**On Mach’s Principle and its implementation into classical mechanics:**

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**abstract:**

In this paper I want to address a shortcoming of classical as well as relativistic mechanics regarding the relationality of the kinetic energy. Going back on an argument given by W. Hofmann in 1904, I claim that the it’s definition for a particle should involve a sum over products of all other masses in the universe and their relative velocities

\[ T_i \propto \sum_j m_i m_j f(r_{ij}) v_{ij}^2, \]

where \( f \) is some function possibly depending on the relative distance. I show that an implementation of such a relational kinetic energy already leads to classical mechanics fully incorporating Mach’s principle. It is then based on a completely Galilean invariant Lagrangian which incorporates an induction of an isotropic inertial mass by distant matter as well as the phenomenology of gravitoelectromagnetism. I will show, that a Lorentz-type force equation can be obtained. The gravitational constant does not have to be put into the equations a priori any more but arises in a natural way as a “gravitational scalar”, whose local value is just G. I will briefly discuss the same argument in relativistic mechanics and show that the corresponding interpretation would be that of an emergent space-time induced by matter present in the universe.

**Keywords:** Mach’s principle, gravitational constant, Galilean invariance, relationality, kinetic energy, origin of inertia, Gravitoelectromagnetism, isotropy of inertia, emergent space-time

1. **Introduction:**

Physics should always ask why the laws of nature are the way they are, and not just settle for describing them. In history it has always been this way how progress was made: An insight why a wide range of laws are the way they are, so to say what is “behind them”. Here I agree with Alexander Unzicker that especially the elimination of constants plays a central role since their appearance is always tied to opaque concepts which are postulated without explanation. The maybe best example for this is the connection between light and electromagnetism. Before Maxwell, light and electromagnetism were two different concepts existing independently from one another. On the one side, there was the concept of an electromagnetic field together with the constants \( \mu_0, \varepsilon_0 \) (from which the latter is, by the way, unexplained until today), on the other there was the concept of light together with \( c \), the speed of light. Both of them had to be postulated since none of them could be derived from the other. This changed when Maxwell was able to show that light was an electromagnetic wave, rendering the latter concept obsolete since it was included in electromagnetism. On the other hand this fact expressed itself in the relation

\[ c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}, \]

effectively also rendering one constant of nature obsolete. This is a pattern in physics which has occurred many times in the past and there is no reason why it should be any different in modern physics. History has shown that major breakthroughs were always tied to explaining one of these numbers, respectively being able to derive one previously postulated concept from another. In this context, in the author’s opinion there exists an unused opportunity to achieve exactly this regarding the gravitational constant G: Mach’s principle, which is until today implemented neither in classical physics nor in general relativity (even though features of it are actually present in general relativity).
In this paper I want to show a way how to do this in classical mechanics and, in this way, obtain the gravitational constant from the equations themselves instead of putting it into them by hand. The concept which is removed shall turn out to be that of absolute space, which was postulated by Newton and is still present in classical as well as relativistic mechanics through the definition of the kinetic energy. Space will then, at least in classical mechanics, just play the role of a coordinate system in which the equations are formulated but without having any physical impact on them like in the Newtonian formulation, where it defines the notion of “rest”. In the relativistic case, it shall turn out that most likely an implementation of Mach’s principle would lead to space and time being a completely emergent phenomenon from whatever deeper reality.

2. The relationality of the kinetic energy:

We begin with a simple consideration based on an argument put forth by the physicist W. Hofmann in 1904 [1,9]: The common definitions of the kinetic energy and momentum, which read:

\[ T = \frac{1}{2} mv^2 \]  \hspace{1cm} (2.1)

and:

\[ \vec{p} = m \vec{v} \]  \hspace{1cm} (2.2)

respectively for a particle of mass m and velocity v, must be incomplete, since they contain two absolute elements, namely \( m \) and \( \vec{v} \). As was stated by Hofmann, this can be seen by a simple thought experiment: Suppose two particles flying within an otherwise empty cosmos with a relative velocity \( \vec{v}_{12} = \vec{v}_1 - \vec{v}_2 \) towards each other. Then observer 1, co-moving with particle 1, ascribes to the system the kinetic energy and momentum:

\[ T_1 = \frac{1}{2} m_2 \cdot v_{12}^2 \]  \hspace{1cm} (2.3)

\[ \vec{p}_1 = m_2 \vec{v}_{12} \]  \hspace{1cm} (2.4)

while observer 2, co-moving with particle 2, ascribes:

\[ T_2 = \frac{1}{2} m_1 v_{12}^2 \]  \hspace{1cm} (2.5)

\[ \vec{p}_2 = m_1 \vec{v}_{12} \]  \hspace{1cm} (2.6)

Upon (inelastic) collision, both observers are able to measure the real values for both momentum and energy, which obviously disagrees with at least one of the expected ones. There are now two possible ways of resolving this problem. One is that the two masses enter symmetrically into the kinetic energy and momentum. Since both must be additive, it is also clear that the dependence on both masses must be linear, leaving the only possible choice:

\[ T_1 = T_2 = \frac{1}{2} m_1 m_2 f(r_{12}) \vec{v}_{12}^2 \]  \hspace{1cm} (2.7)

\[ \vec{p}_1 = \vec{p}_2 = m_1 m_2 f(r_{12}) \vec{v}_{12} \]  \hspace{1cm} (2.8)

---

1) As can easily be seen, this (and therefore all further arguments) also apply to the relativistic kinetic energy since only the dependence of the velocities themselves differ from the Newtonian expressions, not the dependences on the masses as well as the velocities being absolute velocities.
where \( f(r_{12}) \) accounts for a possible distance dependence. From dimensional reasons, it can be seen that:

\[
f(r_{12}) = \frac{G}{c^2 r_{12}}
\]  

(2.9)

must hold. Now, for a cosmos with more than one particle, (2.7) and (2.8) imply that the kinetic energy as well as the momentum have to be dependent on all other particles in the universe; in equations:

\[
T_i = \frac{1}{2} \sum_{j \neq i} \frac{G m_i m_j}{r_{ij}} \beta_{ij}^2 
\]  

(2.10)

\[
\vec{p}_i = \sum_{j \neq i} \frac{G m_i m_j}{c r_{ij}} \vec{\beta}_{ij}
\]  

(2.11)

This is the Machian resolution of the problem since both quantities are defined with respect to all other particles in the universe. The only other possibility implies the assumption of an absolute background, with respect to which both observers are able to measure their “absolute” velocities. The kinetic energy of the particles would then read:

\[
T_i = \frac{1}{2} m_i (\vec{v}_i - \vec{v}_0)^2
\]  

(2.12)

\[
\vec{p}_i = m_i (\vec{v}_i - \vec{v}_0)
\]  

(2.13)

where \( \vec{v}_0 \) is the velocity of this absolute background relative to the chosen observer. This is the Newtonian viewpoint: An absolute space which globally defines rest. The above equations reduce to (2.1) and (2.2) if the observer is at rest relative to it ( \( \vec{v}_0 = 0 \) ); this is the case which is normally assumed in Newtonian mechanics when describing a system. One can simply see that this would also resolve the problem in the above though experiment since both observers would agree on the total energy being:

\[
T = T_1 + T_2 = \frac{m_1}{2} (\vec{v}_0^{(1)})^2 + \frac{m_2}{2} (\vec{v}_0^{(2)})^2 = \frac{m_1}{2} (\vec{v}_{12} - \vec{v}_0^{(1)})^2 + \frac{m_2}{2} (\vec{v}_0^{(1)})^2
\]  

(2.14)

where \( \vec{v}_0^{(1)} \) and \( \vec{v}_0^{(2)} \) are the velocities relative to the absolute background as seen by observer 1 and 2 respectively. However, there are three objections against this resolution. The first two are of empirical nature. The first one is, that for both particles being able to agree on (2.14), it is necessary for them to being able to directly interact via a measurement with the absolute (background) space. But until today, no such direct measurement of space has been successfully performed. The second one is that if such an absolute background globally defines what “rest” means, it will make itself noticeable according to eq. (2.13): An inertial reference frame could be only a such which is un-accelerated relative to the background. Now, since most of the local systems (e.g. galaxies) throughout the universe are accelerated relative to one another, at most one of these can agree with the background system. In all others, there would occur non-inertial contributions to, for example, the movement of the stars which is clearly not observed. We will see in section 4, that, in order to correctly obtain the Newtonian formulation of mechanics, described by a Lagrangian of the form:

\[
L_k = \frac{m_k}{2} \vec{v}_k^2 - V_k
\]  

(2.15)
the kinetic energy needs to have the form (2.10). The last objection is of theoretical nature: If we compare the formulation according to Mach (2.10), (2.11) and the one according to Newton (2.12), (2.13) one can see that the latter needs to postulate the absolute background system which the former one doesn’t. There, this system is induced by something which is present anyway: the particles themselves. Consequently, according to Occam’s razor, the Machian alternative is much more likely to be realized in nature as the Newtonian one.

3. A Lagrangian of the universe:

From this we can already construct a Lagrange function of the universe by adding the potential energy:

$$V = \frac{1}{2} \sum_{j \neq i} G m_i m_j / r_{ij}$$

(3.1)

Since now in L, both terms T and V depend on G and the Lagrange function remains invariant under rescaling, we can leave it out and write:

$$L = T - V = \frac{1}{2} \sum_{j \neq i} m_i m_j / r_{ij} (1 + \beta_{ij}^2)$$

(3.2)

This is (apart from the factor G) the same as used in the „Inertia free mechanics“ by Hans-Jürgen Treder [2,4]. As we see, this expression does not involve a gravitational constant. We will show in the following section, that it comes out of the equations in a natural way.

4. Gravitoelectromagnetism:

We now want to show that expression (3.2) includes the phenomena of gravitoelectromagnetism as well as an inertia induction according to Mach’s principle. The inertial mass will turn out to be isotropic, despite the Lagrangian being dependent on the relative velocities of the particles. The latter fact will just give rise to gravitomagnetic phenomena.

If we define the gravitomagnetic and electric potentials in the classical way via:

$$\vec{A}_k := \sum_{j \neq k} \frac{m_j}{r_{kj}} \vec{\beta}_j$$

(4.1)

$$\Phi_k := \sum_{j \neq k} \frac{m_j}{r_{kj}}$$

(4.2)

then indeed by using the second binomial identity

$$\vec{\beta}_{kj}^2 = \vec{\beta}_k^2 + \vec{\beta}_j^2 - 2 \langle \vec{\beta}_k, \vec{\beta}_j \rangle$$

and gathering together all terms involving the k th particle, we get for its Lagrangian:

$$L_k = \frac{1}{2} m_k^* \cdot \vec{v}_k^2 + m_k \Phi_k - 2 m_k \langle \vec{A}_k, \vec{\beta}_k \rangle + \sum_{j \neq k} m_j m_{kj} / r_{kj} \beta_{kj}^2$$

(4.3)

The first three terms in this expression differ from the common Lagrangian for a particle in an electromagnetic field only by an additional factor of 2 at the magnetic term (the correct relativistic value is 4) and the fact that an „inertial mass“ is given by’):

2) We shall see that the obtained expressions will also hold in arbitrary accelerated frames of reference. The potential \( V_k \) will, of course, be different in all of these systems.

3) This quantity does not have the dimension of a mass since we cancelled out G in the Lagrangian (3.2). But is does play exactly this role, since we could have let G remain in the equations and then see later that it cancels out when calculating the gravitational acceleration. Just like the normal mass does due to the equivalence principle.
Expression (4.4) is isotropic, showing that the relational Lagrangian does not lead to an anisotropic inertial mass. Mach’s principle is satisfied since the inertial mass is determined by distant masses. It can be seen that this, as well as gravitomagnetic phenomena (the 3rd term in (4.3)), are a direct consequence of the kinetic energy having the form (2.10) instead of just being \( \frac{m_k}{2} v_k^2 \). Therefore, an inertia-induction in the sense of Mach’s principle does indeed arise from demanding the Galilean invariance of kinetic energy.

We now want to show how the (local) gravitational constant arises from (4.3). Consider therefore some mass distribution in the foreground, causing a potential \( \delta \Phi \), and the background including all other masses of the universe, causing a potential \( \Phi_0 \). The latter will be assumed to be much farther away than the former and therefore can be considered approximately constant. We write:

\[
\Phi = \Phi_0 + \delta \Phi
\]

If we plug it into the Lagrangian (4.3) and keep in mind that \( \Phi_0 \) is constant, we obtain:

\[
L = \frac{1}{2} m_k (1 + \frac{\delta \Phi}{\Phi_0}) v_k^2 + \frac{1}{2} m_k c^2 (1 + \frac{\delta \Phi}{\Phi_0}) - m_k \frac{c^2}{\Phi_0} (\vec{A}_k \cdot \vec{\beta}_k) + m_k \frac{c^2}{2 \Phi_0} \sum_{j \neq k} \frac{m_j}{r_{kj}} \beta_j^2
\]

(4.5)

Now, in this expression, the classical gravitational potential is just the term \( \frac{1}{2} m_k c^2 \frac{\delta \Phi}{\Phi_0} \). If we compare it with the Newtonian potential:

\[
\frac{1}{2} m_k c^2 \frac{\delta \Phi}{\Phi_0} = G m_k \delta \Phi
\]

we find:

\[
G = \frac{c^2}{2 \Phi_0}
\]

(4.6)

As said, the “gravitational constant” now comes out of the equations naturally. The Lagrangian (4.5) can be written in the following form:

\[
L = \frac{1}{2} m_k^* v_k^2 + \frac{1}{2} m_k^* c^2 - 2 m_k G (\vec{A}_k \cdot \vec{\beta}_k) + \sum_{j \neq k} G m_k \frac{m_j}{r_{kj}} \beta_j^2
\]

(4.7)

\[
m_k^* = m_k (1 + \frac{2 G \delta \Phi}{c^2})
\]

(4.8)

All the terms of (4.7) are also present, although with slightly different numerical factors, in the second order \((1/c^2)\) expansion of general relativity; the Einstein-Infeld-Hoffmann (EIH) equations. Therefore, the Machian formulation based on the kinetic energy (2.10) does already, despite being a classical theory, include much of the phenomena which are currently exclusively present in first post-Newtonian orders of general relativity. These are gravitomagnetism, the induction of inertial mass due to matter present in the vicinity according to (4.8), and a rest energy of \( E_0 = \frac{1}{2} m_0 c^2 \). Especially the second one, the induction of inertial mass, is a purely Machian

4) This, together with eq. (4.4), was already obtained by K. Retzlaff [4].
5) An expression like this was obtained by various authors [5,6].
effect which can be used to test Mach’s principle by terrestrial measurements; we will expound this in section 9.

In (4.8) the „1“ is just the part of the mass which is induced by the background of the universe. As said, the corresponding terms also exist in general relativity. There, the „1“ comes from the Minkowski-metric representing the flat space-time background. Carrying over the Machian interpretation, it would mean that the background space-time is induced by the background mass distribution. This will be further discussed in section 6.

We now determine the equations of motion. We do it in the general case (4.3) by using Euler-Lagrange equations:

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{\mathbf{r}}_k} = \frac{\partial L}{\partial r_k}
\]  

(4.9)

Neglecting the last term, which is quadratic in \( \beta_j \), we obtain:

\[
\frac{2\Phi_k}{c^2} \frac{\partial v_k}{\partial t} = \vec{E}_k - 2\vec{\beta}_k \times \vec{B}_k
\]  

(4.10)

where the gravoelectric and magnetic fields are given by:

\[
\vec{E}_k := \vec{\nabla}_k \Phi_k + \frac{2}{c} \frac{\partial A_k}{\partial t}
\]  

(4.11)

\[
\vec{B}_k := \vec{\nabla}_k \times \vec{A}_k + \vec{\beta}_k \times \vec{\nabla}_k \Phi_k
\]  

(4.12)

If we divide through the factor on the left hand side in eq. (4.10):

\[
\frac{\partial v_k}{\partial t} = \frac{c^2}{2\Phi_k} (\vec{E}_k - 2\vec{\beta}_k \times \vec{B}_k)
\]  

(4.13)

and compare the expressions on the right hand side with the common definitions of the gravitoelectric and magnetic fields, we find:

\[
G = \frac{c^2}{2\Phi(\bar{r})}
\]  

(4.14)

Again, the „gravitational constant“ comes out of the equations in a natural way.

The presence of gravitomagnetic phenomena can be interpreted in a simple way: Since the Lagrangian (3.2) depends on the relative velocities, what “rest” locally means is defined not only by the gravity fields of the surrounding bodies, but also by their accelerations and velocities. This gives rise to gravitomagnetic forces as well as the induction term in (4.11). If the particle is far away from other masses, then the right side of (4.10) is zero, and we re-obtain Newton’s first law:

\[
\frac{\partial v_k}{\partial t} = 0
\]

Also, eq. (4.10) states that a particle is always free falling towards the rest of the universe. This can be seen by expressing eq. (4.10) in the rest-system of the particle. There, \( \vec{v}_k = \vec{v}_k = 0 \) holds and we obtain:

\[
\vec{E}_k = 0
\]  

(4.15)

This, together with the gravitoelectromagnetic equations\(^7\) (4.10-4.12) being valid in an arbitrary accelerated frame, was already used by Sciama to explain inertia and derive expression (4.14) for

\(^7\) Apart from a factor of 2 standing by the vector potential; the relativistic value of this number is 4.
the gravitational constant [5]. However, both were postulated in his work; here, it is just the result of the Euler-Lagrange equations when applied to the Lagrangian (3.2). The Lagrangian (3.2) can be extended to include electromagnetic phenomena as well by simply adding the electromagnetic potential divided by G (since we rescaled the whole Lagrangian (3.2) by this factor):

\begin{equation}
V_{el} = \frac{1}{2} \sum_{i \neq j} \frac{q_i q_j}{4 \pi \varepsilon_0 G r_{ij}} \left(1 - \langle \vec{\beta}_i, \vec{\beta}_j \rangle \right)
\end{equation}

This yields:

\begin{equation}
L = \frac{1}{2} \sum_{j \neq i} \frac{m_i m_j}{r_{ij}} \left(1 + \vec{\beta}_i \cdot \vec{\beta}_j \right) - \eta \left( \frac{m_e}{e} \right)^2 \frac{q_i q_j}{r_{ij}} \left(1 - \langle \vec{\beta}_i, \vec{\beta}_j \rangle \right)
\end{equation}

where \( \eta \) is the relative strength of electric and gravitational force between two protons; the first dimensionless number of Dirac:

\begin{equation}
\eta := \frac{1}{4 \pi \varepsilon_0 G} \left( \frac{e}{m_e} \right)^2 \approx 10^{42}
\end{equation}

Despite being able to eliminate the gravitational constant, this one still enters the equations without explanation. Since it already pops up in a classical Lagrangian, one expects it to have an explanation which has no need of quantum mechanics, or relativity. This awaits further discussion.

5. Explanation of G:

The derived equations allow us to explain, what G actually is, and why gravity is such a weak force. We consider a particle in its rest frame. There we can write for the velocities of the other particles in the universe \( \vec{\beta}_j = -\beta_{k,0} + \beta_{j,0} \) with \( \beta_{k,0} \), \( \beta_{j,0} \) being the velocities of the particles relative to the rest frame defined by all masses. We then have, according to (4.13), in the rest frame of the particle:

\begin{align}
m_k \cdot \vec{\nabla}_k \Phi_k + \frac{2 m_k}{c} \frac{\partial \vec{A}_{k,0}}{\partial t} &= \frac{2 m_k}{c^2} \frac{\partial v_{k,0}^-}{\partial t} \tag{5.1} \\
p_k^- = \sum_{j \neq k} \frac{m_j \beta_{j,0}}{r_{ij}} \tag{5.2}
\end{align}

From eq. (5.1), we can see that the induced mass arises from the gravitoelectric induction effect caused by the whole universe. When the particle is accelerating relative to it, it sees a time-varying mass current and therefore experiences an induction force, which is opposed to its acceleration. The particle does now accelerate exactly by an amount that the induction effect cancels the gravitation, according to eq. (4.15). This results in the expression (4.14) for the gravitational scalar, which is consequently the inverse of the strength of this induction effect. This also explains why gravitational acceleration is so small: Since there is such an enormous amount of matter in the universe, already very tiny accelerations lead to a large (opposed) induction force which is sufficient to cancel the gravitational force. If there was considerably less matter, gravity would be predicted to be much stronger. E.g. if the universe consisted only of the milky way, then gravity would be roughly \( 10^{7} \) times stronger than it is in our universe, at least if \( c \) would keep its known value in such a situation. This was already pointed out by Sciama [5]. That an acceleration relative to all other masses does cause such an induction effect, as opposed to the currently accepted formulation of Newtonian mechanics, is a direct consequence of the kinetic energy and momentum being defined relative to all other masses in the universe, according to (2.10).
We can also approximately calculate the value of the (local) gravitational constant from (4.14) by neglecting a possible foreground contribution. For the contribution of the background we have for an approximately homogeneous universe:

\[ \Phi_0 = \int_{K, c \in \mathbb{R}^3} \rho_i d^3 r_i \approx 4\pi \rho_0^R \int_0^R r \, dr = \frac{3}{2} \frac{M_u}{R_u} \]

(5.3)

where \( M_u, R_u \) are (visible) mass and radius of the observable universe. We obtain:

\[ G_0 \approx \frac{1}{3} \frac{R_u c^2}{M_u} \]

(5.4)

Indeed, inserting \( M_u \approx 10^{53} \text{ kg} \), \( R_u \approx 4 \cdot 10^{26} \text{ m} \) and \( c \approx 3 \cdot 10^8 \frac{\text{m}}{\text{s}} \) we obtain:

\[ G_0 \approx 8 \cdot 10^{-11} \frac{m^3}{\text{kg s}^2} \]

which is an agreement to a very good accuracy since the mass and the radius of the universe are only known by orders of magnitude, and the approximation of a homogeneous universe, if at all, only holds on the largest scales. The mere fact that such a relation holds to such an accuracy is another reason in favour of every theory providing a relation like (5.2) since it is most unlikely that its numerical validity is just a mere coincidence, instead of the result of a deeper rooted mechanism (for example Mach’s principle).

6. A relativistic discussion:

We now make an attempt to discuss the relativistic case of the Hofmann argument. This turns out to be somewhat complicated since one needs to find expressions which are relational and Lorentz invariant at the same time. Astonishingly, for velocities such an expression does indeed exist, namely:

\[ \beta_{ij} = \frac{\sqrt{\beta_i^2 - (\vec{\beta}_i \times \vec{\beta}_j)^2}}{1 - (\vec{\beta}_i, \vec{\beta}_j)} \]

(6.1)

which is relational and at the same time Lorentz invariant. On the other hand, for spatial quantities such an expression is impossible to construct. This already hints into a direction that rapidity should be a more fundamental quantity than space and time. We later want to argue (section 7) that space and time are emergent phenomena in the sense of Mach’s principle that „without matter there is (at least) no space“ and maybe also no time.

If we now, analogously to what is done in the canonic transition to relativistic mechanics, replace:

\[ m_i m_j \frac{v_{ij}^2}{2} \rightarrow -m_i m_j c^2 \sqrt{1 - \beta_{ij}^2} \]

(6.2)

we have indeed:

\[ m_i m_j \frac{1}{r_{ij}} \sqrt{1 - \beta_{ij}^2} \approx m_i m_j \frac{1}{2 c^2 r_{ij}} v_{ij}^2 \]

---

8) Here, one can see that if the universe indeed consisted only of the milky way we had \( M_u \approx 10^{41} \text{ kg} \) and \( R_u \approx 10^{21} \text{ m} \), and therefore roughly (ignoring it’s completely different, non-isotropic structure when seen from the earth) \( G \approx 6 \cdot 10^{-4} \frac{m^3}{\text{kg s}^2} \), \( 10^7 \) times the known value.
which is just the non-relativistic expression (2.10). Now, also \( \frac{1}{r_{ij}} dt \) has to be replaced by some relational quantity which is an invariant, so that the whole action remains invariant. Here exists a problem that it is not possible to find such a quantity in space and time coordinates. We can only find a quantity which is Lorentz-invariant and contains, in lowest order, the relational quantity \( \frac{1}{r_{ij}} \). To find it, we make use of:

\[
\text{inv} = \phi_{\mu} dx^\mu = c \ dt \phi_i - \langle \bar{d}x_i, \bar{A}_j \rangle = c \ dt \left( \phi_i - \langle \bar{\beta}_i, \bar{A}_j \rangle \right)
\] (6.3)

to replace:

\[
\frac{1}{r_{ij}} \rightarrow \frac{1}{r_{ij}} (1 - \langle \bar{\beta}_i, \bar{\beta}_j \rangle)
\]

in eq. (2.10) to get a Lorentz invariant action. Then, last but not least, we have to take into account retardation by replacing (each for the i th particle) every quantity:

\[
X_j \rightarrow [X_j] := X_j(t - \frac{r_{ij}}{c})
\]

Inserting (6.2) into (2.10) and using the identity:

\[
\sqrt{1 - \beta_{ij}^2} = \frac{\sqrt{1 - \beta_i^2} \cdot \sqrt{1 - \beta_j^2}}{1 - \langle \bar{\beta}_i, \bar{\beta}_j \rangle}
\] (6.4)

we obtain:

\[
T = - \sum_{i \neq j} m_i m_j \frac{r_{ij}}{r_{ij}} \sqrt{1 - \beta_i^2} \cdot \sqrt{1 - \beta_j^2}
\] (6.5)

\[
V = - \nu [X_j(t - \frac{r_{ij}}{c})] - \nu \sum_{i \neq j} m_i m_j \frac{r_{ij}}{r_{ij}} (1 - \langle \bar{\beta}_i, \bar{\beta}_j \rangle)
\] (6.6)

If we now use the Euler-Lagrange equations to calculate the generalised momentum, we obtain:

\[
\bar{P}_k = \frac{\partial L}{\partial \dot{v}_k} = \frac{2 [\Phi_k^{(\text{rel})}]}{c^2} \frac{m_k \dot{v}_k}{\sqrt{1 - \beta_k^2}} - 2 \nu \frac{m_k}{c} [\bar{A}_k]
\] (6.7)

with:

\[
\Phi_k^{(\text{rel})} = \sum_{j \neq i} m_j \frac{r_{ij}}{r_{ij}} \sqrt{1 - \beta_j^2}
\] (6.8)

Eq. (6.7) is the familiar expression for the relativistic momentum, apart from the fact that the inertial mass is again, like in the non-relativistic case, induced by distant matter according to Mach’s principle:

\[
m_k^* = m_k \frac{2 [\Phi_k^{(\text{rel})}]}{c^2}
\] (6.9)

It can be seen that there occurs an additional factor \( \sqrt{1 - \beta_j^2} \) as well as retardation compared to (4.4). If one neglects retardation, the inertial mass is still isotropic. Taking retardation into account, the inertial mass becomes anisotropic, in disagreement with observation. We will discuss this issue in section 7.2.
The Lagrangian obtained from (6.5) and (6.6) can be made to agree with the second order EIH-Lagrangian by setting \( \nu = \frac{4}{3} \). Expanding up to second order and doing the same calculation with back- and foreground mass distribution, as was done in section 4 leads to:

\[
L_k = \frac{1}{2} m_k \left( 1 + \frac{3G \delta \Phi_k}{c^2} \right) v_k^2 + \frac{1}{3} m_k c^2 + G \delta \Phi_k \left( -\frac{7}{2} m_k G \langle \vec{A}_k, \vec{b}_k \rangle + \frac{1}{2} \sum_{j \neq k} G m_j \frac{m_k}{r_{kj}} (3 \beta_j^2 - \langle \vec{b}_j, \vec{n}_{ij} \rangle \cdot \langle \vec{b}_j, \vec{n}_{ij} \rangle) \right)
\]

(6.10)

which agrees with the EIH-Lagrangian up to terms of second order v/c (apart from a constant). The absolute terms (which are proportional to a single absolute mass \( m_k \)) are induced here, as already in the non-relativistic case, by the background of the universe. In the following section, we want to discuss what this means in the framework of GR.

7. Mach’s principle and emergent space-time:

In the previous sections we saw how the implementation of Mach’s principle gave rise to a background induced by distant matter. We now want to briefly sketch what interpretation this would have within the framework of general relativity. In GR, due to the different formulation, the terms proportional to an absolute mass arise due to the presence of a background metric \( \eta_{\mu,\nu} \). This can be seen by looking at linearised gravity where the metric is written as [8]:

\[
g_{\mu,\nu} = \eta_{\mu,\nu} + h_{\mu,\nu}
\]

(7.1)

with:

\[
\eta_{\mu,\nu} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix}
\]

(7.2)

\[
h_{\mu,\nu} := h_{\mu,\nu} - h^{\mu}_{\nu} \eta_{\mu,\nu} = \int_{\mathbb{R}^3} \frac{[T_{\mu,\nu}]}{R} d^3 r,
\]

(7.3)

The Lagrangian is given (up to second order in v/c) by:

\[
L_k = -mc^2 \sqrt{\left( 1 - \frac{2G \delta \Phi_k}{c^2} \right) - 4 \langle \vec{b}_k, \vec{A}_k \rangle - \beta_k^2 \left( 1 + \frac{2G \delta \Phi_k}{c^2} \right)}
\]

(7.4)

The \( \beta \)'s in (7.4) arises from the background metric \( \eta_{\mu,\nu} \). Considering how they where obtained in the derivation in section 4, let us assume that here they are also the result of a constant background, and use (4.6) to write:

\[
\frac{2G \delta \Phi}{c^2} = \frac{\delta \Phi}{\Phi_0}
\]

again assuming \( \Phi_0 \approx \text{const} \). Then by using that the Lagrangian stays invariant under multiplication by a constant, we obtain:
By using Euler-Lagrange equations we obtain (for the sake of clarity we approximate 
\[ L = -mc^2 \sqrt{\Phi_0 - \delta \Phi_k - 4 \langle \vec{\beta}_k, \vec{A}_k \rangle - \beta_k^2 (\Phi_0 + \delta \Phi_k)} \]  
(7.5)

By using Euler-Lagrange equations we obtain (for the sake of clarity we approximate 
\[ \frac{1}{\sqrt{\Phi_0 - \delta \Phi_k - 2 \langle \vec{\beta}_i, \vec{A}_i \rangle - \beta_k^2 \Phi_k}} \approx \frac{1}{\sqrt{\Phi_0}} \]  , it changes with respect to what we want to show) :

\[ \frac{d}{dt} \left( \frac{2m_k \Phi_k}{c^2} \vec{v}_k - \frac{2m_k}{c} \vec{A}_k \right) = m_k \vec{V} \left( \delta \Phi_k (1 + \beta_k^2) + 2 \langle \vec{\beta}_k, \vec{A}_k \rangle \right) \]  
(6.7)

It can be seen that now, analogously to eq.(4.4), the „inertial mass“ is given by:

\[ m_\ast = \frac{2m_k \Phi_k}{c^2} \]  

Now eq. (7.5) implies that:

\[ \eta_{\mu, \nu} = \begin{bmatrix} \Phi_0 & 0 & 0 & 0 \\ 0 & -\Phi_0 & 0 & 0 \\ 0 & 0 & -\Phi_0 & 0 \\ 0 & 0 & 0 & -\Phi_0 \end{bmatrix} \]  
(7.7)

With this we now have:

\[ \lim_{M \to 0} \eta_{\mu, \nu} = 0 \]  
(7.8)

stating Mach’s principle in the form that space and time are phenomena arising from the presence of matter in the universe.

Expression (7.7) would still leave the light cone invariant; all points satisfying \( ds = 0 \) also satisfy \( ds' = 0 \) after a Lorentz transformation. The speed of light is therefore still the same as measured by any observer. But expression (7.7) is not completely invariant, since only the mentioned points behave this way under Lorentz transformation. From an experimental point there is no need to require this complete invariance anyway. However, since what we do here is only a sketch in linear theory, it may be possible that in a complete theory of emergent space-time, the diagonal elements of (7.7) are Lorentz-invariants containing \( \Phi_0 \) as the non-relativistic limit.

In the general case, general relativity states that for a spherical mass distribution, we have:

\[ ds^2 = c^2 dt^2 \left( 1 - \frac{2G \delta \Phi}{c^2} \right) - dx^2 \frac{1}{1 - \frac{2G \delta \Phi}{c^2}} \]  
(7.9)

Since the term multiplied by \( dx^2 \) determines the behaviour of some unit length, it can be seen that each mass „generates“ isotropically a certain amount of space in its vicinity, making it „roomier“. Therefore it is more accurate to say that masses in general relativity do not actually curve, but create space. Again, this implies that the „1“ here, which corresponds to the flat background metric, represents the space created by all other masses in the universe. It is to be expected, since the gravity field of the universe is a very strong field, that the contribution of non-linear effects to the local space-time background will dominate. But exactly these non-linear effects are the ones which cannot be correctly described by general relativity since there occur two singularities in the Schwarzschild metric, each at \( r = r_g \) and \( r = 0 \). The latter is already present in Newtonian mechanics and shows that when approaching the mentioned limit, either other effects must occur which are not considered, or the present effects are not described properly. In either case, the relevant non-linear effects which play an important role in the creation of the background space-time are not included properly.
8. Isotropy of inertia:

We’ve seen that, according to (4.4), the inertial mass is given by:

\[ m_k^* = m_k \frac{2 \Phi_k}{c^2} \]

Since this expression is a scalar, it follows that in the classical theory, inertial mass is isotropic despite the Lagrangian being relational. This fact is required by the high precision measurements in the Hugh-Drever experiments which state (to present date) an upper bound for possible mass anisotropies of \( \frac{\Delta m}{m} \leq 10^{-34} \) [9]. The situation of inertial mass being isotropic does not change in the relativistic discussion as long as retardation is neglected. In this case, the inertial mass is still given by the above expression, just with:

\[ \Phi_k^{(rel)} = \sum_{j \neq i} \frac{m_j}{r_{kj}} \sqrt{1 - \beta_j^2} \]

instead of with \( \Phi_k \), which is still isotropic since no scalar products occur. However, the situation does change when retardation effects are taken into account. If we expand this expression around \( t_{ret} = t \) to second order and use that the Lagrangian stays invariant under adding a total time derivative, we obtain:

\[ \Phi_k^{(rel)} \propto \sum_{j \neq i} \frac{m_j}{r_{kj}} \langle \vec{\beta}_k, \vec{\beta}_j \rangle \]

stating that inertia is dependent on the angle between the particles’ velocity and the velocities of the masses in the universe. This leads, assuming one mass in the vicinity of the particle to be dominating, to a relative anisotropy of magnitude:

\[ \frac{\delta m}{m} \sim \frac{G M}{2 R c^2} \left| \vec{\beta}_k \right| \left| \vec{\beta}_j \right| \]

where \( \text{M} \) is the disturbing body’s mass, \( \beta \) its velocity relative to the chosen frame. E.g. for an electron in a hydrogen atom, the anisotropy caused by the galactic centre would be

\[ \frac{\delta m}{m} \sim 10^{-16} \]

which is clearly above the upper limit. Indeed, this is problematic anyway when trying to implement Mach’s principle and retardation at the same time: Since gravitational action travels at a finite speed \( c \), which state of the universe exactly does determine a particles’ inertia? Again, the same term as (8.1) is also present in general relativity [7]. Therefore it is also affected by the problem of the anisotropic mass9). This problem is widely ignored, probably because no apparent solution exists. Considering how the anisotropy arose in the relativistic discussion in section 5, it is suggestive that in general relativity too it arises due to retardation effects. In the author’s opinion, this is an argument in favour of gravity being an action-at-a-distance: If, to whatever reason, retardation effects would effectively cancel out, it would both solve the problem what state would determine a particles inertia as well as remove the anisotropy terms. This awaits further discussion and will necessarily be part of a relativistic implementation of Mach’s principle.

9) This was already mentioned by Treder [3] and Retzlaff [4].
9. Variations of the effective local strength of gravity:

According to what was said in the previous sections, mechanics built on Mach’s principle would predict that different measurements of the gravitational constant cannot yield exactly the same value unless the distances to the surrounding masses stay constant during the measurements. Indeed, the Lagrangian (4.3) predicts that the strength of gravity in the close vicinity of masses is weaker and in the greater distance of them stronger as compared to Newtonian theory. To see this, we stay in the situation given in section 4. We then have, according to (5.1), for the acceleration of some test particle in lowest order (the higher order terms have no impact on the statement):

\[ \frac{\partial \vec{v}_k}{\partial t} = G \cdot \hat{n} \delta \Phi_k \]  

(9.1)

\[ G = \frac{1}{1 + \frac{2 G_0 \delta \Phi}{c^2}} \]  

(9.2)

Since this is exactly the same expression which one obtains from the EIH-Lagrangian (up to a factor of 3/2 and the fact that the gravitational constant is defined as \( G_0 \) in above equation), all the following statements also apply to general relativity. One can simply derive now that an observer at some point \( \vec{r}_0 \) (e.g. the Earth) within a foreground mass distribution who performs a measurement of gravity’s strength \( G_0 \) and interpolates according to Newton’s law gets the acceleration wrong by:

\[ \delta \vec{a} = (G(\vec{r}) - G(\vec{r}_0)) \hat{n} \Phi \]  

(9.3)

or by Taylor expanding \( G \) up to first order in:

\[ \delta \vec{a} = G_0 \cdot \frac{\delta \Phi(\vec{r}_0) - \delta \Phi(\vec{r})}{\Phi_0} \hat{n} \Phi = \frac{\delta \Phi(\vec{r}_0) - \delta \Phi(\vec{r})}{\Phi_0} \vec{a}_N \]  

(9.4)

where \( \vec{a}_N \) is the gravitational acceleration expected by Newton’s law. From this equation can be seen the following: The observer who extrapolates Newton’s law of gravitation would, by doing so, overestimate gravity’s strength in the closer vicinity of masses than he found himself in, and underestimate it at a greater distance from them.

Let us now look what eq. (9.4) would imply for the strength of the gravitational surface acceleration of the Earth. Since the Earth orbits the Sun, we have \( \delta \Phi = \Phi_E + \Phi_S \) (the contributions of other planets are negligible) and for the surface acceleration of the Earth, \( \vec{a}_N \approx -G_0 \frac{M_E}{R_E^2} \hat{g} \). The potential of the sun is given by \( \Phi_S = \frac{M_S}{r} \), where \( r \) is the distance between the Earth and the Sun.

Plugging this into (9.4) together with:

\[ r = a \frac{(1 - e^2)}{1 + e \cdot \cos(\theta)} \]

where \( a = 1 AU \) is the great half axis of the Earth-Sun system and \( e \) the eccentricity of the Earth’s orbit, we obtain:

\[ \delta \vec{g} = \frac{2 G_0 M_S}{a c^2} \frac{e}{1 - e^2 (\cos(\theta_0) - \cos(\theta))} \vec{g}_0 \]  

(9.5)
Here we have again inserted (4.6). $\theta_0$ is the angle corresponding to the Earth’s position where the initial measurement was made. According to (8.5), the strength of the Earth’s surface acceleration will vary periodically over a year, having its minimum value when the earth is in its perihelion. The relative magnitude of this effect is:

$$\frac{2G_0 M_S}{ac^2} e \approx 10^{-10}$$

The same effect does occur in every system whose distance changes over time relative to a massive object nearby. Since this effect occurs due to inertial mass being determined by gravitation of other masses according to eq. (4.8), it’s a direct expression of Mach’s principle. A confirmation of this effect would therefore also be a confirmation of this principle. Finally, we want to note that also the perihelion shift of Mercury is caused by the mass induction term in general relativity, which is, as already mentioned, of the same form as (4.8), just with a factor of 3/2 standing by the second addend. Since these terms are clearly of Machian nature, also the precession of Mercury can already be regarded as a strong evidence for Mach’s principle. However, it is not mandatory since other terms (e.g. the first relativistic correction to the kinetic energy) also yield a contribution to the precession, and therefore this observation does not necessarily imply the presence of terms like (4.8).

10. Conclusions:

I have shown a way how to implement Mach’s principle into classical mechanics, and argued that kinetic energy should be redefined as was proposed by Hofmann. It would cover a variety of phenomena already in classical mechanics, which are currently solely attributed to general relativity, namely those which are of Machian nature. Above all, it explains the nature of one arbitrary number, the gravitational constant. I therefore think that classical mechanics should be formulated in this way. Consequently, also with respect to relativistic theories, it would be worthwhile to stretch beyond general relativity: It does not explain the gravitational constant which is, to my understanding, a result of not correctly incorporating Mach’s principle. Apart from not being able to explain the isotropy of inertia, it also causes some well known problems, especially in cosmology and has singular solutions, which must not appear in any complete theory: Nature is regular everywhere.

If we are to understand phenomena like dark matter or dark energy, the only way we can hope to one day succeed is to get an understanding of why the laws of physics are the way they are. As said at the beginning, I think the constants of nature play a key role in understanding; therefore they have to be questioned in particular. By finding more expressions like the one linking $c$ and the electromagnetic constants, or the one obtained here for $G$, it will be possible to explain all dimensional constants in the laws of nature, without setting them to unity, which is a faulty practice anyway since it just ignores the fact that there is something to explain. The dimensionless quantities which then remain may ultimately provide a link between physics and pure mathematics.

References:


[4] Klaus Retzlaff, Projekt Machsches Prinzip (Trägheitsfreie Mechanik) der Astronomischen Gesellschaft Magdeburg e.V.


