lines make with each other at their intersection point is a right angle, then the other three angles are also right angles; the two sides of each angle are perpendicular to each other; and each straight line is perpendicular to the other.

\(\triangleright\) It is an immediate consequence of [Df. 24, Prs. 17, 5]. \(\square\)
Corollary 48 The two opposite rotations that superpose the two sides of a straight angle are equal to each other.

It is an immediate consequence of [Crs. 46, 47]. □

Proposition 18 If the two adjacent angles that a straight line makes with another intersecting straight line at their unique intersection point are different from each other, then the one is acute and the other is obtuse.

Let $D$ be the unique intersection point of two straight lines $AB$ and $CD$ [Cr. 19].

$DA$, $DC$ and $DB$ are straight lines [Cr. 14] adjacent at $D$ [Df. 4],

where they make two adjacent angles $\alpha$ and $\beta$ [Cr. 39] that sum two right angles [Pr. 4].

If $\alpha \neq \beta$ one of them, for instance $\alpha$, will be less than the other, $\beta$ [Ps. B].

The angle $\alpha$ must be less than a right angle, otherwise, and being $\alpha < \beta$, the sum $\alpha + \beta$ would be greater than two right angles, which is impossible [Pr. 4].

So, $\alpha$ must be an acute angle. For the same reason, $\beta$ must be an obtuse angle [Df. 24]. □

Proposition 19 (Euclid’s Proposition 17) Any two angles of a triangle sum less than two right angles.

Let $ABC$ be a triangle [Pr. 6].

![Figure 20 – Pr. 19](image)

Produce the side $BC$ from $C$ by any given length to a point $D$ [Cr. 16].

$CB$ and $CA$ are adjacent at $C$ [Dfs. 27, 28];

$CB$ and $CD$ are adjacent at $C$ [Cr. 16];

$C$ is the only common point of $AC$ and $BD$, otherwise $CB$ and $CA$ would be in straight line [Cr. 18], which is not the case [Df. 28].

So, $CA$ and $CD$ are adjacent at $C$ [Df. 4].

Consider the exterior angle $\delta$ [Df. 27].

It holds $\beta < \delta$ [EBI, 16].

Hence, $\beta + \gamma < \delta + \gamma$ [Ps. B].

And being $\delta + \gamma$ a straight angle [Pr. 4],

which equals two right angles [Cr. 46],

we conclude that $\beta$ and $\gamma$ sum less than two right angles. The same argument proves that any other couple of angles of $ABC$ sum less than two right angles. □

Proposition 20 From a point, whether or not in straight line with a given straight line, only one perpendicular can be drawn to the given straight line.

Let $AB$ be a straight line [Cr. 23]

and $P$ any point not in straight line with $AB$ [Cr. 20].

$P$ will be a non-common point of one of the sides, for example $Pl_1$ of $AB$ [Ax. 6, Df. 13].

A perpendicular $PQ$ from $P$ to $AB$ can be drawn [Pr. 16].
Assume a second perpendicular \( PR \) from \( P \) to \( AB \) can be drawn. We would have a triangle \( PQR \) [Pr. 6] with two right angles, \( \rho \) and \( \rho' \), which is impossible [Pr. 19].

\( PR \) is then impossible. Let now \( E \) be any point of \( AB \), whether or not produced. Draw the perpendicular \( EF \) to \( AB \) from \( E \) [Pr. 15]

and assume a second perpendicular \( EG \) from \( E \) to \( AB \) can be drawn in the same side \( Pl_1 \) of \( AB \) [Cr. 27].

They will adjacent at \( E \) where they make and angle \( \alpha > 0 \), if not they would be superposed in a unique straight line [Df. 21, Cr. 36].

Being \( EF, EG \) and \( EB \) straight lines adjacent at \( E \) [Cr. 35],

\( \alpha \) and \( \rho'_1 \) are adjacent angles [Cr. 39]

and \( \rho_1 \) is the union angle of them [Df. 22, Pr. 3].

Therefore, \( \rho_1 > \rho'_1 \) [Pr. 3],

which is impossible [Pr. 17].

So, the second perpendicular \( EG \) to \( AB \) in \( Pl_1 \) is impossible. A perpendicular \( EH \) from \( E \) to \( AB \) in \( Pl_2 \) can only be adjacent at \( E \) to \( EF \) because all points of \( EF \) and \( EH \), except \( E \), are non-common points respectively of \( Pl_1 \) and \( Pl_2 \) [Ax. 6, Df. 13].

So, \( EF, EB \) and \( EH \) can only be three adjacent straight lines [Cr. 35]

that make at their common endpoint \( E \) two adjacent angles \( \rho_1 \) and \( \rho_2 \) [Cr. 39]

whose union angle is the straight angle \( \rho_1 + \rho_2 \) [Cr. 46].

So then, \( EF \) and \( EH \) make a unique straight line [Cr. 40].

Therefore, from a point, whether or not in straight line with a straight line, only one perpendicular to the straight line can be drawn. □

**Proposition 21** The distance from a given point not in straight line with a given straight line to the given straight line is the length of the perpendicular from the given point to the given straight line, produced if necessary. And that distance is unique.
and $P$ a point not in straight line with $AB$ [Cr. 20].

From $P$ draw the perpendicular $PQ$ to $AB$ [Pr. 16].

Let $R$ be any point of $AB$, whether or not produced, different from $Q$ [Cr. 1].

Join $P$ and $R$ [Cr. 15].

$P$ is not in straight line with $R$ and $Q$ [Df. 11].

Therefore, $P$, $R$ and $Q$ define a triangle $PRQ$ [Pr. 6].

Since $\rho$ is a right angle [Df. 24],

$\rho$ is the greatest angle of $PRQ$ [Pr. 19].

And the side $PR$ is greater than the side $PQ$ [Pr. 11].

Since the distance between two points is unique [Cr. 31],

$R$ is any point of $AB$, whether or not produced, different from $Q$, and $PQ$ is less than $PR$, $PQ$ is the shortest of the distances [Df. 14]

between $P$ and any point in $AB$, whether or not produced. So, the length of the perpendicular $PQ$ is the distance from the point $P$ to the straight line $AB$ [Df. 15],

and this distance is unique [Pr. 20, Cr. 31]. $\Box$

Hereafter, a perpendicular to a straight line drawn from a point that is not in straight line with that straight line, will be drawn by producing the straight line if necessary [Pr. 16]. And, unless otherwise indicated, when considering more than one perpendicular to a given straight line, all of them will be assumed to be in the same side of the given straight line [Ax. 6, 16].

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**Proposition 22** Draw three points on the same side of a given straight line, two of them equidistant and two of them non-equidistant from the given line. Draw a straight line non-parallel to the given straight line.

▷ Through two points $C$ and $D$ of a given straight line $AB$ [Cr. 1]

draw the perpendiculars $CE$ and $DF$ to $AB$ [Pr. 15].

All points of $CE$ and $DF$ are on the same side of $AB$ [Cr. 26].

Take any point $P$ in $CE$ [Cr. 1];

in $DF$ take a point $Q$ such that $DQ = CP$ [Pr. 1];

and in $DQ$ take any point $R$ [Cr. 1].

It holds $DR < DQ$ [Cr. 13].

Join $P$ and $R$ [Cr. 15].

$P$ and $Q$ are equidistant from $AB$ [Pr. 21],

$P$ and $R$ are non-equidistant from $AB$ [Pr. 21],

$P$, $Q$ and $R$ are in the same side of $AB$; and $PR$ is not parallel to $AB$ [Df. 17, Pr. 21],

and it is in the same side of $AB$ [Cr. 26]. $\Box$

From now on, all of points equidistant from a straight line, whether or not in another straight line, will be assumed to be in the same side of the straight line and at a distance from the straight line greater than zero.

**Proposition 23** (Khayyam-Cataldi’s Axiom extended) All segments of a given straight line in the same side of a second straight line have the same distancing direction with respect to the second straight line as the given straight line. And if the endpoints of the given straight line are equidistant from the second straight line then the given straight line is parallel to the second straight line, being all points of the given straight line non-common points of the same side of the second straight line.

▷ (Fig. 23, left.) Let $l$ be a straight line in a plane $Pl$ [Cr. 23],

and $A$ and $B$ any two non-common points in the same side of $l$ in $Pl$ [Cr. 26],

so that $A$ and $B$ are non-equidistant from $l$ [Pr. 22].

Draw the perpendiculars $AP$ and $BQ$ respectively from $A$ and $B$ to $l$ [Pr. 16],
and assume* \( AP < BQ \). Join \( A \) and \( B \) [Cr. 15].

The points \( A \) and \( B \) define a distancing direction, from \( A \) to \( B \), of the straight line \( AB \) [Dfs. 1, 16] with respect to the straight line \( l \) [Df. 15].

All segments of \( AB \) must have the same distancing direction with respect to \( l \) as \( AB \), otherwise there would be at least one segment whose distancing direction with respect to \( l \) would be opposite to that of \( AB \) [Dfs. 16, 1].

And then, either the endpoints of that segment are given before drawing \( AB \), which is not the case [Ax. 5, Cr. 15],
or they are unknown before drawing \( AB \), in which case they could only be a consequence of the operation, as such an operation, of drawing \( AB \), which is impossible [Df. 4, Ax. 1, Cr. 15],
or the straight line \( AB \) cannot be drawn, which is also impossible [Ax. 5, Cr. 15].

Assume now (Fig. 23, right.) that \( A \) and \( B \) are equidistant from \( l \) [Pr. 22].

Join \( A \) with \( B \) [Cr. 15].

Let \( R \) be any point between \( A \) and \( B \) [Crs. 5, 4],

and assume its distance to \( l \) [Pr. 21]
is different from the equidistance of \( A \) and \( B \). The segments \( AR \) and \( RB \) [Cr. 5] would have different distancing directions with respect to \( l \) [Df. 16, Pr. 21],

So, either the point \( R \) and the distancing directions of \( AR \) and of \( RB \) with respect to \( l \) are given before drawing \( AB \), which is not the case [Ax. 5, Cr. 15],
or they are unknown before drawing \( AB \), in which case they could only be a consequence of the operation, as such an operation, of drawing \( AB \), which is impossible [Df. 4, Ax. 1, Cr. 15],
or the straight line \( AB \) cannot be drawn, which is also impossible [Ax. 5, Cr. 15].

Therefore, \( R \) can only be at the same distance from \( CD \) as \( A \) and \( B \). Consequently, \( AB \) is parallel to \( l \) [Df. 17].

And being \( A \) and \( B \) non common points in the same side of \( l \), all points of \( AB \) are non common points of the same side of \( l \) [Ax. 6, Df. 13]. □

Though a straight line could be considered parallel to itself by a zero equidistance, hereafter only parallel straight lines whose equidistance is greater than zero will be considered.

**Proposition 24** (A variant of Tacquet’s Axiom 11) *If a straight line is parallel to another straight line, then the perpendicular from any point of any of the two straight lines to the other straight line is also perpendicular to the first straight line.*

\[\text{Figure 24 – Pr. 24}\]

Let \( AB \) be a straight line parallel to another straight line \( CD \) [Pr. 23].

All points of \( AB \) are at the same distance greater than zero from \( CD \) [Df. 17].
From a point $P$ of $AB$ draw the perpendicular $PE$ to $CD$ [Pr. 16].

Draw the perpendicular from $E$ to $AB$ [Pr. 16]

and assume it is not $EP$ but $EF$. From $F$ draw the perpendicular $FG$ to $CD$ [Pr. 16].

It will be different from $FE$, otherwise there would be two perpendiculars to $CD$ from the same point $E$, namely $PE$ and $FE$, which is impossible [Pr. 20].

Consider the triangle $FGE$ [Prs. 23, 6].

The right angle $\rho$ [Df. 24]

is the greatest angle of $FGE$ [Pr. 19].

So $EF$ is greater than $FG$ [Pr. 11],

and $FG$ is equal to $PE$ because $AB$ is parallel to $CD$ [Df. 17].

In consequence, the shortest distance from $E$ to $AB$ would not be the length of the perpendicular $EF$, but that of $EP$ [Ps. B],

which is impossible [Pr. 21].

So, $EP$ is also perpendicular to $AB$. Let now $Q$ be any point in $CD$. Draw the perpendicular $QH$ to $AB$ [Pr. 16].

Assume the perpendicular from $H$ to $CD$ is not $HQ$ but $HJ$. It has just been proved that $HJ$ is also perpendicular to $AB$. So, there would be two different perpendiculars, $HJ$ and $HQ$, to $AB$ from the same point $H$, which is impossible [Pr. 20].

Hence, the perpendicular $QH$ is also perpendicular to $CD$. □

Proposition 25 A straight line parallel to another given straight line can only be produced as a straight line parallel to the given straight line.

Let $AB$ be a straight line parallel to another straight line $CD$ [Pr. 23],

and $PQ$ any given finite distance [Df. 14, Pr. 13].

Draw the perpendicular $BE$ from $B$ to $CD$ [Pr. 16].

In $CD$ and in the direction from $C$ to $D$ take a point $F$ such that $EF = PQ$ [Pr. 1].

From $F$ draw the perpendicular $FG$ to $CD$ [Pr. 15].

Take a point $B'$ in $FG$ such that $B'F = BE$ [Pr. 1].

Join $B$ and $B'$ [Cr. 15].

$BB'$ is parallel to $CD$ [Pr. 23].

And $BE$ is perpendicular to $AB$ and to $BB'$ through their common endpoint $B$ [Pr. 24].

$AB$ and $BB'$ are, then, the two sides of a straight angle [Cr. 46]

and they make the straight line $AB'$ [Cr. 40],

which is parallel to $CD$ [Pr. 23].

Assume now $BB' \neq EF$, for instance $BB' > EF$ [Ps. A].

Take a point $H$ in $BB'$ such that $BH = EF$ [Pr. 1].

Join $H$ and $F$ [Cr. 15].

$BE$ is parallel to $HF$ [Pr. 23];
CF is perpendicular to HF [Pr. 24];
and HF is perpendicular to CF [Cr. 47].
So, if $BB' \neq EF$ there would be two different perpendiculars, $B'F$ and $HF$, to $CF$ from the same point $F$, which is impossible [Pr. 20].
In consequence $BB' = EF$. So, $BB'$ is the only production of $AB$ from $B$ by the given length $EF = PQ$ [Cr. 16],
and it is parallel to $CD$ [Pr. 23].
The same argument applies to the endpoint $A$. □

**Proposition 26** (Posidonius-Geminus’ Axiom) If two points of a given straight line are equidistant from a second straight line, then the given straight line is parallel to the second straight line.

Let $AB$ and $CD$ be two straight lines such that two points $P$ and $Q$ of $AB$ are equidistant from $CD$ [Pr. 23].
The segment $PQ$, which is the only straight line joining $P$ and $Q$ [Cr. 15],
is parallel to $CD$ [Pr. 23].
If $PA$ were not parallel to $CD$, the straight line $PQ$ [Cr. 14] could be produced from $Q$ by a length $PA$ as a straight line $PA$ [Cr. 16] non parallel to $CD$, which is impossible [Pr. 25].
The same applies to $QB$. $AB$ is then parallel to $CD$. □

**Proposition 27** If a straight line is parallel to another straight line, this second straight line is also parallel, and by the same equidistance, to the first straight line.

Let $AB$ be a straight line parallel to another straight line $CD$ [Cr. 26].
Let $E$ and $F$ be any two points of $CD$ [Cr. 1].
From $E$ and from $F$ draw the respective perpendiculars $EG$ and $FH$ to $AB$ [Pr. 16].
These perpendiculars are also perpendicular to $CD$ [Pr. 24].
So, the distance from $E$ to $AB$ is the same as the distance from $G$ to $CD$ [Pr. 21];
and the distance from $F$ to $AB$ is the same as the distance from $H$ to $CD$ [Pr. 21].
Since $AB$ is parallel to $CD$, the distances to $CD$ from $G$ and $H$ are equal to each other [Df. 17].
Hence, the distances to $AB$ from $E$ and $F$ are also equal to each other [Ps. B].
$E$ and $F$ are, then, two points in $CD$ at the same distance from $AB$. Therefore, $CD$ is parallel to $AB$ [Cr. 26],
and by the same equidistance $GE$. □

**Proposition 28** To draw a straight line parallel to a given straight line through a given point not in straight line with the given straight line.

Let $CD$ be a straight line [Cr. 23],
and $P$ a point not in straight line with $CD$ [Cr. 20].
From $P$ draw the perpendicular $PQ$ to $CD$ [Pr. 16].
Take any point $R$ in $CD$ different from $Q$ [Cr. 1].
From $R$ draw the perpendicular $RS$ to $CD$ [Pr. 15].
And in $RS$ take a point $T$ such that $RT = QP$ [Pr. 1].
Join $P$ and $T$ [Cr. 15] and produce $PT$ respectively from $P$ and from $T$ to any two points $A$ and $B$ [Cr. 16].
The straight line $AB$ has two points, $P$ and $T$, equidistant from $CD$. Therefore, $AB$ is a parallel to $CD$ [Cr. 26]
through the point $P$. □

**Proposition 29** (Playfair’s Axiom 11) Through a given point not in straight line with a given straight line, one, and only one, parallel to the given straight line can be drawn.
Let $AB$ be a straight line \[Cr. 23\]
and $P$ a point not in straight line with $AB$ \[Cr. 20\].
Through $P$ a parallel $CD$ to $AB$ can be drawn \[Pr. 28\].
Assume that through $P$ more than one parallel to $AB$ can be drawn. From $P$ draw the perpendicular $PQ$ to $AB$ \[Pr. 16\].
$PQ$ is also perpendicular from $P$ to each of the assumed parallels to $AB$ \[Pr. 24\].
And each of these assumed parallels would be a different perpendicular to $PQ$ through the same point $P$ \[Cr. 47\],
which is impossible \[Pr. 20\].
Therefore, through a given point not in straight line with a given straight line, one \[Pr. 28\], and only one, parallel to a given straight line can be drawn. □

**Proposition 30** For any given straight line and through different points, a number of parallels to the given straight line greater than any given number can be drawn.

Let $AB$ be a straight line \[Cr. 23\]
and $CD$ another straight line that intersects $AB$ at any point $P$ of $AB$ \[Crs. 19, 25\].
$CD$ has a number of points greater than any given number $n$ \[Cr. 1\],
none of which, except $P$, is in straight line with $AB$, even arbitrarily producing $AB$ and $CD$ \[Cr. 19\].
Through each of those $n$ points of $CD$ one, and only one, parallel to $AB$ can be drawn \[Pr. 28, 29\]. Therefore, it is possible to draw a number greater than any given number of parallels to a given straight line. □

**Proposition 31** If two straight lines have a common perpendicular, then they are parallel to each other.

Let $AB$ be a straight line in the same side of another straight line $CD$ \[Cr. 27\].
From a point $P$ of $AB$ draw the perpendicular $PQ$ to $CD$ \[Pr. 16\].
If $PQ$ is also perpendicular to $AB$, then $AB$ must be parallel to $CD$, otherwise through $P$ a parallel $EF$ to $CD$ could be drawn \[Pr. 28\].
$PQ$ would be perpendicular to $EF$ \[Pr. 24\],
and $EF$ would be perpendicular to $PQ$ \[Cr. 47\],
and there would be two perpendicular to $PQ$, namely $AB$ and $EF$, through the same point $P$, which is impossible \[Pr. 20\]. □

**Proposition 32** Two parallel straight lines cannot intersect.

Assume two parallel straight lines $AB$ and $CD$ \[Cr. 30\]
intersect at a point $P$. From a point $Q$ of $AB$ different from $P$ \[Cr. 1\] draw the perpendicular $QR$ to $CD$ \[Pr. 16\].
$QR$ is also perpendicular to $AB$ \[Pr. 24\].
$PQ$ and $PR$ would be two perpendiculars to $QR$ \[Cr. 47\]
through the same point $P$, which is impossible \[Pr. 20\]. □

**Note.** The fact that two parallel straight lines cannot intersect with each other, does not imply that non parallel straight lines have to intersect, as Posidonius defended. His pupil Geminus of Rhodes discovered the flaw \[3, p. 40, 190\], \[1, pp. 58-59\].

**Proposition 33** If a common transversal cuts two straight lines and makes with them equal the angles of a couple of alternate angles, or of corresponding angles, then the two angles of each couple of alternate angles, and of corresponding angles, are also equal. And the interior angles of the same side of the transversal sum two right angles. If the interior angles of the same side of the transversal sum two right angles, the two angles of each couple of alternate angles, and of corresponding angles, are equal to each other.

Let $AB$ and $CD$ be two straight lines that are intersected by a common transversal $EF$ \[Df. 25, Cr. 30\] at $P$ and $Q$ respectively. On the one hand we have: $\alpha_i = \alpha_e$; $\beta_i = \beta_e$; $\gamma_i = \gamma_e$; $\delta_i = \delta_e$ \[Pr. 5\].
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On the other, and being $\rho$ a right angle: $\rho + \rho = \alpha_c + \beta_c = \alpha_i + \beta_i = \gamma_i + \delta_i = \gamma_e + \delta_e = \alpha_e + \beta_e + \alpha_i = \gamma_i + \delta_e = \delta_i + \gamma_e$ [Pr. 4].

So, if $\alpha_i = \gamma_i$, and being $\alpha_i = \alpha_e$ and $\gamma_i = \gamma_e$, we immediately get $\alpha_e = \gamma_e; \alpha_i = \gamma_i; \alpha_e = \gamma_i$ [Ps. A].

A similar argument proves that the two angles of any other couple of alternate angles, or of corresponding angles [Df. 26], are equal to each other. In addition, from $\alpha_i + \beta_i = \rho + \rho$ [Pr. 4], $\alpha_i = \gamma_i$ and $\beta_i = \delta_i$, it follows $\gamma_i + \delta_i = \rho + \rho; \alpha_i + \delta_i = \rho + \rho$ [Ps. A].

On the other hand, if $\alpha_i + \delta_i = \rho + \rho$, and being $\rho + \rho = \gamma_i + \delta_i$ [Pr. 4], we immediately get $\alpha_i + \delta_i = \gamma_i + \delta_i$ [Ps. B].

Therefore, $\alpha_i = \gamma_i$ [Ps. B], and the same argument above proves that the two angles of each couple of alternate angles, and of corresponding angles, are equal. □

**Proposition 34** *A common transversal makes with two parallel straight lines equal the two angles of each couple of alternate angles and of corresponding angles.*

Let $AB$ and $CD$ be any two parallel straight lines [Cr. 30].

And $EF$ any common transversal [Df. 25, Cr. 30] that cuts them at $P$ and $Q$ respectively [Cr. 30].

If $EF$ is perpendicular to $AB$, it is also perpendicular to $CD$ [Pr. 24], and the eight angles it makes with $AB$ and $CD$ at its corresponding intersection points are right angles [Cr. 47], in which case the two angles of each couple of alternate and of each couple of corresponding angles are equal [Pr. 17, Df. 26].

If $EF$ is not perpendicular to $AB$, draw the perpendicular $PR$ to $CD$ [Pr. 16], which is also perpendicular to $AB$ [Pr. 24].

And $AB$ and $CD$ are perpendicular to $PR$ [Cr. 47].

Therefore $\rho_1$ and $\rho_2$ are right angles [Df. 24].

Take a point $S$ in $PB$ such that $PS = RQ$ [Pr. 1].

Join $S$ and $Q$ [Cr. 15].

$SQ$ and $PR$ are parallel [Cr. 26, Pr. 27].
Hence, $AB$ and $CD$ are perpendicular to $SQ$ [Pr. 24], and $SQ$ is perpendicular to $AB$ and to $CD$ [Cr. 47].

Therefore, $\rho_3$ and $\rho_4$ are right angles [Df. 24, Cr. 47]. And being $AB$ parallel to $CD$, it holds $PR = SQ$ [Df. 17].

Consider the triangles $PRQ$ and $PQS$ [Prs. 23, 6].

The three sides of $PRQ$ are equal to the three sides of $PQS$. So, $\alpha = \alpha'$ [Pr. 10]. Therefore, the two angles of any other couple of alternate angles, and of corresponding angles [Df. 26], are also equal [Pr. 33]. □

**Proposition 35** If a common transversal makes with two straight lines equal the angles of a couple of alternate angles, then both straight lines are parallel to each other.

![Figure 28 – Pr. 35](image)

Let $EF$ be a common transversal [Df. 25, Cr. 30] that intersects two straight lines $AB$ and $CD$ respectively at $P$ and $Q$, where $EF$ makes with $AB$ and $CD$ equal the two angles of a couple of alternate angles $\alpha$ and $\alpha'$ [Df. 26].

The two interior angles of the same side of $EF$, $\alpha$ and $\beta$ in one side, and $\alpha'$ and $\beta'$ in the other, sum two right angles [Pr. 33]. Therefore, $AB$ and $CD$ cannot intersect each other, otherwise there would be a triangle with two angles that sum to two right angles, which is impossible. [Pr. 19].

Therefore, $AB$ and $CD$ are each on the same side of the other. [Cr. 29]

If $\alpha$ and $\alpha'$ are right angles, $EF$ will be perpendicular to $AB$ and to $CD$ [Cr. 47], and $AB$ and $CD$ will be parallel [Pr. 31].

If $\alpha$ and $\alpha'$ are not right angles, $EF$ is not perpendicular to $AB$ nor to $CD$ [Df. 24].

In this case, draw the perpendicular $PR$ from $P$ to $CD$ [Pr. 16]. On $PB$ take a point $S$ such that $PS = RQ$ [Pr. 1].

Join $S$ and $Q$ [Cr. 15].

$PR$ and $SQ$ are parallel [Cr. 26, Pr. 27], and then $CD$ is perpendicular to $SQ$ [Pr. 24], and $SQ$ is perpendicular to $CD$ [Cr. 47].

$SQP$ and $PQR$ are triangles [Prs. 23, 6]. They have a a common side $PQ$, and also $PS = RQ$, and $\alpha = \alpha'$. Therefore $PR = SQ$ [Cr. 45].

Since $PR$ and $SQ$ are perpendicular to $CD$, $P$ and $S$ are at the same distance from $CD$ [Pr. 21]. So, $AB$ and $CD$ are parallel to each other [Cr. 26, Pr. 27]. □

**Proposition 36** Two straight lines are parallel to each other if, and only if, a common transversal makes with them two interior angles in the same side of the transversal that sum two right angles.
If a common transversal $EF$ makes with two straight lines $AB$ and $CD$ [Df. 25, Cr. 30] two interior angles $\alpha$ and $\beta$ [Df. 25] on the same side of the transversal [Df. 21] that sum two right angles, then the two angles of any couple of alternate angles $\alpha$ and $\alpha'$ are equal to each other [Pr. 33] and both straight lines are parallel [Pr. 35].

If a transversal cuts two parallel straight lines [Df. 25, Cr. 30], it makes with them equal the two angles of each couple alternate angles, for instance $\alpha$ and $\alpha'$ [Pr. 34] and then the two interior angles of the same side of the transversal [Df. 21] sum two right angles [Pr. 33]. □

**Proposition 37** (Proclus’ Axiom) *If a first straight line is parallel to a second straight line and the second straight line is parallel to a third straight line, then the first straight line is also parallel to the third straight line.*

![Figure 29 – Pr. 37](image)

Let $EF$ be a common transversal of two straight lines $AB$ and $CD$ [Df. 25, Cr. 30], which is parallel to another straight line $EF$ [Cr. 30]. Assume first that $AB$ and $EF$ are in different sides of $CD$ [Ax. 6] (Fig. 29, left).

From any point $P$ of $CD$ draw the perpendicular $PQ$ to $AB$ and the perpendicular $PR$ to $EF$ [Pr. 16].

So, $\rho_1$, $\rho_2$, $\rho_3$ and $\rho_4$ are right angles [Df. 24].

$PQ$ and $PR$ cannot be two different perpendiculars to $CD$ from $P$ [Pr. 20].

So, $QR$ is a unique straight line, which is a common transversal of $AB$ and $EF$, and makes with them in the same side of $QR$ two interior angles $\rho_1$ and $\rho_3$ [Df. 25] that sum two right angles. Therefore $AB$ is parallel to $EF$ [Pr. 36].

If $AB$ and $EF$ are in the same side of $CD$ [Cr. 27] (Fig. 29, right), then draw the perpendicular $PQ$ from any point $P$ in $AB$ to $EF$, and from $Q$ the perpendicular $QR$ to $CD$ [Pr. 16].

So, $\rho_1$ and $\rho_2$ are right angles [Df. 24].

Since $EF$ is parallel to $CD$, $QR$ is also perpendicular to $EF$ [Pr. 24], and $\rho_3$ is a right angle [Df. 24].

And, for the same reasons above, $PR$ is a unique straight line, which is perpendicular to $EF$ through $Q$. And being perpendicular to $CD$, $PR$ is also perpendicular to $AB$ [Pr. 24], and then $\rho_4$ is a right angle [Df. 24].

In consequence, $PQ$ is a transversal of $AB$ and $EF$ that make two interior angles, $\rho_1$ and $\rho_4$ [Df. 25], on the same side of $PQ$ that sum two right angles. So, $AB$ is also parallel to $EF$ [Pr. 36]. □

**Proposition 38** If a common transversal makes with two straight lines two interior angles in the same side of the transversal that sum less (more) than two right angles, the interior angles in the other side of the transversal sum more (less) than two right angles.

Let $EF$ be a common transversal of two straight lines $AB$ and $CD$ [Df. 25, Cr. 30] that cuts them respectively at $P$ and $Q$ and makes with them the interior angles $\alpha$ and $\beta$ [Df. 25]...
on the same side of $EF$ [Ax. 6, Df. 21]
so that $\alpha + \beta < \rho + \rho$, where $\rho$ is a right angle [Pr. 17].
$AB$ and $CD$ are not parallel [Pr. 36].
Let $\gamma$ and $\delta$ be the interior angles that $AB$ and $CD$ make with $EF$ on the other side of $EF$ [Df. 21].
On the one hand we have: $\alpha + \gamma = \beta + \delta = \rho + \rho$ [Pr. 4],
so that $\alpha + \gamma + \beta + \delta = 2\rho + 2\rho$ [Ps. B].
On the other hand $\gamma + \delta \leq \rho + \rho$, otherwise $AB$ and $CD$ would be parallel [Pr. 36].
But if $\gamma + \delta < \rho + \rho$, we would have $\gamma + \delta + \rho < \rho + \rho + \rho + \rho$ [Ps. B];
and being $\alpha + \beta < \rho + \rho$, we also have $\alpha + \beta + \gamma + \delta < \rho + \gamma + \delta$ [Ps. B].
Therefore $\alpha + \beta + \gamma + \delta < \rho + \rho + \rho + \rho$ [Ps. B], which is not the case.
So it must be $\gamma + \delta > \rho + \rho$. A similar argument applies to the case $\alpha + \beta > \rho + \rho$. □

**Proposition 39** All segments with the same length of a given straight line have the same distancing direction
and the same relative distancing with respect to any other non-parallel straight line in the same side of the given
straight line.

\[\text{Figure 31 – Pr. 39}\]

\[\text{Figure 30 – Pr. 38}\]
and $Q$ is not in straight line with $PT$ [Cr. 22].

So, $QPT$ is a triangle [Pr. 6].

For the same reason $SRU$ is also a triangle. $PT$ and $RU$ are parallel to $CD$ [Pr. 36], and then they are parallel to each other [Pr. 37].

Therefore $\alpha = \alpha'$ [Pr. 34].

$QF$ and $SH$ are parallel to each other [Pr. 31], and then $\beta = \beta'$ [Pr. 34].

The triangles $QPT$ and $SRU$ verify: $\alpha = \alpha'$; $\beta = \beta'$ [Pr. 34], and $PQ = RS$. Consequently, $QT = SU$ [Pr. 8].

Being $PE = TF$ [Df. 17] and $QT = QF - TF$ [Cr. 13], it will be $QT = QF - PE$ [Ps. A].

$QT$ is, then, the relative distancing of the segment $PQ$ with respect to $CD$ [Df. 16].

For the same reasons $SU$ is the relative distancing of the segment $RS$ with respect to $CD$. Since $QT = SU$, and $PQ$ and $RS$ are any two segments of $AB$ with the same length, we conclude that all segments of $AB$ with the same length have the same relative distancing with respect to $CD$ [Df. 16] in the same distancing direction [Pr. 23]. □

**Proposition 40** (Khayyām’s Axiom) Two non-parallel straight lines, produced if necessary by a finite length, intersect with each other at a unique point.

\begin{itemize}
  \item Let $AB$ and $CD$ be any two non parallel [Pr. 22] and non-intersecting straight lines [Cr. 30].
  \item If one of them has its two endpoints in different sides of the other, then it intersects the other or a finite production of the other [Cr. 29, Pr. 13].
  \item Assume, then, the two endpoints of each straight line are non-common points of the same side of the other [Cr. 27].
  \item Each straight line will be in the same side of the other [Ax. 6].
  \item Assume the distancing direction of $AB$ with respect to $CD$ [Df. 16] is, for instance*, from $B$ to $A$ [Pr. 39].
  \item In this case, draw from $A$ the perpendicular $AE$ to $CD$ [Pr. 16].
  \item If $AB$ were a segment of $AE$ then $AB$ intersects $CD$ at $E$ [Df. 24] at a finite distance $BE$ [Pr. 13] (Fig. 32, left).
  \item If $AB$ is not a segment of $AE$ (Fig. 32, right), draw though $A$ the parallel $AQ$ to $CD$ [Pr. 28].
  \item $AB$ is not parallel to $AQ$, otherwise $AB$ would be parallel to $CD$ [Pr. 37], which is not the case. Take in $AB$ any point $P_1$, and draw the perpendicular $P_1Q_1$ to $AQ$ [Pr. 16].
  \item $P_1Q_1$ is the relative distancing of $AP_1$ with respect to $AQ$ [Df. 16, Pr. 21].
  \item Being $AE$ and $P_1Q_1$ finite [Pr. 13],
\end{itemize}
there will be a natural number \( n \geq 1 \) such that \( n \) times \( P_1Q_1 \) is greater than \( AE \), otherwise there would be an impossible last natural number. On \( AB \), produced if necessary [Cr. 16],

and from \( P_1 \), take just \( n - 1 \) (4 in Figure 32) successive points \( P_2, P_3, P_4 \ldots P_n \) [Cr. 1]
each separated from the previous one by the same finite distance \( AP_1 \) [Pr. 1].

Since the distances to \( AQ \) from the successive \( P_1, P_2, P_3 \ldots P_n \) increase in the same direction [Pr. 23] and by the same distance \( P_1Q_1 \) [Pr. 39],

and \( n \) times \( P_1Q_1 \) is greater than \( AE \), the distance to \( AQ \) from \( P_n \) is greater than \( AE \), and being zero the distance from \( A \) to \( AQ \) [Df. 14],

there is a point \( F \) in \( AP_n \) whose distance to \( AQ \) is just \( AE \) [Ax. 7].

And \( F \) is unique, if not \( AP_n \) would be parallel to \( AQ \) [Cr. 26],

which is not the case. \( D \) is a non-common point of one side of \( AB \) and then of \( AP_n \) [Dfs. 13, 11].

Join \( D \) and \( F \) [Cr. 15].

\( DF \) is parallel to \( AQ \) [Cr. 26],

and it must be a production of \( CD \) from \( D \) to \( F \) [Cr. 16],

otherwise there would be two different parallels, \( CD \) and \( DF \), to \( AQ \) from the same point \( D \), which is impossible [Pr. 29].

So, \( AB \) and \( CD \) can be produced respectively from \( B \) and from \( D \) to the point \( F \). Since \( A \) is a point of the perpendicular \( AE \) to \( CD \), it is not in straight line with \( CD \) [Cr 22].

So, \( A, E \) and \( F \) define a triangle \( AEF \) [Pr. 6],

of which \( \rho_2 \) is the greater angle [Pr. 19]

and \( AF \) the greater side [Pr. 11].

\( AF \) is finite because it is equal or less than \( n \) times the finite length \( AP_1 \). And \( DF < EF; BF < AF \) [Cr. 13].

Therefore, the lengths of the productions \( DF \) and \( BF \) respectively of \( CD \) and of \( AB \) are both finite, and \( AB \) and \( CD \) intersect at a point \( F \) at a finite distance from their corresponding endpoints \( B \) and \( D \) [Df. 14].

In consequence, and taking into account that \( AB \) and \( CD \) are any two non-parallel and non-intersecting straight lines, we conclude that any two non-parallel and non-intersecting straight lines can be produced by a finite length so that they intersect with each other at a unique point. □

**Proposition 41** (Euclid’s Postulate 5) *If a common transversal makes with two given non-intersecting straight lines two angles in the same side of the transversal that sum less than two right angles, then the given straight lines can be produced in that side of the transversal by a finite length to a unique point where they intersect with each other.*

[Diagram of Proposition 41]
whose sum is less than two right angles [Df. 24].

$AB$ and $CD$ are not parallel to each other [Pr. 36].

Therefore, they can be produced [Cr. 16]

by a finite length to a unique intersection point $R$ [Pr. 40].

$P$, $R$ and $Q$ define a triangle $PRQ$ [Pr. 6]

and $R$ can only be a point in the side of $EF$ where $EF$ makes the angles $\alpha$ and $\beta$ respectively with $AB$ and $CD$, because in the other side [Ax. 6]

the interior angles sum more than two right angles [Pr. 38],

and $PRQ$ would have two angles whose sum is greater than two right angles, which is impossible [Pr. 19]. □

References