Theory of relativity and measurement uncertainty

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Abstract

Relativity of simultaneity is shown to limit the accuracy in simultaneous measurement of position and momentum of a moving body by an inertial observer. The inaccuracy or uncertainty product derived from special relativity theory is found to be similar to the much known quantum mechanical uncertainty relation. However, whether the above two uncertainty relations indicate the same physics is debatable and any conclusion is not attempted here. A modification of this relativistic uncertainty formula under gravitational field is achieved using general relativity principles, and a small gravitational correction term has to be introduced. This gravitational term being small remains negligible in ordinary conditions but becomes appreciable at small time scale.

Keywords: Special theory of relativity; Gravitation; General theory of relativity; Uncertainty principle

Introduction

Even before Albert Einstein, it was well recognized that the synchronization of distant clocks to be a key problem but viewed only as an experimental difficulty[1]. Einstein's special relativity paper[2] in 1905 first pointed out the fundamental problem in synchronizing clocks placed in different inertial frames and defined simultaneity as a relative phenomenon. This relativistic principle is now famously known as relativity of simultaneity. It is often considered as a crucial breakthrough Einstein achieved in his 1905 paper. Briefly, the relativity of simultaneity states that events that are simultaneous in one observer's frame may occur at an interval in another inertial observer's frame. An important consequence of the simultaneity principle, the relativistic time dilation phenomenon, has therefore been verified experimentally[3, 4]. The aim of the present study is to understand how the consideration of relativistic simultaneity may change the accuracy and precision in an otherwise perfect classical measurement. It could be realised through simple mathematics that relative simultaneity between different inertial frames leads to unsurpassable inaccuracy in the measured position (Δx) and momentum (Δp) of a moving body. This new uncertainty relation derivable from special relativity equations would be shown as $\Delta x.\Delta p \geq {}^{h}/_{2}$, which has similar form to the much known Heisenberg's uncertainty relation. However, Heisenberg's uncertainty is a quantum mechanical principle, and points to the restriction on obtaining the simultaneous knowledge of canonically conjugate variables of quantum mechanical systems[5-7]. Originally Heisenberg proposed the uncertainty relation heuristically, pointing to the constraints on joint measurability of variables resulting from a trade-off between accuracy and disturbance. The uncertainty product was derived on the basis of a supposed experiment observing an electron using a γ-ray microscope[5] and a detailed physical picture about the origin of the said inaccuracies was not fully explained. In the Hilbert space the uncertainty relation has been shown to imply, that it is impossible to prepare systems that have both their position probability distribution and their momentum probability distribution sharply concentrated around single values, and the uncertainty product could only be derived under quantum mechanical formalisms[6, 7]. Therefore, identification of the present relativistic uncertainty with that of Heisenberg's relation may not be concluded in a straightforward manner and will not be attempted here.

Whereas special relativity accounts for the Lorentz invariance involving inertial observers, the general relativity theory was proposed by Einstein to account for gravitation[8]. In essence general relativity describes how a massive object causes distortion to the space-time surrounding it. An important consequence of general relativity theory is that clocks run slowly in the curved space-time near massive objects[8-11]. Since firstly we shall derive the uncertainty relation using the concept of relative simultaneity in the flat Minkowsky space-time[12], we shall further seek a simple extension of it under curved space-time from the general relativity principles. It will lead to a modified uncertainty relation having the above mentioned flat space term ($\Delta x.\Delta p \geq h/_2$) and an additional gravitational correction term. When the modified uncertainty relation is evaluated at large cosmological scale, the ratio between the gravitational correction and the flat space term could be estimated to be ~ 10^{-121}, a value similar to the ratio between the observed dark energy density and quantum field theory estimation for the vacuum energy density.

Measurement uncertainty and special theory of relativity

Let us suppose the two inertial frames $\mathcal{K}$ and $\mathcal{K}^{'}$, with spatial origins O and $O^{'}$ and coordinates labels (t,x,y,z) and $(t^{'},x^{'},y^{'},z^{'})$, are attached to an observer/measuring device stationary in the laboratory frame of reference and an electron moving with an uniform relative velocity v ($< c$, the speed of light) respectively. Isotropy of space allows us to orient the spatial axes so that the relative velocity between the frames is along x and $x^{'}$. Considering the relationship between the other coordinates as trivial, we shall ignore them and assume $y = y^{'} = z = z^{'} = 0$, and focus only upon (t,x) and $(t^{'}, x^{'})$. t and $t^{'}$ represent the epochs recorded by similar clocks at rest in $\mathcal{K}$ and $\mathcal{K}^{'}$. The measurement method used by the observer can be arbitrarily chosen as we shall see the results of our analysis are independent of the measurement method. For sake of completeness of description, we may assume the observer to be equipped with a hypothetical noiseless measurement apparatus. Now, to meet both position and momentum measurements, the observer need to carry out the measurement over an interval Δt, during which the electron moves by distance Δx. Let the measurement be described by freely chosen start and end points, which represent two events (x_1, ct_1) and (x_2, ct_2) in the Minkowski space, where $\Delta x = x_2 - x_1$ and $\Delta t = t_2 - t_1$. The intervals Δt and Δx can be chosen arbitrarily small. It is worthwhile to mention here that we take $\Delta t \neq 0$, since $\Delta t = 0$ implies no measurement performed by the observer.

For simultaneous knowledge of the position and momentum of the electron, we must have $\Delta t' = 0$ in the inertial frame attached to the electron. This implies the events (x_1, ct_1) and (x_2, ct_2) occur simultaneously in the electron's frame. Under this simultaneity requirement we will try to find out the limits for values of Δx and Δt in the observer's frame. It is pertinent to note here that in the semi-classical approach adopted for the present analysis, the electron's velocity v should be taken equal to the corresponding group velocity of the associated de Broglie wave packet[13].

In Minkowsky space-time, the relative position and time intervals in the electron's frame are given as[14],

$$\begin{pmatrix} c\Delta t' \\ \Delta x' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} c\Delta t \\ \Delta x \end{pmatrix} \tag{1}$$

Where, $\beta = {}^{v}/_{c}$ and $\gamma = \sqrt{1 - {}^{v^2}/_{c^2}}$. The condition of simultaneity is obtained by letting $\Delta t^{'} = 0$. Hence,

$$\beta \Delta x = c \Delta t \tag{2}$$

or $\Delta x.\Delta \nu = c/\beta$, since the electron can be associated with its de Broglie wave with $\Delta \nu \sim \frac{1}{\Delta t}$, phase velocity $v_{ph} = {}^{c}/_{\beta} = \nu\lambda$, where λ is the wavelength and ν is the frequency. The momentum of the electron is related to the wavelength by the de Broglie formula, $p = {}^{h}/_{\lambda} = {}^{h\nu}/_{v_{ph}}$, where h is the Planck's constant. Taking $\Delta p = h \Delta \nu \beta / c$, we get

$$\Delta x.\Delta p = h \tag{3}$$

Considering this as the lower limit,

$$\Delta x.\Delta p \geq h \tag{4}$$

It is important to mention here that recognizing the reciprocal nature of Lorentz transformations, the above uncertainty relation should also be valid when simultaneous values are sought in the observer's frame. Further, since statistically the errors in measurements can be assumed to be similarly distributed between the events (x_1, ct_1) and (x_2, ct_2), we can write in terms of measurement precisions,

$$\overline{\Delta x}.\overline{\Delta p} \geq h/2 \tag{5}$$

Since virtually in all set of measurements we may work with the precision values, we may conveniently write,

$$\Delta x.\Delta p \geq h/2 \tag{6}$$

Uncertainty relation (6) is of the similar form as the Heisenberg's uncertainty relation. But whether the two relations should refer to the same physical phenomenon is outside the scope of the present paper. However, it is pertinent to mention here that an uncertainty relation involving position and momentum measurement of an electron with a light signal, had been derived earlier by Rosen and Vallera from special relativity principles and they found $\Delta x.\Delta p$ should have values around h with upper and lower bound depending on the electron's velocity[15].

Additionally, since $\Delta x.\Delta \nu = c/\beta$, multiplying by h and using (2) we can also write,

$$\Delta E.\Delta t = \Delta x.\Delta p \geq h/2 \tag{7}$$

Gravitational correction

The above uncertainty relations are valid only in a flat space-time geometry. It will be therefore worthwhile to seek for an extension of (6) and (7) that will be also consistent in curved space-time under a gravitational field. In this effort we will consider only basic general relativistic concepts to obtain a first order approximation. The kinematical effects of special relativity and the general relativistic warping of space-time often expected to be coupled in the situation where a frame moves in a gravitational field in an arbitrary way. Therefore, here we would try to achieve a first order approximation for the extension of (6) by considering the simple case when a neutral particle is moving along a radial trajectory in an isotropic static gravitational field near a spherical mass (*M*). If we consider measurements only performed away from the Schwarzschild radius, the gravitational effects can be suitably described by the Schwarzschild metric[9, 10],

$$\Delta s^2 =-\left(1-2\frac{GM}{rc^2}\right)c^2\Delta t^2+\left(1-2\frac{GM}{rc^2}\right)^{-1}\Delta r^2+r^2(\Delta\theta^2+sin^2\theta\Delta\emptyset^2) \tag{8}$$

We shall ignore the variations introduced in space-time by any quantum effects or from any other origin. These considerations, of course, can be a source of more uncertainty but not important in the present analysis. Our aim is to find out how uncertainty relations (6) and (7) may change when the local observation frame of particle's motion is positioned inside the gravitational field of *M* and the final observer is at a far away place so that a difference of potential exists between them.

A simple analysis comprising of two observers, one positioned locally to the particle and another at a far place, can be carried out to understand the measurement. First, let us assume a stationary observer A (shell observer) is positioned at a radial distance r ($>r^*$, Schwarzschild radius) from *M* and the test particle is freely falling towards *M.* Observer A performs her measurements as the particle passes by. Since in that scenario, both are positioned locally to each other, that is in the same tiny region of space around the massive object, for an infinitesimal measurement interval the particle approximately undergoes inertial motion with respect to observer A. Hence locally we can build Minkowskian coordinate by transforming the metric tensor of general coordinate ($g_{\mu\nu}$) to the metric tensor of inertial coordinate ($\eta_{\mu\nu}$) that is $g_{\mu\nu} \approx \eta_{\mu\nu}$. Hence the observer A's measurement becomes similar as in the case of the flat space-time and the uncertainties are given by (2), say $\Delta r'/\Delta t' = c/\beta$. Next, we consider another observer B, situated at a far away place along the radial coordinate so that the gravitation field is weak at B ($g_{\mu\nu} \neq \eta_{\mu\nu}$) . The measurements performed by observer A are relayed to the far away Observer B, who then looks into the measurement results of observer A, and adjust for her own coordinates. Taking into account the Schwarzschild metric, the observer B will find the earlier measured $\Delta r'$ and $\Delta t'$ should be corrected to Δr and Δt, where

$$\frac{\Delta r}{\Delta t}=\frac{\Delta r'\left(1-2\frac{GM}{rc^2}\right)^{1/2}}{\Delta t'\left(1-2\frac{GM}{rc^2}\right)^{-1/2}}=\frac{c}{\beta}\left(1-2\frac{GM}{rc^2}\right) \tag{9}$$

The corresponding uncertainty becomes,

$$\Delta r.\Delta p \geq \frac{h}{2}\left(1-2\frac{GM}{rc^2}\right) \tag{10}$$

$$\text{or } \Delta r.\Delta p \geq \frac{h}{2}\left(1-\frac{r^*}{r}\right) \tag{11}$$

where the Schwarzschild radius given as, $r^* = {2GM}/{c^2}$. Conditions (10) and (11) represent the modified uncertainty relation in a curved space-time. It is pertinent to note here that considering the coordinate singularities of the Schwarzschild metric at $r = r^*$, (10) and (11) are valid only in the region $r > r^*$. It is also important to mention here that since measurement uncertainties originate from the finite values of $\Delta r'$ and $\Delta t'$ as described in the flat space case, modification of the values of $\Delta r'$ and $\Delta t'$ to Δr and Δt due to the gravitational effects should be adequate to describe the errors in the far away observer's frame. Under ordinary situations, like for an observer carrying out measurements on phenomena occurring on earth's surface from a space station, the change in equation (6) will be only about $h \times 10^{-10}$, and therefore can be easily ignored. Hereafter, we refer to the first term in (10) and (11) as the flat space term and the second as gravitational term.

For the far away observer, from (2), (7) and (10) we also have,

$$\Delta E.\Delta t \geq \frac{h}{2}\left(1-2\frac{GM}{rc^2}\right) \tag{12}$$

This is the modified energy time uncertainty relation in curved space-time.

Now, in the case when an external gravitational field is not present, we may consider the particle to experience its own gravitational field by taking particle's self-mass as M and r

should define the volume within which we expect to find the particle. Since the uncertainties may be taken to be equal to the corresponding absolute values of the variables, we have $r = c\Delta t$ and $\Delta E = Mc^2$. Therefore, we may write (12) as,

$$\Delta E \geq \frac{h}{2\Delta t}\left[1 - \frac{hG}{c^5\Delta t^2} + \left(\frac{hG}{c^5\Delta t^2}\right)^2 - \left(\frac{hG}{c^5\Delta t^2}\right)^3 + \ldots\right] \tag{13}$$

For $\frac{hG}{c^5\Delta t^2} < 1$,

$$\Delta E \geq \frac{h}{2\Delta t}\left(1 + \frac{hG}{c^5\Delta t^2}\right)^{-1} \tag{14}$$

$\frac{hG}{c^5\Delta t^2} = 1$ gives, $\Delta t = \sqrt{hG/c^5} = t_e$(say), which is of the same order as Plank time ($\sqrt{\hbar G/c^5} \approx$ 5.39x10^{-44} s)[16]. The corresponding energy is $\Delta E = \frac{h}{4t_e} = \frac{1}{4}\sqrt{\frac{hc^5}{G}} = E_e$(say). Considering the equality limit, the variation of ΔE with Δt is shown in Fig. 1. It can be seen from Fig. 1 that while for $\Delta t \gg t_e$ the energy uncertainty asymptotically merges with the flat space value, $\Delta E = h/2\Delta t$, the gravitational term becomes appreciable as we approach near $\Delta t = t_e$, and a noticeable difference between the energy curves can be seen.

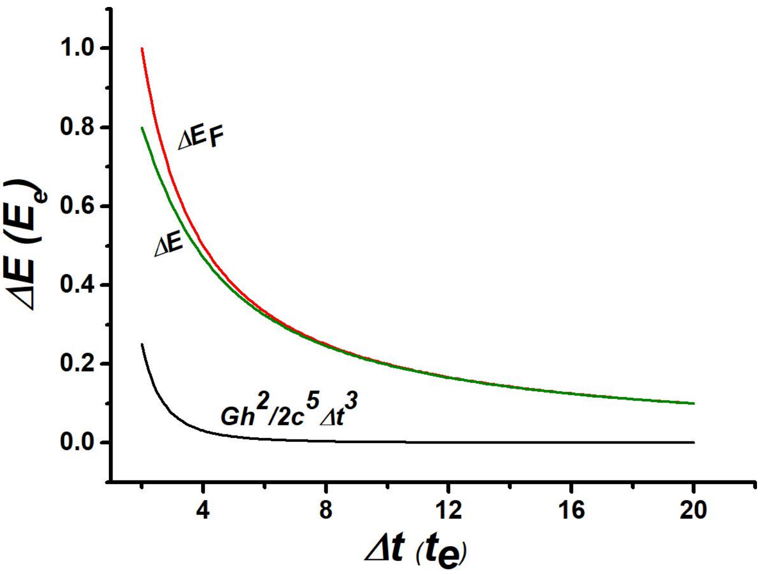

Fig. 1. The variations ΔE and the flat space term, ΔE_F with Δt. The energy and time axes are scaled by E_e and t_e respectively. The total energy ΔE asymptotically merges into flat space term (ΔE_F) for larger Δt but deviates from ΔE_F near $\Delta t = t_e$. The dark line shows the variation of the first gravitational correction term $h^2 G/2c^5\Delta t^3$, which remain negligible at larger Δt but becomes noticeable near $\Delta t = t_e$.

It is also pertinent to note here that since at the cosmological scale, $\frac{hG}{c^5\Delta t^2} \ll 1$, hence (14) can be written as,

$$\Delta E \geq \frac{h}{2\Delta t} - \frac{h^2 G}{2c^5\Delta t^3} \tag{15}$$

A careful inspection of (15) suggests, while the first term in the right hand side represents the energy as seen by a local observer, the second correction term, coming from the curvature of space-time, has an opposing sign. This second term is missing when measurements carried out by any local observer and only appear for an observer situated at a far away place. The opposite sign of the second term suggests that it points to an apparent small loss of energy due to curved space-time when measured from a distance. Therefore, it is reasonable to say, the second term represents a fraction of energy, relatively hidden into the curved space-time fabric over distances to a far away observer and its effects should be best seen at cosmological scale. Hence, considering the ratio between the second and first term,

$$\left(\frac{h^2 G}{c^5 t_0^3}\right)\left(\frac{h}{t_0}\right)^{-1} = \left(\frac{hG}{c^5}\right).\frac{1}{t_0^2} \approx 0.99 \times 10^{-121} \tag{16}$$

Where, $\Delta t \sim t_0 \sim 13.7 \times 10^9$ years[16], the age of our universe in the present epoch, given as,

$$t_0 = \int_0^{t_0} dt = \int_0^1 \frac{da}{aH} \tag{17}$$

H being the scale dependent Hubble parameter and *a* is the scale factor of the universe.

Interestingly, equation (16) is in excellent agreement with the estimated ratio between the observed dark energy density in our present epoch and vacuum energy density calculated from quantum field theory[9, 16].

Conclusion

To conclude our analysis has shown that consideration of simultaneity of relativity during joint measurement of position and momentum of a moving point particle prohibits arbitrary accuracy in the measured quantities and the indeterminacy or uncertainty relation has a similar form as that of quantum mechanical Heisenberg's relation. The uncertainty relation could be extended using simple general relativity principles to account for measurements carried out near a spherical gravitational mass and the modified uncertainty relation comprised of an additional small gravitational term of opposite sign contributed by curved space-time. While under usual circumstances this gravitation term remains small and can be neglected for most of the purposes, it may have implications at small time scale.

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Figure caption:

Fig. 1. The variations ΔE and the flat space term, ΔE_F with Δt are shown. The energy and time axes are scaled by E_e and t_e respectively. ΔE_F is the flat space energy term. The total energy ΔE asymptotically merges into flat space term (ΔE_F) for larger Δt but deviates from ΔE_F near $\Delta t = t_e$. The dark line shows the variation of the first gravitational correction term $Gh^2/2c^5\Delta t^3$, which remain negligible at larger Δt but becomes noticeable near $\Delta t = t_e$.

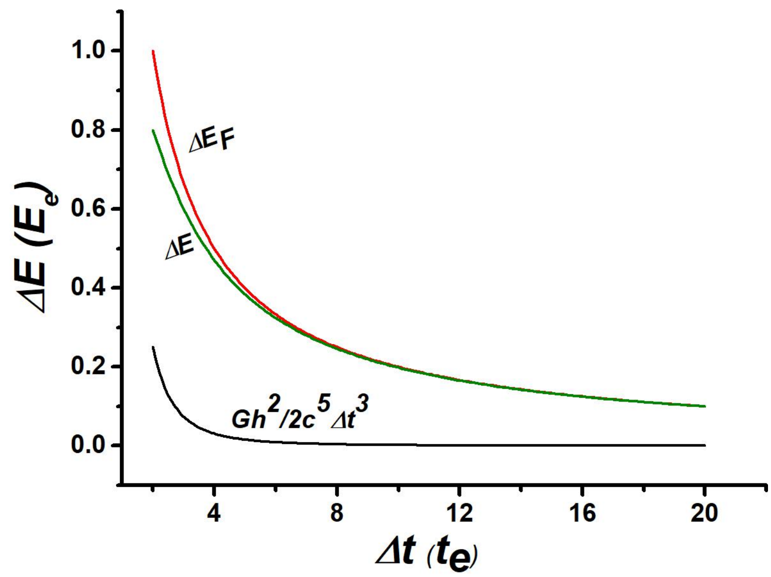

Figure 1