Philosophy of Mathematics and Division by Zero

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Abstract: From the viewpoint of the philosophy of mathematics, we would like to introduce our recent results on the division by zero that has a long and mysterious history.

Key Words: Division by zero, philosophy, division by zero calculus, differential equation, analysis, infinity, discontinuous, point at infinity, Laurent expansion, conformal mapping, stereographic projection, Riemann sphere, horn torus, elementary geometry, zero and infinity, $1/0 = 0/0 = z/0 = \tan(\pi/2) = 0$.

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1 Introduction

First of all, the author feels that we can not discuss on the fundamental ideas of

what is mathematics?,
what is philosophy?
and
what are human beings from these concepts?

However, the author wrote such general concepts in Japanese in

No.81, May 2012(pdf 432kb)
www.jams.or.jp/kaiho/kaiho-81.pdf
Traduzir esta página

Here, we will restrict to the topics of the division by zero from the viewpoint of philosophy of mathematics. For philosophy of mathematics, we think that the problems are on

what is mathematics?,
what is logics in mathematics?
and
what are the relations among mathematics and the universe?

However, we think that the division by zero may be related to philosophy of mathematics.

Here, we can recall the fundamental concepts:

What is INFINITY?
What is ZERO?
What is VOID?
What is NOTHING?
What is the model of the universe and space (time)?

and others.

Incidentally, these fundamental concepts may be related to each others by means of the division by zero with the concept of division by zero calculus. From our results in mathematics for these new concepts, we will be able to see the relations among the above basic concepts. From these viewpoints, we will introduce our results over mathematicians to some general people.

For the long history of division by zero, see [4, 28]. S. K. Sen and R. P. Agarwal [35] quite recentry referred to our paper [10] in connection with division by zero, however, their understandings on the paper seem to be not suitable (not right) and their ideas on the division by zero seem to be traditional, indeed, they stated as the conclusion of the introduction of the book in the following way:

"Thou shalt not divide by zero" remains valid eternally.

However, in [31] we stated simply based on the division by zero calculus that

We Can Divide the Numbers and Analytic Functions by Zero with a Natural Sense.

For the long tradition on the division by zero, people may not be accepted our new results against many clear evidences on our division by zero, however, a physicist stated as follows:

Here is how I see the problem with prohibition on division by zero, which is the biggest scandal in modern mathematics as you rightly pointed out (2017.10.14.08:55).

The common sense on the division by zero with the long and mysterious history is wrong and our basic idea on the space around the point at infinity is also wrong since Euclid. On the gradient or on differential coefficients we have a great missing since $\tan(\pi/2) = 0$. Our mathematics is also wrong in elementary mathematics on the division by zero. In a new and definite sense, we will show and give various applications of the division by zero $0/0 = 1/0 = z/0 = 0$. In particular, we will introduce several fundamental concepts in calculus, Euclidean geometry, analytic geometry, complex analysis and differential equations. On Euclidean geometry and analytic geometry, we will find new fields by the concept of the division by zero. We will give many concrete properties in mathematical sciences from the viewpoint of the division by zero. We will know that the division by zero is our elementary and fundamental mathematics with some new ideas on the universe.

The contents are as follows.

2 Division by zero

The division by zero with the mysterious and long history was indeed trivial and clear as in the followings.

By the concept of the Moore-Penrose generalized solution of the fundamental equation $ax = b$, the division by zero was trivial and clear as $b/0 = 0$ in the **generalized fraction** that is defined by the generalized solution of the equation $ax = b$. Here, the generalized solution is always uniquely determined and the theory is very classical. See [10] for example. However, we can state clearly and directly its essence as follows:

For a complex number α and the associated matrix A, the correspondence

$$\alpha = a_1 + ia_2 \longleftrightarrow A = \begin{pmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{pmatrix}$$

is homomorphism between the complex number field and the matrix field of 2×2.

For any matrix A, there exists a uniquely determined Moore-Penroze generalized inverse $A^\dagger$ satisfying the conditions, for complex conjugate transpose $*$,

$$AA^\dagger A = A,$$

$$A^\dagger AA^\dagger = A^\dagger A,$$

$$(AA^\dagger)^* = AA^\dagger,$$

and

$$(A^\dagger A)^* = A^\dagger A,$$

and it is given by, for $A \neq O$, not zero matrix,

$$A^\dagger = \frac{1}{|a|^2 + |b|^2 + |c|^2 + |d|^2} \cdot \begin{pmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{pmatrix}$$

for

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

If $A = O$, then $A^\dagger = O$.

In general, for a vector $x \in \mathbf{C}^n$, its Moore-Penrose generalized inverse $x^\dagger$ is given by

$$x^\dagger = \begin{cases} 0^* & \text{for } x = 0 \\ (x^*x)^{-1}x^* & \text{for } x \neq 0. \end{cases}$$

Recall the uniqueness theorem by S. Takahasi on the division by zero. See [10, 38] whoes proof is very simple and it is given by three lines:

Proposition 2.1 *Let F be a function from $\mathbf{C} \times \mathbf{C}$ to $\mathbf{C}$ such that*

$$F(a,b)F(c,d) = F(ac,bd)$$

for all

$$a,b,c,d \in \mathbf{C}$$

and

$$F(a,b) = \frac{a}{b}, \quad a,b \in \mathbf{C}, b \neq 0.$$

Then, we obtain, for any $a \in \mathbf{C}$

$$F(a,0) = 0.$$

In the long mysterious history of the division by zero, this proposition seems to be decisive.

Following Proposition 2.1, we should **define**

$$F(b,0) = \frac{b}{0} = 0,$$

and we should consider that for the mapping

$$W = f(z) = \frac{1}{z}, \tag{2.1}$$

the image $f(0)$ of $z = 0$ is $W = 0$ (**should be defined from the form**). This fact seems to be a curious one in connection with our well-established popular image for the point at infinity on the Riemann sphere ([2]). As the representation of the point at infinity on the Riemann sphere by the zero $z = 0$, we will see some delicate relations between 0 and ∞ which show a strong discontinuity at the point of infinity on the Riemann sphere. We did not consider any value of the elementary function $W = 1/z$ at the origin $z = 0$, because we did not consider the division by zero $1/0$ in a good way. Many and many people consider its value at the origin by limiting like $+\infty$ and $-\infty$ or by the point at infinity as ∞. However, their basic idea comes from **continuity** with the common sense or based on the basic idea of Aristotele. – For the related Greece philosophy, see [41, 42, 43]. However, as the division by zero we will consider its value of the function $W = 1/z$ as zero at $z = 0$. We will see that this new definition is valid widely in mathematics and mathematical sciences, see ([14, 18]) for example. Therefore, the division by zero will give great impact to calculus, Euclidean geometry, analytic geometry, complex analysis and the theory of differential equations at an undergraduate level and furthermore to our basic idea for the space and universe.

The simple field structure containing division by zero was established by M. Yamada ([13]) in a natural way. For a simple introduction, H. Okumura [25] discovered the very simple essence that:

To divide by zero is to multiply by zero.

That is, for any complex numbers a and b, the general fraction(division) a/b may be defined as follows; for $b \neq 0$, with its inversion b^{-1}

$$\frac{a}{b} = ab^{-1}$$

and for $b = 0$

$$\frac{a}{b} = ab.$$

Then, the general fractions containing the division by zero form the Yamada field.

For the operator properties of the generalized fractions, see [38].

3 Division by zero calculus

As the number system containing the division by zero, the Yamada field structure is perfect. However, for applications of the division by zero to **functions**, we need the concept of the division by zero calculus for the sake of unique determination of the results and for some important reasons.

For example, for the typical linear mapping

$$W = \frac{z - i}{z + i}, \tag{3.1}$$

it gives a conformal mapping on $\{\mathbf{C} \setminus \{-i\}\}$ onto $\{\mathbf{C} \setminus \{1\}\}$ in one to one and from

$$W = 1 + \frac{-2i}{z - (-i)}, \tag{3.2}$$

we see that $-i$ corresponds to 1 and so the function maps the whole $\{\mathbf{C}\}$ onto $\{\mathbf{C}\}$ in one to one.

Meanwhile, note that for

$$W = (z - i) \cdot \frac{1}{z + i}, \tag{3.3}$$

we should not enter $z = -i$ in the way

$$[(z - i)]_{z=-i} \cdot \left[\frac{1}{z + i}\right]_{z=-i} = (-2i) \cdot 0 = 0. \tag{3.4}$$

However, in many cases, the above two results will have practical meanings and so, we will need to consider many ways for the application of the division by zero and we will need to check the results obtained, in some practical viewpoints. We referred to this delicate problem with many examples in the references.

We will introduce the division by zero calculus. For any Laurent expansion around $z = a$,

$$f(z) = \sum_{n=-\infty}^{-1} C_n (z - a)^n + C_0 + \sum_{n=1}^{\infty} C_n (z - a)^n, \tag{3.5}$$

we **define** the identity

$$f(a) = C_0. \tag{3.6}$$

Note that here, there is no problem on any convergence of the expansion (3.5) at the point $z = a$, because all terms $(z - a)^n$ are zero at $z = a$ for $n \neq 0$, when we use the result $1/0 = 0$.

Apart from the motivation, we define the division by zero calculus by (3.6). With this assumption, we can obtain many new results and new ideas. However, for this assumption we have to check the results obtained whether they are reasonable or not. By this idea, we can avoid any logical problems. – In this point, the division by zero calculus may be considered as a fundamental assumption like an axiom.

This paragraph is important for philosophy and foundation of mathematics. Indeed, whether the definition of the division by zero calculus is a new axiom or a new assumption? Some specialists are interesting in this problem as a problem of mathematical logic. We have a serious problem when we consider it as a new axiom, because we have to check it strictly and it is not simple.

Meanwhile, when we consider it as a new assumption, we have to show its contributions and impact practically. By considering it as an assumption, we are showing its contributions and impact practically. Here, we have to recall:

what is mathematics?

On February 16, 2019 Professor H. Okumura introduced the surprising news in Research Gate:

José Manuel Rodríguez Caballero
Added an answer
In the proof assistant Isabelle/HOL we have $x/0 = 0$ for each number x. This is advantageous in order to simplify the proofs. You can download this proof assistant here: **https://isabelle.in.tum.de/**.

J.M.R. Caballero kindly showed surprisingly several examples to the author by the system that

$$\tan\frac{\pi}{2} = 0,$$

$$\log 0 = 0,$$

$$\exp\frac{1}{x}(x = 0) = 1,$$

and others for the questions of the author. Furthermore, for the presentation at the annual meeting of the Japanese Mathematical Society at the Tokyo Institute of Technology:

March 17, 2019: 9: 45-10: 00 in Complex Analysis Session, *Horn torus models for the Riemann sphere from the viewpoint of division by zero* with [6],

he kindly sent the message to the author as follows:

It is nice to know that you will present your result at the Tokyo Institute of Technology. Please remember to mention Isabelle/HOL, which is a software in which x/0 = 0. This software is the result of many years of research and a millions of dollars were invested in it. If x/0 = 0 was false, all these money was for nothing. Right now, there is a team of mathematicians formalizing all the mathematics in Isabelle/HOL, where x/0 = 0 for all x, so this mathematical relation is the future of mathematics. https://www.cl.cam.ac.uk/ lp15/Grants/Alexandria/

It seems that he wanted to show that there is no any contradiction for our assumption and results.

4 We can divide the numbers and analytic functions by zero

As the simple introduction and logical base on the division by zero we can start with the definition of the division by zero calculus in Section 3. Indeed, we can develop our theory from the definition. In this sense, the division by zero calculus may be considered as a new axiom.

On this logic, the meaning (definition) of

$$\frac{1}{0} = 0$$

is given by $f(0) = 0$ by means of the division by zero calculus for the function $f(z) = 1/z$. Similarly, the definition

$$\frac{0}{0} = 0$$

is given by $f(0) = 0$ by means of the division by zero calculus for the function $f(z) = 0/z$.

In the division by zero, the essential problem was in the sense of the division by zero (**definition**) $z/0$. Many confusions and simple history of division by zero may be looked in [24].

In this section, in order to give the precise meaning of division by zero, we will give a simple and affirmative answer, for a famous rule that we are not permitted to divide the numbers and functions by zero. In our mathematics, **prohibition** is a famous word for the division by zero.

For any analytic function $f(z)$ around the origin $z = 0$ that is permitted to have any singularity at $z = 0$ (of course, any constant function is permitted), we can consider the value, by the division by zero calculus

$$\frac{f(z)}{z^n} \tag{4.1}$$

at the point $z = 0$, for any positive integer n. This will mean that from **the form** we can consider it as follows:

$$\frac{f(z)}{z^n} \mid_{z=0} . \tag{4.2}$$

For example,

$$\frac{e^x}{x^n} \mid_{x=0} = \frac{1}{n!}.$$

This is the definition of our division by zero (general fraction). In this sense, we can divide the numbers and analytic functions by zero. For $z \neq 0$, $\frac{f(z)}{z^n}$ means the usual division of the function $f(z)$ by z^n.

The content of this subsection was presented by [31].

In [34], we gave a general concept of the division by calculus for an arbitrary function, however, for simplication of the contents, we do not refer to it.

5 General division and usual division

Since the native division by zero $z/0$ in the sense that from $z/0 = X$ to $z = X \cdot 0$ is impossible for $z \neq 0$, we introduced its sense by the division by zero calculus. However, in our many formulas in mathematics and mathematical sciences we can see that they have the natural senses; that is for (4.2), we have:

$$\frac{f(z)}{z^n} \mid_{z=0} = \frac{f(0)}{0^n}.$$

However, this is, in general, not valid. Indeed, for the function $f(z) = \sin z$, we have

$$\frac{\sin z}{z} \mid_{z=0} = \frac{\sin 0}{0} = \frac{0}{0} = 0,$$

however, we have, by the division by zero calculus

$$\frac{\sin z}{z} \mid_{z=0} = 1.$$

For the functions $f(z) = 1/z$ and $g(z) = zf(z)$, we have $f(0) = 0$ and $g(0) = 1$ by the division by zero calculus, but we have another result in this way $g(0) = 0 \times f(0) = 0 \times 0 = 0$.

Here, we will show typical examples. See also [12, 14, 23, 26] for many examples.

5.1 Examples of $0/0 = 0$

The conditional probability $P(A|B)$ for the probability of A under the condition that B happens is given by the formula

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

If $P(B) = 0$, then, of course, $P(A \cap B) = 0$ and from the meaning, $P(A|B) = 0$ and so, $0/0 = 0$.

For the distance d of the centers of the inscribed circle and circumscribed circle, we have the Euler formula

$$r = \frac{1}{2}R - \frac{d^2}{2R}.$$

If $R = 0$, then we have $d = 0$ and

$$0 = 0 - \frac{0}{0}.$$

For the second curvature

$$K_2 = \left((x'')^2 + (y'')^2 + (z'')^2\right)^{-1} \cdot \begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix},$$

if $(x'')^2 + (y'')^2 + (z'')^2 = 0$; that is, for the case of lines, then $0 = 0/0$.

In a Hilbert space H, for a fixed member v and for a given number d we set

$$V = \{y \in H; (y, v) = d\}$$

and for fixed $x \in H$

$$d(x, V) := \frac{|(x, v) - d|}{\|v\|}.$$

If $v = 0$, then, $(y, v) = 0$ and d has to zero. Then, since $H = V$, we have

$$0 = \frac{0}{0}.$$

5.2 Examples of $1/0 = 0$

For constants a and b satisfying

$$\frac{1}{a} + \frac{1}{b} = k, \quad (\neq 0, \text{const.})$$

the function

$$\frac{x}{a} + \frac{y}{b} = 1$$

passes the point $(1/k, 1/k)$. If $a = 0$, then, by the division by zero, $b = 1/k$ and $y = 1/k$; this result is natural.

We will consider the line $y = m(x-a)+b$ through a fixed point (a, b); $a, b > 0$ with its gradient m. We set $A(0, -am + b)$ and $B(a - (b/m), 0)$ that are common points with the line and both lines $x = 0$ and $y = 0$, respectively. Then,

$$\overline{AB}^2 = (-am + b)^2 + \left(a - \frac{b}{m}\right)^2.$$

If $m = 0$, then $A(0, b)$ and $B(a, 0)$, by the division by zero, and furthermore

$$\overline{AB}^2 = a^2 + b^2.$$

Then, the line AB is a corresponding line between the origin and the point (a, b). Note that this line has only one common point with both lines $x = 0$ and $y = 0$. Therefore, this result will be very natural in a sense. – Indeed, we can understand that the line $\overline{AB}$ is broken into two lines $(0, b) - (a, b)$ and $(a, b) - (a, 0)$, suddenly. Or, the line AB is one connecting the origin and the point (a, b).

The general line equation through fixed point (a, b) with its gradient m is given by

$$y = m(x - a) + b \tag{5.1}$$

or, for $m \neq 0$

$$\frac{y}{m} = x - a + \frac{b}{m}.$$

By $m = 0$, we obtain the equation $x = a$, by the division by zero. This equation may be considered as the cases $m = \infty$ and $m = -\infty$, and these cases may be considered by the strictly right logic with the division by zero.

By the division by zero, we can consider the equation (5.1) as a general line equation.

For the Newton's formula; that is, for a C^2 class function $y = f(x)$, the curvature K at the origin is given by

$$K = \lim_{x \to 0} \left| \frac{x^2}{2y} \right| = \left| \frac{1}{f''(0)} \right|,$$

we have for $f''(0) = 0$,

$$K = \frac{1}{0} = 0.$$

Recall the formula

$$b_n = \frac{1}{\pi} \int_0^{2\pi} x \sin nx dx = -\frac{2}{n},$$

for

$$n = \pm 1, \pm 2, ..., ...$$

Then, for $n = 0$, we have

$$b_0 = -\frac{2}{0} = 0.$$

5.3 Trigonometric functions

In order to see how elementary of the division by zero, we will see the division by zero in trigonometric functions as the fundamental object. Even the cases of triangles and trigonometric functions, we can derive new concepts and results.

Even the case

$$\tan x = \frac{\sin x}{\cos x},$$

we have the identity, for $x = \pi/2$

$$0 = \frac{1}{0}.$$

Note that from the inversion of the both sides

$$\cot x = \frac{\cos x}{\sin x},$$

for example, we have, for $x = 0$,

$$0 = \frac{1}{0}.$$

We will consider a triangle ABC with $BC = a, CA = b, AB = c$. Let θ be the angle of the side BC and the bisector line of A. Then, we have the identity

$$\tan\theta = \frac{c+b}{c-b}\tan\frac{A}{2}, \quad b < c.$$

For $c = b$, we have

$$\tan\theta = \frac{2b}{0}\tan\frac{A}{2}.$$

Of course, $\theta = \pi/2$; that is,

$$\tan\frac{\pi}{2} = 0.$$

Here, we used

$$\frac{2b}{0} = 0$$

and we did not consider that by the division by zero calculus

$$\frac{c+b}{c-b} = 1 + \frac{2b}{c-b}$$

and for $c = b$

$$\frac{c+b}{c-b} = 1.$$

In the Napier's formula

$$\frac{a+b}{a-b} = \frac{\tan(A+B)/2}{\tan(A-B)/2},$$

there is no problem for $a = b$ and $A = B$.

We have the formula

$$\frac{a^2+b^2-c^2}{a^2-b^2+c^2} = \frac{\tan B}{\tan C}.$$

If $a^2 + b^2 - c^2 = 0$, then by the Pythagorean theorem $C = \pi/2$. Then,

$$0 = \frac{\tan B}{\tan \frac{\pi}{2}} = \frac{\tan B}{0}.$$

Meanwhile, for the case $a^2 - b^2 + c^2 = 0$, $B = \pi/2$, and we have

$$\frac{a^2 + b^2 - c^2}{0} = \frac{\tan \frac{\pi}{2}}{\tan C} = 0.$$

Let H be the perpendicular leg of A to the side BC and let E and M be the mid points of AH and BC, respectively. Let θ be the angle of EMB ($b > c$). Then, we have

$$\frac{1}{\tan \theta} = \frac{1}{\tan C} - \frac{1}{\tan B}.$$

If $B = C$, then $\theta = \pi/2$ and $\tan(\pi/2) = 0$.

Thales' theorem

We consider a triangle BAC with $A(-1, 0), C(1, 0), \angle BOC = \theta; O(0, 0)$ on the unit circle. Then, the gradients of the lines AB and CB are given by

$$\frac{\sin \theta}{\cos \theta + 1}$$

and

$$\frac{\sin \theta}{\cos \theta - 1},$$

respectively. We see that for $\theta = \pi$ and $\theta = 0$, they are zero, respectively.

For many similar formulas, see [33].

5.4 Examples of Ctesibios and E. Torricelli

As a typical case, we recall

Ctesibios (BC. 286-222): *We consider a flow tube with some fluid. Then, when we consider some cut with a plane with area S and with velocity v of the fluid on the plane, by continuity, we see that for any cut plane, $Sv = C$; C : constant. That is,*

$$v = \frac{C}{S}.$$

When S tends to zero, the velocity v tends to infinity. However, for $S = 0$, the flow stops and so, $v = 0$. Therefore, this example shows the division by zero $C/0 = 0$ clearly. Of course, in the situation, we have $0/0 = 0$, trivially.

We can find many and many similar examples, for example, in Archimedes' principle and Pascal's principle.

We will state one more example:

E. Torricelli (1608 -1646): *We consider some water tank and the initial high $h = h_0$ for $t = 0$ and we assume that from the bottom of the tank with a hole of area A, water is fall down. Then, by the law with a constant k*

$$\frac{dh}{dt} = -\frac{k}{A}\sqrt{h},$$

we have the equation

$$h(t) = \left(\sqrt{h_0} - \frac{k}{2A}\right)^2.$$

Similarly, of course, for $A = 0$, we have

$$h(t) = h_0.$$

5.5 Bhāskara's example – sun and shadow

We will consider the circle such that its center is the origin and its radius R. We consider the point S (sun) on the circle such that $\angle SOI = \theta$; $O(0,0), I(R,0)$. For fixed $d > 0$, we consider the common point $(-L, -d)$ of two lines OS and $y = -d$. Then we obtain the identity

$$L = \frac{R\cos\theta}{R\sin\theta}d,$$

([7], page 77.). That is the length of the shadow of the segment of $(0,0) - (0,-d)$ onto the line $y = -d$ of the sun S.

When we consider $\theta \to +0$ we see that, of course

$$L \to \infty.$$

Therefore, Bhāskara considered that

$$\frac{1}{0} = \infty. \tag{5.2}$$

Even nowadays, our mathematics and many people consider so.

However, for $\theta = 0$, we have S=I and we can not consider any shadow on the line $y = -d$, so we should consider that $L = 0$; that is

$$\frac{1}{0} = 0. \tag{5.3}$$

Furthermore, for $R = 0$; that is, for S=O, we see its shadow is the point $(0, -d)$ and so $L = 0$ and

$$L = \frac{0 \cos \theta}{0 \sin \theta} d = 0;$$

that is

$$\frac{0}{0} = 0.$$

This example shows that the division by zero calculus is not almighty.

Note that both identities (5.2) and (5.3) are right in their senses. Depending on the interpretations of $1/0$, we obtain INFINITY and ZERO, respectively.

6 Division by zero calculus, applications

We will see several typical results of the division by zero calculus.

6.1 Double natures of the zero point $z = 0$

Any line on the complex plane arrives at the point at infinity and the point at infinity is represented by zero. That is, a line, indeed, contains the origin; the true line should be considered as the sum of a usual line and the origin. We can say that it is a compactification of the line and the compacted point is the point at infinity, however, it is represented by $z = 0$. Later, we will see this property by analytic geometry and the division by zero calculus in many situations.

However, for the general line equation

$$ax + by + c = 0,$$

by using the polar coordinates $x = r\cos\theta, y = r\sin\theta$, we have

$$r = \frac{-c}{a\cos\theta + b\sin\theta}.$$

When $a\cos\theta + b\sin\theta = 0$, by the division by zero, we have $r = 0$; that is, we can consider that the line contains the origin. We can consider so, in the natural sense. We can define so as a line with the compactification and the representation of the point at infinity - the ideal point.

For the envelop of the lines represented by, for constants m and a fixed constant $p > 0$,

$$y = mx + \frac{p}{m}, \tag{6.1}$$

we have the function, by using an elementary ordinary differential equation,

$$y^2 = 4px. \tag{6.2}$$

The origin of this parabolic function is excluded from the envelop of the linear functions, because the linear equations do not contain the y axis as the tangential line of the parabolic function. Now recall that, by the division by zero, as the linear equation for $m = 0$, we have the function $y = 0$, the x axis.

– This function may be considered as a function with zero gradient and passing the point at infinity; however, the point at infinity is represented by 0, the origin; that is, the line may be considered as the x axis. Furthermore, then we can consider the x axis as a tangential line of the parabolic function, because they are gradient zero at the point at infinity. –

Furthermore, we can say later that the x axis $y = 0$ and the parabolic function have the zero gradient at **the origin**; that is, in the reasonable sense the x axis is a tangential line of the parabolic function.

Indeed, we will see the surprising property that the gradient of the parabolic function at the origin is zero.

Anyhow, by the division by zero, the envelop of the linear functions may be considered as the whole parabolic function containing the origin.

When we consider the limiting of the linear equations as $m \to 0$, we will think that the limit function is a parallel line to the x axis through the point at infinity. Since the point at infinity is represented by zero, it will become the x axis.

Meanwhile, when we consider the limiting function as $m \to \infty$, we have the y axis $x = 0$ and this function is a native tangential line of the parabolic function. From these two tangential lines, we see that the origin has **double natures**; one is the continuous tangential line $x = 0$ and the second is the discontinuous tangential line $y = 0$.

In addition, note that the tangential point of (6.2) for the line (6.1) is given by

$$\left(\frac{p}{m}, \frac{2p}{m}\right)$$

and it is $(0, 0)$ for $m = 0$.

We can see that the point at infinity is reflected to the origin; and so, the origin has the double natures; one is the native origin and another is the reflected one of the point at infinity.

6.2 Difficulty in Maple for specialization problems

For the Fourier coefficients a_n

$$a_n = \int t \cos n\pi t dt = \frac{\cos n\pi t}{n^2\pi^2} + \frac{t}{n\pi} \cos n\pi t,$$

we obtain, by the division by zero calculus,

$$a_0 = \frac{t^2}{2}.$$

Similarly, for the Fourier coefficients a_n

$$a_n = \int t^2 \cos n\pi t dt = \frac{2t}{\pi^2 n^2} \cos n\pi t - \frac{2}{n^3\pi^3} \sin n\pi t + \frac{t^2}{n\pi} \sin n\pi t,$$

we obtain

$$a_0 = \frac{t^3}{3}.$$

For the Fourier coefficients a_k of a function

$$\frac{a_k \pi k^3}{4}$$

$$= \sin(\pi k) \cos(\pi k) + 2k^2\pi^2 \sin(\pi k) \cos(\pi k) + 2\pi(\cos(\pi k))^2 - \pi k,$$

for $k = 0$, we obtain, by the division by zero calculus, immediately

$$a_0 = \frac{8}{3}\pi^2.$$

We have many such examples.

6.3 Ratio

On the real x line, we fix two different points $P_1(x_1)$ and $P_2(x_2)$ and we will consider the point, with a real number r

$$P(x; r) = \frac{x_1 + rx_2}{1 + r}.$$

If $r = 1$, then the point $P(x; 1)$ is the mid point of two points P_1 and P_2 and for $r > 0$, the point P is on the interval (x_1, x_2). Meanwhile, for $-1 < r < 0$, the point P is on $(-\infty, x_1)$ and for $r < -1$, the point P is on $(x_2, +\infty)$. Of course, for $r = 0$, $P = P_1$. We see that when r tends to $+\infty$ and $-\infty$, P tends to the point P_2. We see the pleasant fact that by the division by zero calculus, $P(x, -1) = P_2$. For this fact we see that for all real numbers r correspond to all real line points.

In particular, we see that in many text books at the undergraduate course the formula is stated as a parameter representation of the line through two pints P_1 and P_2. However, if we do not consider the case $r = -1$ by the division by zero calculus, the classical statement is not right, because the point P_2 can not be considered.

6.4 Discontinuity in geometrical meanings

The division by zero calculus shows an interesting discontinuity in the sense of geometrical properties. We will show typical cases.

- The area $S(x)$ surrounded by two x, y axes and the line passing a fixed point $(a, b), a, b > 0$ and a point $(x, 0)$ is given by

  $$S(x) = \frac{bx^2}{2(x - a)}.$$

  For $x = a$, we obtain, by the division by zero calculus, the very interesting value

  $$S(a) = ab.$$

- For example, for fixed point $(a, b); a, b > 0$ and fixed a line $y = (\tan\theta)x, 0 < \theta < \pi$, we will consider the line $L(x)$ passing two points

(a, b) and $(x, 0)$. Then, the area $S(x)$ of the triangle surrounded by three lines $y = (\tan\theta)x$, $L(x)$ and the x axis is given by

$$S(x) = \frac{b}{2}\frac{x^2}{x - (a - b\cot\theta)}.$$

For the case $x = a - b\cot\theta$, by the division by zero calculus, we have

$$S(a - b\cot\theta) = b(a - b\cot\theta).$$

Note that this is the area of the parallelogram through the origin and the point (a, b) formed by the lines $y = (\tan\theta)x$ and the x axis.

- We consider the circle

  $$h(x^2 + y^2) + (1 - h^2)y - h = 0$$

  through the points $(-1, 0)$, $(1, 0)$ and $(0, h)$. If $h = 0$, then we have

  $$y = 0.$$

  However, from the equation

  $$x^2 + y^2 + \left(\frac{1}{h} - h\right)y - 1 = 0,$$

  by the division by zero, we have an interesting result

  $$x^2 + y^2 = 1.$$

- We consider the regular triangle with the vertices

  $$(-a/2, \sqrt{3}a/2), (a/2, \sqrt{3}a/2).$$

  Then, the area $S(h)$ of the triangle surrounded by the three lines that the line through $(0, h + \sqrt{3}a/2)$ and

  $$(-a/2, \sqrt{3}a/2),$$

  the line through $(0, h + \sqrt{3}a/2)$ and $(a/2, \sqrt{3}a/2)$ and the x- axis is given by

  $$S(h) = \frac{\left(h + (\sqrt{3}/2)a\right)^2}{2h}.$$

  Then, by the division by zero calculus, we have, for $h = 0$,

  $$S(0) = \frac{\sqrt{3}}{2}a^2.$$

7 Derivatives of functions

On derivatives, we obtain new concepts, from the division by zero calculus. At first, we will consider the fundamental properties. From the viewpoint of the division by zero, when there exists the limit, at x

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \infty \tag{7.1}$$

or

$$f'(x) = -\infty, \tag{7.2}$$

both cases, we can write them as follows:

$$f'(x) = 0. \tag{7.3}$$

This property was derived from the fact that the gradient of the y axis is zero; that is,

$$\tan \frac{\pi}{2} = 0. \tag{7.4}$$

We will look this fundamental result by elementary functions. For the function

$$y = \sqrt{1 - x^2},$$

$$y' = \frac{-x}{\sqrt{1 - x^2}},$$

and so,

$$[y']_{x=1} = 0, \quad [y']_{x=-1} = 0.$$

Of course, depending on the context, we should refer to the derivatives of a function at a point from the right hand direction and the left hand direction.

Here, note that, for $x = \cos \theta, y = \sin \theta$,

$$\frac{dy}{dx} = \frac{dy}{d\theta} \left(\frac{dx}{d\theta} \right)^{-1} = -\cot \theta.$$

Note also that from the expansion

$$\cot z = \frac{1}{z} + \sum_{\nu=-\infty, \nu \neq 0}^{+\infty} \left(\frac{1}{z - \nu\pi} + \frac{1}{\nu\pi} \right) \tag{7.5}$$

or the Laurent expansion

$$\cot z = \sum_{n=-\infty}^{\infty} \frac{(-1)^n 2^{2n} B_{2n}}{(2n)!} z^{2n-1},$$

we have

$$\cot 0 = 0.$$

The differential equation

$$y' = -\frac{x}{y}$$

with a general solution

$$x^2 + y^2 = a^2$$

is satisfied for all points of the solutions by the division by zero. However, the differential equations

$$x + yy' = 0, \quad y' \cdot \frac{y}{x} = -1$$

are not satisfied for the points $(-a, 0)$ and $(a, 0)$.

In many and many textbooks, we find the differential equations, however, they are not good in this viewpoint.

For the function $y = \log x$,

$$y' = \frac{1}{x}, \tag{7.6}$$

and so,

$$[y']_{x=0} = 0. \tag{7.7}$$

For the elementary ordinary differential equation

$$y' = \frac{dy}{dx} = \frac{1}{x}, \quad x > 0, \tag{7.8}$$

how will be the case at the point $x = 0$? From its general solution, with a general constant C

$$y = \log x + C, \tag{7.9}$$

we see that

$$y'(0) = \left[\frac{1}{x}\right]_{x=0} = 0, \tag{7.10}$$

that will mean that the division by zero $1/0 = 0$ is very natural.

In addition, note that the function $y = \log x$ has infinite order derivatives and all values are zero at the origin, in the sense of the division by zero calculus.

However, for the derivative of the function $y = \log x$, we have to fix the sense at the origin, clearly, because the function is not differentiable in the usual sense, but it has a singularity at the origin. For $x > 0$, there is no problem for (7.8) and (7.9). At $x = 0$, we see that we can not consider the limit in the usual sense. However, for $x > 0$ we have (7.9) and

$$\lim_{x \to +0} (\log x)' = +\infty. \tag{7.11}$$

In the usual sense, the limit is $+\infty$, but in the present case, in the sense of the division by zero, we have the identity

$$\left[(\log x)'\right]_{x=0} = 0$$

and we will be able to understand its sense graphically.

By the new interpretation for the derivative, we can arrange the formulas for derivatives, by the division by zero. The formula

$$\frac{dx}{dy} = \left(\frac{dy}{dx}\right)^{-1} \tag{7.12}$$

is very fundamental. Here, we considered it for a local one to one correspondence of the function $y = f(x)$ and for nonvanishing of the denominator

$$\frac{dy}{dx} \neq 0. \tag{7.13}$$

However, if a local one to one correspondence of the function $y = f(x)$ is ensured like the function $y = x^3$ around the origin, we do not need the assumption (7.13). Then, for the point $dy/dx = 0$, we have, by the division by zero,

$$\frac{dx}{dy} = 0.$$

This will mean that the function $x = g(y)$ has the zero derivative and the tangential line at the point is a parallel line to the y- axis. In this sense the formula (7.12) is valid, even the case $dy/dx = 0$.

8 Differerential equations

From the viewpoint of the division by zero calculus, we will see many incompleteness mathematics, in particular, in the theory of differential equations at an undergraduate level; indeed, we have considered our mathematics around an isolated singular point for analytic functions, however, we did not consider mathematics at the singular point itself. At the isolated singular point, we considered our mathematics with the limiting concept, however, the limiting value to the singular point and the value at the singular point of the function are, in general, different. By the division by zero calculus, we can consider the values and differential coefficients at the singular point. From this viewpoint, we will be able to consider differential equations even at singular points. We find many incomplete statements and problems in many undergraduate textbooks. In this section, we will point out the problems in concrete ways by examples.

This section is an arrangement of the paper [27] with new materials.

8.1 Missing a solution

For the differential equation

$$2xydx - (x^2 - y^2)dy = 0,$$

we have a general solution with a constant C

$$x^2 + y^2 = 2Cy.$$

However, we are missing the solution $y = 0$. By this expression

$$\frac{x^2 + y^2}{C} = 2y,$$

for $C = 0$, by the division by zero, we have the missing solution $y = 0$.

For the differential equation

$$x(y')^2 - 2yy' - x = 0,$$

we have the general solution

$$C^2x^2 - 2Cy - 1 = 0.$$

However, $x = 0$ is also a solution, because

$$xdy^2 - 2ydydx - xdx^2 = 0.$$

From

$$x^2 - \frac{2y}{C} - \frac{1}{C^2} = 0,$$

by the division by zero, we obtain the solution.

For the differential equation

$$2y = xy' - \frac{x}{y'},$$

we have the general solution

$$2y = Cx^2 - \frac{1}{C}.$$

For $C = 0$, we have the solution $y = 0$, by the division by zero.

8.2 Differential equations with singularities

For the differential equation

$$y' = -\frac{y}{x},$$

we have the general solution

$$y = \frac{C}{x}.$$

From the expression

$$xdy + ydx = 0,$$

we have also the general solution

$$x = \frac{C}{y}.$$

Therefore, there is no problem for the origin. Of course, $x = 0$ and $y = 0$ are the solutions.

For the differential equation

$$y' = \frac{2x - y}{x - y}, \tag{8.1}$$

we have the beautiful general solution with constant C

$$2x^2 - 2xy + y^2 = C. \tag{8.2}$$

By the division by zero calculus we see that on the whole points on the solutions (8.2) the differential equation (8.1) is satisfied. If we do not consider the division by zero, for $y = x(\neq 0)$, we will have a serious problem. However, for $x = y \neq 0$, we should consider that $y' = 0$, not by the division by zero calculus, but by $1/0 = 0$.

8.3 Continuation of solution

We will consider the differential equation

$$\frac{dx}{dt} = x^2 \cos t. \tag{8.3}$$

Then, as the general solution, we obtain, for a constant C

$$x = \frac{1}{C - \sin t}.$$

For $x_0 \neq 0$, for any given initial value (t_0, x_0) we obtain the solution satisfying the initial condition

$$x = \frac{1}{\sin t_0 + \frac{1}{x_0} - \sin t}. \tag{8.4}$$

If

$$\left| \sin t_0 + \frac{1}{x_0} \right| < 1,$$

then the solution has many poles and L. S. Pontrjagin stated in his book that the solution is disconnected at the poles and so, the solution may be considered as infinitely many solutions.

However, from the viewpoint of the division by zero, the solution takes the value zero at the singular points and the derivatives at the singular points are all zero; that is, the solution (8.4) may be understood as one solution.

Furthermore, by the division by zero, the solution (8.4) has its sense for even the case $x_0 = 0$ and it is the solution of (8.3) satisfying the initial condition $(t_0, 0)$.

We will consider the differential equation

$$y' = y^2.$$

For $a > 0$, the solution satisfying $y(0) = a$ is given by

$$y = \frac{1}{\frac{1}{a} - x}.$$

Note that the solution satisfies on the whole space $(-\infty, +\infty)$ even at the singular point $x = \frac{1}{a}$, in the sense of the division by zero, as

$$y'\left(\frac{1}{a}\right) = y\left(\frac{1}{a}\right) = 0.$$

8.4 Singular solutions

We will consider the differential equation

$$(1 - y^2)dx = y(1 - x)dy.$$

By the standard method, we obtain the general solution, for a constant C $(C \neq 0)$

$$\frac{(x-1)^2}{C} + y^2 = 1.$$

By the division by zero, for $C = 0$, we obtain the singular solution

$$y = \pm 1.$$

For the simple Clairaut differential equation

$$y = px + \frac{1}{p}, \quad p = \frac{dy}{dx},$$

we have the general solution

$$y = Cx + \frac{1}{C}, \tag{8.5}$$

with a general constant C and the singular solution

$$y^2 = 4x.$$

Note that we have also the solution $y = 0$ from the general solution, by the division by zero $1/0 = 0$ from $C = 0$ in (8.5).

8.5 Solutions with singularities

1). We will consider the differential equation

$$y' = \frac{y^2}{2x^2}.$$

We will consider the solution with an isolated singularity at a point a taking the value $-2a$ in the sense of division by zero.

First, by the standard method, we have the general solution, with a constant C

$$y = \frac{2x}{1 + 2Cx}.$$

From the singularity, we have, $C = -1/2a$ and we obtain the desired solution

$$y = \frac{2ax}{a - x}.$$

Indeed, from the expansion

$$\frac{2ax}{a - x} = -2a - \frac{2a^2}{x - a},$$

we see that it takes $-2a$ at the point a in the sense of the division by zero calculus. This function was appeared in ([13]).

2). We will consider the singular differential equation

$$\frac{d^2y}{dx^2} + \frac{3}{x}\frac{dy}{dx} - \frac{3}{x^2}y = 0. \tag{8.6}$$

By the series expansion, we obtain the general solution, for any constants a, b

$$y = \frac{a}{x^3} + bx. \tag{8.7}$$

We see that by the division by zero

$$y(0) = 0, y'(0) = b, y''(0) = 0.$$

The solution (8.7) has its sense and the equation is satisfied even at the origin. The value $y'(0) = b$ may be given arbitrary, however, in order to determine the value a, we have to give some value for the regular point $x \neq 0$. Of course, we can give the information at the singular point with the Laurent coefficient a, that may be interpreted with the value at the singular point zero. Indeed, the value a may be considered at the value

$$[y(x)x^3]_{x=0} = a.$$

3). Next, we will consider the Euler differential equation

$$x^2 \frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = 0.$$

We obtain the general solution, for any constants a, b

$$y = \frac{a}{x} + \frac{b}{x^2}.$$

This solution is satisfied even at the origin, by the division by zero and furthermore, all derivatives of the solution of any order are zero at the origin.

4). We will note that as the general solution with constants C_{-2}, C_{-1}, C_0

$$y = \frac{C_{-2}}{x^2} + \frac{C_{-1}}{x} + C_0,$$

we obtain the nonlinear ordinary differential equation

$$x^2 y''' + 6xy'' + 6y' = 0.$$

5). For the differential equation

$$y' = y^2(2x - 3),$$

we have the special solution

$$y = \frac{1}{(x-1)(2-x)}$$

on the interval $(1, 2)$ with the singularities at $x = 1$ and $x = 2$. Since the general solution is given by, for a constant C,

$$y = \frac{1}{-x^2 + 3x + C},$$

we can consider some conditions that determine the special solution.

8.6 Solutions with an analytic parameter

For example, in the ordinary differential equation

$$y'' + 4y' + 3y = 5e^{-3x},$$

in order to look for a special solution, by setting $y = Ae^{kx}$ we have, from

$$y'' + 4y' + 3y = 5e^{kx},$$

$$y = \frac{5e^{kx}}{k^2 + 4k + 3}.$$

For $k = -3$, by the division by zero calculus, we obtain

$$y = e^{-3x} \left(-\frac{5}{2}x - \frac{5}{4} \right),$$

and so, we can obtain the special solution

$$y = -\frac{5}{2}xe^{-3x}.$$

For example, for the differential equation

$$y'' + a^2 y = b \cos \lambda x,$$

we have a special solution

$$y = \frac{b}{a^2 - \lambda^2} \cos \lambda x.$$

Then, for $\lambda = a$ (reasonance case), by the division by zero calculus, we obtain the special solution

$$y = \frac{bx \sin(ax)}{2a} + \frac{b \cos(ax)}{4a^2}.$$

We can find many examples.

8.7 Special reductions by division by zero of solutions

For the differential equation

$$y'' - (a+b)y' + aby = e^{cx}, c \neq a, b; a \neq b,$$

we have the special solution

$$y = \frac{e^{cx}}{(c-a)(c-b)}.$$

If $c = a (\neq b)$, then, by the division by zero calculus, we have

$$y = \frac{xe^{ax}}{a-b}.$$

If $c = a = b$, then, by the division by zero calculus, we have

$$y = \frac{x^2 e^{ax}}{2}.$$

For the differential equation

$$m\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + kx = 0,$$

we obtain the general solution, for $\gamma^2 > 4mk$

$$x(t) = e^{-\alpha t}\left(C_1 e^{\beta t} + C_2 e^{-\beta t}\right)$$

with

$$\alpha = \frac{\gamma}{2m}$$

and

$$\beta = \frac{1}{2m}\sqrt{\gamma^2 - 4mk}.$$

For $m = 0$, by the division by zero calculus we obtain the reasonable solution $\alpha = 0$ and $\beta = -k/\gamma$.

We will consider the differential equation, for a constant K

$$y' = Ky.$$

Then, we have the general solution

$$y(x) = y(0)e^{Kt}.$$

For the differential equation

$$y' = Ky\left(1 - \frac{y}{R}\right),$$

we have the solution

$$y = \frac{y(0)e^{Kt}}{1 + \frac{y(0)(e^{Kt}-1)}{R}}.$$

If $R = 0$, then, by the division by zero, we obtain the previous result, immediately.

For the differential equation

$$x''(t) = -g + k(x'(t))^2$$

satisfying the initial conditions

$$x(0) = 0, x'(0) = V,$$

we have

$$x'(t) = -\sqrt{\frac{g}{k}}\tan(\sqrt{kg}t - \alpha),$$

with

$$\alpha = \tan^{-1}\sqrt{\frac{k}{g}}V$$

and the solution

$$x(t) = \frac{1}{k}\log\frac{\cos\left(\sqrt{kg}t - \alpha\right)}{\cos\alpha}.$$

Then we obtain for $k = 0$, by the division by zero calculus

$$x'(t) = -gt + V$$

and

$$x(t) = -\frac{1}{2}gt^2 + Vt.$$

We can find many and many such examples. However, note the following fact.

For the differential equation

$$y''' + a^2y' = 0,$$

we obtain the general solution, for $a \neq 0$

$$y = A \sin ax + B \cos ax + C.$$

For $a = 0$, from this general solution, how can we obtain the corresponding solution

$$y = Ax^2 + Bx + C,$$

naturally?

For the differential equation

$$y' = ae^{\lambda x} y^2 + afe^{\lambda x} y + \lambda f,$$

we obtain a special solution, for $a \neq 0$

$$y = -\frac{\lambda}{a} e^{-\lambda x}.$$

For $a = 0$, from this solution, how can we obtain the corresponding solution

$$y = \lambda f x + C,$$

naturally?

9 Euclidean spaces and division by zero calculus

In this section, we will see the division by zero properties on the Euclidean spaces. Since the impact of the division by zero and division by zero calculus is widely expanded in elementary mathematics, here, elementary topics will be introduced as the first stage.

9.1 Broken phenomena of figures by area and volume

The strong discontinuity of the division by zero around the point at infinity will appear as the destruction of various figures. These phenomena may be looked in many situations as the universe one. However, the simplest cases are disc and sphere (ball) with their radius $1/R$. When $R \to +0$, the areas and volumes of discs and balls tend to $+\infty$, respectively, however, when $R = 0$, they are zero, because they become the half-plane and half-space, respectively. These facts may be also looked by analytic geometry, as we see later. However, the results are clear already from the definition of the division by zero.

The behavior of the space around the point at infinity may be considered by that of the origin by the linear transform $W = 1/z$ (see [2]). We thus see that

$$\lim_{z \to \infty} z = \infty, \tag{9.1}$$

however,

$$[z]_{z=\infty} = 0, \tag{9.2}$$

by the division by zero. Here, $[z]_{z=\infty}$ denotes the value of the function $W = z$ at the topological point at the infinity in one point compactification by Aleksandrov. The difference of (9.1) and (9.2) is very important as we see clearly by the function $W = 1/z$ and the behavior at the origin. The limiting value to the origin and the value at the origin are different. For surprising results, we will state the property in the real space as follows:

$$\lim_{x \to +\infty} x = +\infty, \quad \lim_{x \to -\infty} x = -\infty,$$

however,

$$[x]_{+\infty} = 0, \quad [x]_{-\infty} = 0.$$

Of course, two points $+\infty$ and $-\infty$ are the same point as the point at infinity. However, $\pm$ will be convenient in order to show the approach directions. In [14], we gave many examples for this property.

In particular, in $z \to \infty$ in (9.1), ∞ represents the topological point on the Riemann sphere, meanwhile ∞ in the left hand side in (9.1) represents the limit by means of the ϵ - δ logic. That is, for any large number M, when we take for some large number N, we have, for $|z| > N$, $|z| > M$.

9.2 Parallel lines

We write lines by

$$L_k : a_k x + b_k y + c_k = 0, k = 1, 2.$$

The common point is given by, if $a_1 b_2 - a_2 b_1 \neq 0$; that is, the lines are not parallel

$$\left(\frac{b_1 c_2 - b_2 c_1}{a_1 b_2 - a_2 b_1}, \frac{a_2 c_1 - a_1 c_2}{a_1 b_2 - a_2 b_1} \right).$$

By the division by zero, we can understand that if $a_1b_2 - a_2b_1 = 0$, then the common point is always given by

$$(0, 0),$$

even two lines are the same.

We write a line by the polar coordinate

$$r = \frac{d}{\cos(\theta - \alpha)},$$

where $d = \overline{OH} > 0$ is the distance of the origin O and the line such that OH and the line is orthogonal and H is on the line, α is the angle of the line OH and the positive x axis, and θ is the angle of OP ($P = (r, \theta)$ on the line) from the positive x axis. Then, if $\theta - \alpha = \pi/2$; that is, OP and the line is parallel and P is the point at infinity, then we see that $r = 0$ by the division by zero calculus; the point at infinity is represented by zero and we can consider that the line passes the origin, however, it is in a discontinuous way.

This will mean simply that any line arrives at the point at infinity and the point is represented by zero and so, for the line we can add the point at the origin. In this sense, we can add the origin to any line as the point of the compactification of the line. This surprising new property may be looked in our mathematics globally.

The distance d from the origin to the line determined by the two planes

$$\Pi_k : a_kx + b_ky + c_kz = 1, k = 1, 2,$$

is given by

$$d = \sqrt{\frac{(a_1 - a_2)^2 + (b_1 - b_2)^2 + (c_1 - c_2)^2}{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}}.$$

If the two lines are coincident, then, of course, $d = 0$. However, if two planes are parallel, by the division by zero, $d = 0$. This will mean that any plane contains the origin as in a line.

9.3 Tangential lines and $\tan \frac{\pi}{2} = 0$

We looked the very fundamental and important formula $\tan \frac{\pi}{2} = 0$ in Section 6. In this subsection, for its importance we will furthermore see its geometrical meanings.

We consider the high $\tan\theta \left(0 \leq \theta \leq \frac{\pi}{2}\right)$ that is given by the common point of two lines $y = (\tan\theta)x$ and $x = 1$ on the (x, y) plane. Then,

$$\tan\theta \longrightarrow \infty; \quad \theta \longrightarrow \frac{\pi}{2}.$$

However,

$$\tan\frac{\pi}{2} = 0,$$

by the division by zero. The result will show that, when $\theta = \pi/2$, two lines $y = (\tan\theta)x$ and $x = 1$ do not have a common point, because they are parallel in the usual sense. However, in the sense of the division by zero, parallel lines have the common point $(0, 0)$. Therefore, we can see the result $\tan\frac{\pi}{2} = 0$ following our new space idea.

We consider general lines represented by

$$ax + by + c = 0, \quad a'x + b'y + c' = 0.$$

The gradients are given by

$$k = -\frac{a}{b}, k' = -\frac{a'}{b'},$$

respectively. In particular, note that if $b = 0$, then $k = 0$, by the division by zero.

If $kk' = -1$, then the lines are orthogonal; that is,

$$\tan\frac{\pi}{2} = 0 = \pm\frac{k - k'}{1 + kk'},$$

which shows that the division by zero $1/0 = 0$ and orthogonality meets in a very good way.

Furthermore, even in the case of polar coordinates $x = r\cos\theta, y = r\sin\theta$, we can see the division by zero

$$\tan\frac{\pi}{2} = \frac{y}{0} = 0.$$

The division by zero may be looked even in the rotation of the coordinates.

We will consider a 2 dimensional curve

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

and a rotation defined by

$$x = X\cos\theta - Y\sin\theta, \quad y = X\sin\theta + Y\cos\theta.$$

Then, we write, by inserting these (x, y)

$$AX^2 + 2HXY + BY^2 + 2GX + 2FY + C = 0.$$

Then,

$$H = 0 \Longleftrightarrow \tan 2\theta = \frac{2h}{a-b}.$$

If $a = b$, then, by the division by zero,

$$\tan\frac{\pi}{2} = 0, \quad \theta = \frac{\pi}{4}.$$

For $h^2 > ab$, the equation

$$ax^2 + 2hxy + by^2 = 0$$

represents 2 lines and the angle θ made by two lines is given by

$$\tan\theta = \pm\frac{2\sqrt{h^2 - ab}}{a+b}.$$

If $h^2 - ab = 0$, then, of course, $\theta = 0$. If $a + b = 0$, then, by the division by zero, $\theta = \pi/2$ from $\tan\theta = 0$.

For a hyperbolic function

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1; \quad a, b > 0$$

the angle θ made by two asymptotic lines $y = \pm(b/a)x$ is given by

$$\tan\theta = \frac{2(b/a)}{1 - (b/a)^2}.$$

If $a = b$, then $\theta = \pi/2$ from $\tan\theta = 0$.

We consider the unit circle with its center at the origin on the (x, y) plane. We consider the tangential line for the unit circle at the point that is the common point of the unit circle and the line $y = (\tan\theta)x$ $\left(0 \le \theta \le \frac{\pi}{2}\right)$.

Then, the distance R_θ between the common point and the common point of the tangential line and x-axis is given by

$$R_\theta = \tan\theta.$$

Then,

$$R_0 = \tan 0 = 0,$$

and

$$\tan\theta \longrightarrow \infty; \quad \theta \longrightarrow \frac{\pi}{2}.$$

However,

$$R_{\pi/2} = \tan\frac{\pi}{2} = 0.$$

This example shows also that by the stereographic projection mapping of the unit sphere with its center at the origin $(0,0,0)$ onto the plane, the north pole corresponds to the origin $(0,0)$.

In this case, we consider the orthogonal circle C_{R_θ} with the unit circle through at the common point and the symmetric point with respect to the x-axis with its center $((\cos\theta)^{-1},0)$. Then, the circle C_{R_θ} is as follows:

C_{R_0} is the point $(1,0)$ with curvature zero, and $C_{R_{\pi/2}}$ (that is, when $R_\theta = \infty$, in the common sense) is the y-axis and its curvature is also zero. Meanwhile, by the division by zero calculus, for $\theta = \pi/2$ we have the same result, because $(\cos(\pi/2))^{-1} = 0$.

The points $(\cos\theta, 0)$ and $((\cos\theta)^{-1},0)$ are the symmetric points with respect to the unit circle, and the origin corresponds to the origin.

In particular, the formal calculation

$$\sqrt{1 + R_{\pi/2}^2} = 1$$

is not good. The identity $\cos^2\theta + \sin^2\theta = 1$ is valid always, however $1 + \tan^2\theta = (\cos\theta)^{-2}$ is not valid formally for $\theta = \pi/2$.

This equation should be written as

$$\frac{\cos^2\theta}{\cos^2\theta} + \tan^2\theta = (\cos\theta)^{-2},$$

that is valid always.

Of course, as analytic functions, in the sense of the division by zero calculus, the identity is valid for $\theta = \pi/2$.

From the point at
$$x = \frac{1}{\cos\theta}$$
when we look the unit circle, we can see that the length $L(x)$ of the arc that we can see is given by
$$L(x) = 2\cos^{-1}\frac{1}{x}.$$
For $\theta = \pi/2$ that is for $x = 0$ we see that $L(x) = 0$.

We fix $B(0,1)$ and let $\angle ABO = \theta$ with $A(\tan\theta, 0)$. Let H be the point on the line BA such that two lines OH and AB are orthogonal. Then we see that
$$AH = \frac{\sin^2\theta}{\cos\theta}.$$
Note that for $\theta = \pi/2$, $AH = 0$.

On the point $(p,q)(0 \leq p,q \leq 1)$ on the unit circle, we consider the tangential line $L_{p,q}$ of the unit circle. Then, the common points of the line $L_{p,q}$ with x-axis and y-axis are $(1/p, 0)$ and $(0, 1/q)$, respectively. Then, the area S_p of the triangle formed by three points $(0,0)$, $(1/p,0)$ and $(0,1/q)$ is given by
$$S_p = \frac{1}{2pq}.$$
Then,
$$p \longrightarrow 0; \quad S_p \longrightarrow +\infty,$$
however,
$$S_0 = 0$$
(H. Michiwaki: 2015.12.5.). We denote the point on the unit circle on the (x,y) plane with $(\cos\theta, \sin\theta)$ for the angle θ with the positive real line. Then, the tangential line of the unit circle at the point meets at the point $(R_\theta, 0)$ for $R_\theta = [\cos\theta]^{-1}$ with the x-axis for the case $\theta \neq \pi/2$. Then,
$$\theta\left(\theta < \frac{\pi}{2}\right) \to \frac{\pi}{2} \Longrightarrow R_\theta \to +\infty,$$
$$\theta\left(\theta > \frac{\pi}{2}\right) \to \frac{\pi}{2} \Longrightarrow R_\theta \to -\infty,$$
however,
$$R_{\pi/2} = \left[\cos\left(\frac{\pi}{2}\right)\right]^{-1} = 0,$$

by the division by zero. We can see the strong discontinuity of the point $(R_\theta, 0)$ at $\theta = \pi/2$ (H. Michiwaki: 2015.12.5.).

The line through the points $(0, 1)$ and $(\cos\theta, \sin\theta)$ meets the x axis with the point $(R_\theta, 0)$ for the case $\theta \neq \pi/2$ by

$$R_\theta = \frac{\cos\theta}{1 - \sin\theta}.$$

Then,

$$\theta \left(\theta < \frac{\pi}{2}\right) \to \frac{\pi}{2} \Longrightarrow R_\theta \to +\infty,$$

$$\theta \left(\theta > \frac{\pi}{2}\right) \to \frac{\pi}{2} \Longrightarrow R_\theta \to -\infty,$$

however,

$$R_{\pi/2} = 0,$$

by the division by zero. We can see the strong discontinuity of the point $(R_\theta, 0)$ at $\theta = \pi/2$.

Note also that

$$\left[1 - \sin\left(\frac{\pi}{2}\right)\right]^{-1} = 0.$$

9.4 Length of tangential lines

We will consider the inversion $A(1/x, 0)$ of a point $X(x, 0), 0 < x < 1$ with respect to the unit circle with its center the origin. Then the length $T(x)$ of the tangential line AB $(B(x, \sqrt{1 - x^2}))$ is given by

$$T(x) = \frac{1}{x}\sqrt{1 - x^2}.$$

For $x = 0$, by the division by zero calculus, we have

$$T(0) = 0$$

that was considered as $+\infty$.

We will consider a function $y = f(x)$ of C^1 class on the real line. We consider the tangential line through $(x, f(x))$

$$Y = f'(x)(X - x) + f(x).$$

Then, the length (or distance) $d(x)$ between the point $(x, f(x))$ and $\left(x - \frac{f(x)}{f'(x)}, 0\right)$ is given by, for $f'(x) \neq 0$

$$d(x) = |f(x)|\sqrt{1 + \frac{1}{f'(x)^2}}.$$

How will be the case $f'(x^*) = 0$? Then, the division by zero shows that

$$d(x^*) = |f(x^*)|.$$

Meanwhile, the x axis point $(X_t, 0)$ of the tangential line at (x, y) and y axis point $(0, Y_n)$ of the normal line at (x, y) are given by

$$X_t = x - \frac{f(x)}{f'(x)}$$

and

$$Y_n = y + \frac{x}{f'(x)},$$

respectively. Then, if $f'(x) = 0$, we obtain the reasonable results:

$$X_t = x, \quad Y_n = y.$$

9.5 Our life figure

As an interesting figure which shows an interesting relation between 0 and infinity, we will consider a sector Δ_α on the complex $z = x + iy$ plane

$$\Delta_\alpha = \left\{ |\arg z| < \alpha; 0 < \alpha < \frac{\pi}{2} \right\}.$$

We will consider a disc inscribed in the sector Δ_α whose center $(k, 0)$ with its radius r. Then, we have

$$r = k \sin \alpha.$$

Then, note that as k tends to zero, r tends to zero, meanwhile k tends to $+\infty$, r tends to $+\infty$. However, by our division by zero calculus, we see that immediately

$$[r]_{r=\infty} = 0.$$

On the sector, we see that from the origin as the point 0, the inscribed discs are increasing endlessly, however their final disc reduces to the origin suddenly - it seems that the whole process looks like our life in the viewpoint of our initial and final.

9.6 H. Okumura's example

The suprising example by H. Okumura will show a new phenomenon at the point at infinity.

On the sector Δ_α, we shall change the angle and we consider a fixed circle $C_a, a > 0$ with its radius a inscribed in the sectors. We see that when the circle tends to $+\infty$, the angles α tend to zero. How will be the case $\alpha = 0$? Then, we will not be able to see the position of the circle. Surprisingly enough, then C_a is the circle with its center at the origin 0. This result is derived from the division by zero calculus for the formula

$$k = \frac{a}{\sin \alpha}.$$

The two lines $\arg z = \alpha$ and $\arg z = -\alpha$ were tangential lines of the circle C_a and now they are the positive real line. The gradient of the positive real line is of course zero. Note here that the gradient of the positive y axis is zero by the division by zero calculus that means $\tan \frac{\pi}{2} = 0$. Therefore, we can understand that the positive real line is still a tangential line of the circle C_a.

This will show some great relation between zero and infinity. We can see some mysterious property around the point at infinity.

These two subsections were taken from [14].

9.7 Interpretation by analytic geometry

We write lines by

$$L_k : a_k x + b_k y + c_k = 0, k = 1, 2, 3.$$

The area S of the triangle surrounded by these lines is given by

$$S = \pm \frac{1}{2} \cdot \frac{\triangle^2}{D_1 D_2 D_3},$$

where $\triangle$ is

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

and D_k is the co-factor of $\triangle$ with respect to c_k. $D_k = 0$ if and only if the corresponding lines are parallel. $\triangle = 0$ if and only if the three lines are parallel or they have a common point. We can see that the degeneracy (broken) of the triangle may be stated by $S = 0$ beautifully, by the division by zero.

Similarly we write lines by

$$M_k : a_{k1}x + a_{k2}y + a_{3k} = 0, k = 1, 2, 3.$$

The area S of the triangle surrounded by these lines is given by

$$S = \frac{1}{A_{11}A_{22}A_{33}} \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix}$$

where A_{kj} is the co-factor of a_{kj} with respect to the matrix $[a_{kj}]$. We can see that the degeneracy (broken) of the triangle may be stated by $S = 0$ beautifully, by the division by zero.

For a function

$$S(x, y) = a(x^2 + y^2) + 2gx + 2fy + c, \tag{9.3}$$

the radius R of the circle $S(x, y) = 0$ is given by

$$R = \sqrt{\frac{g^2 + f^2 - ac}{a^2}}.$$

If $a = 0$, then the area πR^2 of the disc is zero, by the division by zero. In this case, the circle is a line (degenerated).

The center of the circle (9.3) is given by

$$\left(-\frac{g}{a}, -\frac{f}{a} \right).$$

Therefore, the center of a general line

$$2gx + 2fy + c = 0$$

may be considered as the origin $(0, 0)$, by the division by zero.

We consider the functions

$$S_j(x, y) = a_j(x^2 + y^2) + 2g_jx + 2f_jy + c_j.$$

The distance d of the centers of the circles $S_1(x,y)=0$ and $S_2(x,y)=0$ is given by

$$d^2 = \frac{g_1^2+f_1^2}{a_1^2} - 2\frac{g_1g_2+f_1f_2}{a_1a_2} + \frac{g_2^2+f_2^2}{a_2^2}.$$

If $a_1 = 0$, then by the division by zero

$$d^2 = \frac{g_2^2+f_2^2}{a_2^2}.$$

Then, $S_1(x,y)=0$ is a line and its center is the origin $(0,0)$. Therefore, the result is very reasonable.

This subsection was taken from [11]. For more examples see it.

10 Analytic functions and division by zero calculus

The values of analytic functions at isolated singular points were given by the coefficients C_0 of the Laurent expansions (the first coefficients of the regular part) as the division by zero calculus. Therefore, their property may be considered as arbitrary ones by any sift of the image complex plane. Therefore, we can consider the values as zero in any Laurent expansions by shifts, as normalizations. However, if by another normalizations, the Laurent expansions are determined, then the values will have their senses. We will firstly examine such properties for the Riemann mapping function.

Let D be a simply-connected domain containing the point at infinity having at least two boundary points. Then, by the celebrated theorem of Riemann, there exists a uniquely determined conformal mapping with a series expansion

$$W = f(z) = C_1 z + C_0 + \frac{C_{-1}}{z} + \frac{C_{-2}}{z^2} + \dots, \quad C_1 > 0, \tag{10.1}$$

around the point at infinity which maps the domain D onto the exterior $|W|>1$ of the unit disc on the complex W plane. We can normalize (10.1) as follows:

$$\frac{f(z)}{C_1} = z + \frac{C_0}{C_1} + \frac{C_{-1}}{C_1 z} + \frac{C_{-2}}{C_1 z^2} + \dots.$$

Then, this function $\frac{f(z)}{C_1}$ maps D onto the exterior of the circle of radius $1/C_1$ and so, it is called the **mapping radius** of D. See [3, 39]. Meanwhile, from the normalization

$$f(z) - C_0 = C_1 z + \frac{C_{-1}}{z} + \frac{C_{-2}}{z^2} + \dots,$$

by the natural shift C_0 of the image plane, the unit circle is mapped to the unit circle with center C_0. Therefore, C_0 may be called as **mapping center** of D. The function $f(z)$ takes the value C_0 at the point at infinity in the sense of the division by zero calculus and now we have its natural sense by the mapping center of D. We have considered the value of the function $f(z)$ as infinity at the point at infinity, however, practically it was the value C_0. This will mean that in a sense the value C_0 is the farthest point from the point at infinity or the image domain with the strong discontinuity.

The properties of mapping radius were investigated deeply in conformal mapping theory like estimations, extremal properties and meanings of the values, however, it seems that there is no information on the property of mapping center. See many books on conformal mapping theory or analytic function theory. See [39] for example.

From the fundamental Bierberbach area theorem, we can obtain the following inequality:

For analytic functions on $|z| > 1$ with the normalized expansion around the point at infinity

$$g(z) = z + b_0 + \frac{b_1}{z} + \cdots$$

that are univalent and take no zero point,

$$|b_0| \leq 2.$$

In our sense

$$g(\infty) = b_0.$$

See [16], Chapter V, Section 8 for the details.

11 The Descartes circle theorem

We recall the famous and beautiful theorem ([8, 36]):

Theorem (Descartes). *Let C_i $(i = 1, 2, 3)$ be circles touching to each other of radii r_i. If a circle C_4 touches the three circles, then its radius r_4 is given by*

$$\frac{1}{r_4} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \pm 2\sqrt{\frac{1}{r_1 r_2} + \frac{1}{r_2 r_3} + \frac{1}{r_3 r_1}}. \tag{11.1}$$

As well-known, circles and lines may be looked as the same ones in complex analysis, in the sense of stereographic projection and with many reasons. Therefore, we will consider whether the theorem is valid for line cases and point cases for circles. Here, we will discuss this problem clearly from the division by zero viewpoint. The Descartes circle theorem is valid except for one case for lines and points for the three circles and for one exception case, we can obtain very interesting results, by the division by zero calculus.

We can consider all cases for the Descartes theorem for lines and point circles.

For the details, see the paper [19].

12 Horn torus models and division by zero calculus – a new world

We recall the essence of the paper [6] for horn torus models.

We will consider the three circles represented by

$$\xi^2 + \left(\zeta - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2,$$

$$\left(\xi - \frac{1}{4}\right)^2 + \left(\zeta - \frac{1}{2}\right)^2 = \left(\frac{1}{4}\right)^2, \tag{12.1}$$

and

$$\left(\xi + \frac{1}{4}\right)^2 + \left(\zeta - \frac{1}{2}\right)^2 = \left(\frac{1}{4}\right)^2.$$

By rotation on the space (ξ, η, ζ) on the (x, y) plane as in $\xi = x, \eta = y$ around ζ axis, we will consider the sphere with $1/2$ radius as the Riemann sphere and the horn torus made in the sphere.

The stereographic projection mapping from (x, y) plane to the Riemann sphere is given by

$$\xi = \frac{x}{x^2 + y^2 + 1},$$

$$\eta = \frac{y}{x^2 + y^2 + 1},$$

and

$$\zeta = \frac{x^2 + y^2}{x^2 + y^2 + 1}.$$

Of course,

$$\xi^2 + \eta^2 = \zeta(1 - \zeta),$$

and

$$x = \frac{\xi}{1 - \zeta}, y = \frac{\eta}{1 - \zeta}, \tag{12.2}$$

([2]).

The mapping from (x, y) plane to the horn torus is given by

$$\xi = \frac{2x\sqrt{x^2 + y^2}}{(x^2 + y^2 + 1)^2},$$

$$\eta = \frac{2y\sqrt{x^2 + y^2}}{(x^2 + y^2 + 1)^2},$$

and

$$\zeta = \frac{(x^2 + y^2 - 1)\sqrt{x^2 + y^2}}{(x^2 + y^2 + 1)^2} + \frac{1}{2}.$$

This Puha mapping has a simple and beautiful geometrical correspondence. At first for the plane we consider the stereographic mapping to the Riemann sphere and next, we consider the common point of the line connecting the point and the center (0,0,1/2) and the horn torus. This is the desired point on the horn torus for the plane point.

The inversion is given by

$$x = \xi \left(\xi^2 + \eta^2 + \left(\zeta - \frac{1}{2} \right)^2 - \zeta + \frac{1}{2} \right)^{(-1/2)} \tag{12.3}$$

and

$$y = \eta \left(\xi^2 + \eta^2 + \left(\zeta - \frac{1}{2} \right)^2 - \zeta + \frac{1}{2} \right)^{(-1/2)} . \tag{12.4}$$

For the properties of horn torus with physical applications, see [5].

12.1 Conformal mapping from the plane to the horn torus with a modified mapping

W. W. Däumler discovered a surprising conformal mapping from the extended complex plane to the horn torus model (2018.8.18):

https://www.horntorus.com/manifolds/conformal.html
and
https://www.horntorus.com/manifolds/solution.html

We can represent the direct Däumler mapping from the z plane onto the horn torus as follows (V. V. Puha: 2018.8.28.22:31):

With

$$\phi = 2\cot^{-1}(-\log|z|), \quad z = x + yi, \tag{12.5}$$

$$\xi = \frac{x \cdot (1/2)(\sin(\phi/2))^2}{\sqrt{x^2 + y^2}},$$

$$\eta = \frac{y \cdot (1/2)(\sin(\phi/2))^2}{\sqrt{x^2 + y^2}},$$

and

$$\zeta = -\frac{1}{4}\sin\phi + \frac{1}{2}.$$

We have the inversion formula from the horn torus to the x, y plane:

$$x = \frac{\xi}{\sqrt{\xi^2 + \eta^2}} \exp \pm \left\{ \frac{\sqrt{\zeta - (\xi^2 + \eta^2 + \zeta^2)}}{\sqrt{\xi^2 + \eta^2 + \left(\zeta - \frac{1}{2}\right)^2}} \right\} \tag{12.6}$$

and

$$y = \frac{\eta}{\sqrt{\xi^2 + \eta^2}} \exp \pm \left\{ \frac{\sqrt{\zeta - (\xi^2 + \eta^2 + \zeta^2)}}{\sqrt{\xi^2 + \eta^2 + \left(\zeta - \frac{1}{2}\right)^2}} \right\} . \tag{12.7}$$

.

12.2 New world and absolute function theory

We will discuss on Däumler's horn torus model from some fundamental viewpoints.

First of all, note that in the Puha mapping and the Däumler mapping, and even in the classical stereographic mapping, we find the division by zero $1/0 = 0/0 = 0$. See [6] for the details.

12.2.1 What is the number system?

What are the numbers? What is the number system? For these fundamental questions, we can say that the numbers are complex numbers $\mathbf{C}$ and the number system is given by the Yamada field with the simple structure **as a field containing the division by zero.**

Nowadays, we have still many opinions on these fundamental questions, however, this subsection excludes all those opinions as in the above.

12.2.2 What is the natural coordinates?

We represented the complex numbers $\mathbf{C}$ by the complex plane or by the points on the Riemann sphere. On the complex plane, the point at infinity is the ideal point and for the Riemann sphere representation, we have to accept the **strong discontinuity**. From these reasons, the numbers and the numbers system should be represented by the Däumler's horn torus model that is conformally equaivalent to the extended complex plane.

12.2.3 What is a function? What is the graph of a function?

A function may be considered as a mapping from a set of numbers into a set of numbers.

The numbers are represented by Däumler's horn torus model and so, we can consider that a function, in particular, an analytic function can be considered as a mapping from Däumler's horn torus model into Däumler's horn torus model.

12.2.4 Absolute function theory

Following the above considerings, for analytic functions when we consider them as the mappings from Däumler's horn torus model into Däumler's horn torus model we would like to say that it is an **absolute function theory.**

For the classical theory of analytic functions, discontinuity of functions at singular points will be the serious problems and the theory will be quite different from the new mathematics, when we consider the functions on the Däumler's horn torus model. Even for analytic function theory on bounded domains, when we consider their images on Däumler's horn torus model, the results will be very interesting.

12.2.5 New mathematics and future mathematicians

The structure of Däumler's horn torus model is very involved and so, we will need some computer systems like MATHEMATICA and Isabelle/HOL system for our research activity. Indeed, for the analytical proof of the conformal mapping of Däumler, we had to use MATHEMATICA, already. Here, we will be able see some future of mathematicans.

13 WHAT IS THE ZERO?

The zero 0 as the complex number or real number is given clearly by the axioms by the complex number field and real number field, respectively. For this fundamental idea, we should consider the **Yamada field** containing the division by zero. The Yamada field and the division by zero calculus will arrange our mathematics, beautifully and completely; this will be our real and complete mathematics.

Standard value

The zero is a center and stand point (or bases, a standard value) of the coordinates - here we will consider our situation on the complex or real 2 dimensional spaces. By stereographic projection mapping or the Yamada field, the point at infinity $1/0$ is represented by zero. The origin of the coordinates and the point at infinity correspond with each other.

As the standard value, for the point $\omega_n = \exp\left(\frac{\pi}{n}i\right)$ on the unit circle $|z| = 1$, for $n = 0$:

$$\omega_0 = \exp\left(\frac{\pi}{0}i\right) = 1, \quad \frac{\pi}{0} = 0.$$

For the mean value

$$M_n = \frac{x_1 + x_2 + ... + x_n}{n},$$

we have

$$M_0 = 0 = \frac{0}{0}.$$

Fruitful world

For example, in very general partial differential equations, if the coefficients or terms are zero, we have some simple differential equations and the extreme case is all terms zero; that is, we have the trivial equation $0 = 0$; then its solution is zero. When we consider the converse, we see that the zero world is a fruitful one and it means some vanishing world. Recall the Yamane phenomena, the vanishing result is very simple zero, however, it is the result from some fruitful world. Sometimes, zero means void or nothing world, however, it will show some change as in the Yamane phenomena.

From 0 to 0; 0 means all and all are 0

As we see from our life figure, a story starts from the zero and ends to the zero. This will mean that 0 means all and all are 0, in a sense. The zero is a mother of all.

Impossibility

As the solution of the simplest equation

$$ax = b \tag{13.1}$$

we have $x = 0$ for $a = 0, b \neq 0$ as the standard value, or the Moore-Penrose generalized inverse. This will mean in a sense, the solution does not exist; to solve the equation (13.1) is impossible. We saw for different parallel lines or different parallel planes, their common point is the origin. Certainly they have the common point of the point at infinity and the point at infinity is represented by zero. However, we can understand also that they have no solutions, no common points, because the point at infinity is an ideal point.

We will consider the point P at the origin with starting at the time $t = 0$ with velocity $V > 0$ and the point Q at the point $d > 0$ with velocity $v > 0$. Then, the time of coincidence P=Q is given by

$$T = \frac{d}{V - v}.$$

When $V = v$, we have, by the division by zero, $T = 0$. This zero represents impossibility. We have many such situations.

We will consider the simple differential equation

$$m\frac{d^2x}{dt^2} = 0, m\frac{d^2y}{dt^2} = -mg \tag{13.2}$$

with the initial conditions, at $t = 0$

$$\frac{dx}{dt} = v_0 \cos\alpha, \quad \frac{dy}{dt} = v_0 \sin\alpha; \quad x = y = 0.$$

Then, the highest high h, arriving time t, the distance d from the starting point at the origin to the point $y(2t) = 0$ are given by

$$h = \frac{v_0^2 \sin\alpha}{2g}, \quad d = \frac{v_0^2 \sin 2\alpha}{g}$$

and

$$t = \frac{v_0 \sin\alpha}{g}.$$

For the case $g = 0$, we have $h = d = t = 0$. We considered the case that they are infinity; however, our mathematics means zero, which shows impossibility.

These phenomena were looked in many cases on the universe; it seems that God does not like the infinity.

As we stated already in the Bhāskara's example – sun and shadow

Zero represents void or nothing

On ZERO, the authors S. K. Sen and R. P. Agarwal [35] published its history and many important properties. See also R. Kaplan [9] and E. Sondheimer and A. Rogerson [37] on the very interesting books on zero and infinity.

India has a great tradition on ZERO, VOID and INFINITY and they are familiar with those concepts.

Meanwhile, Europian (containing the USA) people do not like such basic ideas and they are not familiar with them.

14 Void and nothing

It seems that void or nothing may be considered as in ZERO that for any situation, when we remove the same one, we can consider the lest is void or nothing. Therefore, void or nothing may be considered as in ZERO. However, is it possible physically? The physical problem will be too difficult to realize it and we think that void or nothing is based on our idea, not practical one.

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