About
Structure of a connected Quaternion-JULIA-Set
and
Symmetries of a related JULIA-Network

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Udo E. Steinemann.
Findeisen-Str. 5/7
D-71665 Vaihingen/Enz
Germany
udo.steinemann@t-online.de
A. Abstract.

If a variable is replaced by its square and subsequently enlarged by a constant during a number of iteration-steps in quaternion-space, a network of (3) sets will be built gradually. As long as for the iteration-constant certain conditions are fulfilled, the network will consist of: an unbounded set (escape-set) with trajectories escaping to infinity during course of the iteration, a bounded set (prisoner-set) with trajectories tending to a sink-point and a further bounded one (JULIA-set) with a fixed-point as repeller having a repulsive effect on all points of both the other sets. The iteration will continue until the attracting sink-point of prisoner-set and the repelling fixed-point on JULIA-set have been found. This situation is reached if predecessor- and successor-state of the iteration became equal. The fixed-point-condition provisionally formulated in general terms of quaternions, can be separated into (3) sub-conditions. When heeding the HAMILTONian-rules for interactions of the imaginary sub-spaces of the quaternion-space, each sub-condition will be appropriate for one imaginary sub-spaces and independently debatable. Knowledge of fixed-points from this fundamental network will one enable to study the structure of a connected JULIA-set.

The iteration will start from (1) on real-axis, this is not a restriction on generality because an appropriate scaling on real-axis can always be archived this way. It will become obvious, that the fixed-points in prisoner- and JULIA-set will depend on the iteration-constant only. Thus (16) different constants chosen appropriately will enable to arrange (16) fixed-points of JULIA-sets in the square-points of a hyper-cube and thereby together with the JULIA-sets to build a related JULIA-network. The symmetry-properties of this related JULIA-network can be studied on base of a hyper-cube’s symmetry-group extended by some additional considerations.
1. Introduction.

In the following attention is applied to the results of an iteration, which takes place in quaternion-space (a space with hyper-cubes with its space-elements) a layout of this is given next:

Each hyper-cube:
- Is surrounded by (8) cubes each one with (6) surfaces. Thus all together, cubes will have (48) surfaces.
- Because the cubes will share surfaces, only (24) surfaces will have to be counted effectively.

The quaternion-space is spanned by a real unit-vector (e) vertical to a tripod of imaginary unit-vectors \{i, j, d\}. Among these reference-vectors the HAMILTONian rules must hold:

1.1. \[
e^2 = (-i^2) = (-j^2) = (-d^2) = 1
\]
\[
[ij = (-ji) = d] \land [jd = (dj) = i] \land [di = (id) = j].
\]

Any point in the space is given by:
- \(Q = eQ_0 + iQ_1 + jQ_2 + dQ_3\) \(\Leftrightarrow\) \((Q = \text{quaternion-variable}) \land ([Q_0, Q_1, Q_2, Q_3] = \text{real components})\).

A sequence:

1.2. \(Q \rightarrow Q^2 + (N = N_0 + iN_1 + jN_2 + dN_3)^2 + N \rightarrow \ldots \Rightarrow (N = \text{constant}) \land ([N_0, N_1, N_2, N_3] = \text{real components})\)

iteratively executed is to considered next, where when noting the HAMILTONian rules (1.1.) the following relations between \(Q\) and \(Q^2\) must hold:

\begin{tabular}{|c|c|}
\hline
\textbf{Derivation 1.1.} & \\
\hline
\(Q = eQ_0 + iQ_1 + jQ_2 + dQ_3\) & \(\bullet\) \\
\hline
\(Q^2 = (eQ_0 + iQ_1 + jQ_2 + dQ_3)^2\) & \(\downarrow\) \\
\hline
\(Q^2 = e^2Q_0^2 + i^2Q_1^2 + j^2Q_2^2 + d^2Q_3^2 + i2Q_0Q_1 + j2Q_0Q_2 + d2Q_0Q_3 + j(iQ_2Q_1 + dQ_1Q_3) + d(iQ_3Q_1 + jQ_3Q_1)\) & \(\bullet\) \\
\hline
\end{tabular}

\begin{tabular}{|c|c|}
\hline
\textbf{leads to} & \(\bullet\) \\
\end{tabular}

\begin{tabular}{|c|c|}
\hline
\textbf{leads to} & with \\
\hline
\(e^2 = (-i^2) = (-j^2) = (-d^2) = 1\) & \(\downarrow\) \\
\hline
\(i \cdot j = (-j \cdot i) = d\) & \(\bullet\) \\
\hline
\(j \cdot d = (-d \cdot j) = i\) & \\
\hline
\(d \cdot i = (-i \cdot d) = j\) & \\
\hline
\end{tabular}

\[Q^2 = Q_0^2 - Q_1^2 - Q_2^2 - Q_3^2 + i2Q_0Q_1 + j2Q_0Q_2 + d2Q_0Q_3 + dQ_1Q_2 - jQ_1Q_3 - dQ_2Q_3 + iQ_3Q_1 + jQ_3Q_2 - iQ_3Q_2\]

\begin{tabular}{|c|c|}
\hline
\textbf{leads to} & \\
\hline
\end{tabular}

\[Q^2 = Q_0^2 + i2Q_0Q_1 + Q_1^2 + Q_2^2 + Q_3^2\]

\[Q^2 = j2Q_0Q_1 + Q_0^2 + Q_1^2 + Q_3^2\]

\[Q^2 = d2Q_0Q_1 + Q_0^2 + Q_1^2 + Q_3^2\]

\[Q^2 = i2Q_0Q_1 + Q_0^2 + Q_1^2 + Q_3^2\]

\[ Q_0^2 + d^2 Q_0^3 Q_0^2 - Q_0^3 = -2Q_0^2 \]

leads to
\[ Q^2 = (Q_0 + i Q_1)^2 + (Q_0 + j Q_2)^2 + (Q_0 + k Q_3)^2 - 2Q_0^2 \]

leads to
\[ \text{with} \]
\[ \begin{align*}
Q_{i_i} &= Q_0 + i Q_1 \land Q_{j_j} &= Q_0 + j Q_2 \land Q_{k_k} &= Q_0 + k Q_3 \\
Q &= (Q_0 + i Q_1) + (Q_0 + j Q_2) + (Q_0 + k Q_3) - 2Q_0
\end{align*} \]

Without restriction on generality due to a free choice of an appropriate scaling on the \( e \)-axis, \( Q_0 = 1 \) can be assumed for \((1^\circ)\) and thus one may further write:

\[ \begin{align*}
1^3. \quad &\ [(P = Q_0 + Q_1 + Q_2 - 2) \rightarrow (P^2 = Q_i^2 + Q_j^2 + Q_k^2 - 2) + N]^2 + N \rightarrow \ldots \quad \Rightarrow \quad N = N_{\phi} + N_{\phi'} + N_{\phi''}
\end{align*} \]

This iteration will run until its predecessor- and successor-state become equal. When certain restrictions on \((N)\) are observed, a network of \((3)\) connected sets will be generated:

- An unbounded escape-set with trajectories escaping to infinity in execution-time of the iteration,
- A bounded prisoner-set with trajectories tending to a sink-point while the iteration is going on and
- A bounded JULIA-set with a fractal structure formed by points acting as repellers against all points of both the other sets.

At the moment iteration stops, \((2)\) fixed-point have been generated:

- A repeller-point \((H_{11})\) on JULIA-set and
- A attractive sink-point \((H_{21})\) in prisoner-set.

From sequence \((1^\circ - 3.)\) the following condition for the fixed-points must hold:

\[ Q_i^2 + Q_j^2 + Q_k^2 - Q_i - Q_j - Q_k + N_0 + iN_1 + jN_2 + dN_3 = 0. \]

This will result in the \((2)\) fixed-point-solutions \((H_{1L2L})\) with their components:

\[ \begin{align*}
[Q_i &\leftarrow Q_{i_i}] \land [Q_j &\leftarrow Q_{j_j}] \land [Q_k &\leftarrow Q_{k_k}] \\
\end{align*} \]

Thus equation \((1^\circ - 3.)\) can now be re-written as:

\[ \begin{align*}
H_i^2 + H_j^2 + H_k^2 - H_i - H_j - H_k + N_0 + iN_1 + jN_2 + dN_3 &= 0,
\end{align*} \]

which under \((N_0 = N_{\phi} + N_{\phi'} + N_{\phi''})\) can be separated into:

\[ \begin{align*}
1^4. \quad &H_i^2 - H_i + N_{\phi} + iN_1 = 0 \\
1^5. \quad &H_j^2 - H_j + N_{\phi'} + jN_2 = 0 \\
1^6. \quad &H_k^2 - H_k + N_{\phi''} + dN_3 = 0.
\end{align*} \]

### 2. About the Structure of a connected Quaternion-JULIA-Set.

Searching for the fixed-points of an appropriate network (escape-, prisoner- and JULIA-set) seems to be a good way to enter the discussion on the structure of a connected JULIA-set. For further discussions an invariance of forward- and backward-iterations relative to the repelling fixed-point is of major interest.

Instead trying to find the fixed-points directly their projections in complex planes \((e^i \bar{Q}) \land (e^i \bar{L}) \land (e^i d))\) (obtained via solutions of equations \((1^\circ - 4. - 1^\circ - 6.)\) are used preliminarily in order to specify them indirectly.

#### 2.1. Fixed-Points from Interaction \((1^\circ 3.\) of Sequence \((1^\circ 1.)\).

From e.g. \([1 \& 2]\) is known, that a network with complex escape- prisoner- and JULIA-set can be obtained, when a sequence like:

\[ ([h = e\ell_0 + i\ell_1] \rightarrow h^2 + [\ell = e\ell_0 + i\ell_1])^2 + \ell \rightarrow ((h^2 + \ell)^2 + \ell)^2 + \ell \rightarrow \ldots \quad \Rightarrow \quad ([h = \text{variable}] \land [\ell = \text{constant}]). \]

is executed recursively and the iteration finally stops due to equality of its predecessor— and successor— state. This complex network will have properties comparable with the network specified from (1^3.) with the exception, it only exists in complex plane. For this complex network it has become obvious, there is a structural dichotomy. Depending on the sequence—constant (\( \ell \)) both prisoner— and JULIA—set may behave differently:

- For a specific \( \ell \)—set, the complex prisoner— and JULIA—set are connected (each on consists of one piece only) and the prisoner—set possesses a fixed—point as sink, while the JULIA—set has a fixed—point as a repeller for the point—sets of the prisoner—set and escape—set as well.
- In case of an alternate \( \ell \)—set prisoner— and JULIA—set will become CANTOR—sets, which means, they appear completely disconnected.

B. B. MANDELBROT [3] had the idea of picturing this dichotomy in a set of parameters (\( \ell \)) varying in the complex plane. This leads directly to the MANDELBROT—set:

He coloured each point in the plane of \( \ell \)—values black or white depending on whether the associated JULIA—sets respectively turned out to be one piece or dust.

What now a question about the characters of the complex solutions from equations (1^4.—1^6.) is concerned, it must be identified, that they are subjected to the same dichotomy as those in case of (2.1^1.). Solutions of (1^4.—1^6.) only will become fixed—points, if the complex components \((Nr0+iN1) \land (N+0+jN2) \land (N+0+dN3)\) within (1^3.) are extracted from the black part of the MANDELBROT—set.

2.1.1. Conditions to find Components of Fixed-Points.

Under these conditions (1^4.) leads to the preliminary solutions:

- \( H_{14} = \frac{1}{2} \pm \frac{1}{4} (1-4N=0-i4N1) \).

This can be further evaluated by settings:

- \( 1-4N=0-i4N1 = (u-ix)^2 = u^2 - i2ux + x^2 \)

and leads via a fourth—degree—equation for (u) to the following solutions of (u) and (x):

- \( u = \pm \left( \sqrt{2} \right) \left( 1-2N=0+i4N1 \right)^{\frac{1}{2}} \)
- \( x = \pm \left( \sqrt{2} \right) \left( 1+2N=0+i4N1 \right)^{\frac{1}{2}} \)

and finally to:

2.1.1. \( H_{14} = \frac{1}{2} \pm \frac{1}{4} (1-4N=0+i4N1) \).

The attracting or repelling property of the fixed—points is in essence the derivation of the sequence for (P) at the locations of \( H_{14} \). This derivation can be calculated in the same way as for the real case. A fixed—point is attractive, if the absolute value of the derivation at fixed—point location is (<1), it is repelling if (>1). Therefore one obtains:

- \( |2H_{14}| > 1 \rightarrow H_{14} \) is repelling and thus a point on corresponding JULIA—set.
- \( |2H_{14}| < 1 \rightarrow H_{14} \) is attracting point and thus a sink in the corresponding prisoner—set.

More details about the derivations can be found in the deviation—scheme (2.1.1^1.).
Derivation 2.1.1-1.

\[
\begin{align*}
\mathbf{H}_{1(1,2)}^2 - \mathbf{H}_{1(1,2)} + i \mathbf{N}_{10} + i \mathbf{N}_1 &= 0 \\
\text{leads to} & \quad \downarrow \\
\mathbf{H}_{1(1,2)} &= \frac{1}{2} \pm \frac{i}{2} \left(1 - 4 \mathbf{N}_{10} - i 4 \mathbf{N}_1 \right)^k \\
\text{leads to} & \quad \downarrow \\
1 - 4 \mathbf{N}_{10} - i 4 \mathbf{N}_1 &= (u - i x)^2 = u^2 - i 2ux + x^2 \\
\mathbf{H}_{1(1,2)} &= \frac{1}{2} + \frac{i}{2} u q + i \frac{1}{2} x \\
\text{leads to} & \quad \downarrow \downarrow \\
2\mathbf{H}_{1(1)} &= \left(1 + u \right)^2 + x^2 \\
\text{leads to} & \quad \downarrow \downarrow \\
2\mathbf{H}_{1(2)} &= \left(1 - u \right)^2 + x^2 \\
\text{leads to} & \quad \downarrow \downarrow \\
1 - 4 \mathbf{N}_{10} = u^2 + 4 \mathbf{N}_1^2 / u^2 \\
\text{leads to} & \quad \downarrow \downarrow \\
(4 \mathbf{N}_1 = 2ux) \rightarrow (2 \mathbf{N}_1 / u = x) \\
1 - 4 \mathbf{N}_{10} = u^2 + 4 \mathbf{N}_1^2 / u^2 \\
\text{leads to} & \quad \downarrow \downarrow \\
u^4 - (1 - 4 \mathbf{N}_{10})u^2 + 4 \mathbf{N}_1^2 = 0 \\
\text{leads to} & \quad \downarrow \downarrow \\
u^2 = \frac{1}{2} - 2 \mathbf{N}_{10} + \left(\frac{1}{2} - 2 \mathbf{N}_{10}^2 - 4 \mathbf{N}_1^2 \right)^k \\
\text{leads to} & \quad \downarrow \downarrow \\
u = \pm \left(\frac{1}{2} - 2 \mathbf{N}_{10} + \left(\frac{1}{2} - 2 \mathbf{N}_{10}^2 - 4 \mathbf{N}_1^2 \right)^k \right)^k \\
\text{leads to} & \quad \downarrow \downarrow \\
x = \pm 2 \mathbf{N}_1 / \left(\frac{1}{2} - 2 \mathbf{N}_{10} + \left(\frac{1}{2} - 2 \mathbf{N}_{10}^2 - 4 \mathbf{N}_1^2 \right)^k \right)^k \\
\text{leads to} & \quad \downarrow \downarrow \\
\mathbf{H}_{1(1,2)} = \frac{1}{2} \\
\text{leads to} & \quad \downarrow \downarrow \\
\mathbf{H}_{1(1,2)} = \frac{1}{2} - \left(\frac{1}{2} - 4 \mathbf{N}_{10} + \left(\frac{1}{2} - 2 \mathbf{N}_{10}^2 - 4 \mathbf{N}_1^2 \right)^k \right)^k \\
\left[ u > 0 \right] \rightarrow [0 < 2\mathbf{H}_{1(1)} = (1 + |u|)\left[1 + 4\mathbf{N}_1^2 / u^2 (1 + |u|)^2 \right] > 1] \\
\left[ u < 0 \right] \rightarrow [0 < 2\mathbf{H}_{1(2)} = (1 - |u|)\left[1 + 4\mathbf{N}_1^2 / u^2 (1 - |u|)^2 \right] < 1] \\
\left[ u > 0 \right] \rightarrow \left[0 > 2\mathbf{H}_{1(1)} = -(1 - |u|)\left[1 + 4\mathbf{N}_1^2 / u^2 (1 - |u|)^2 \right] < -1 \right] \\
\left[ u < 0 \right] \rightarrow \left[0 > 2\mathbf{H}_{1(2)} = -(1 + |u|)\left[1 + 4\mathbf{N}_1^2 / u^2 (1 + |u|)^2 \right] > -1 \right] \\
\text{leads to} & \quad \downarrow \downarrow \\
0 < \mathbf{H}_{1(1)} > 1 \\
0 < \mathbf{H}_{1(2)} < 1 \\
\text{leads to} & \quad \downarrow \downarrow \\
\mathbf{H}_{1(1)} : \text{Component associated with repeller-point on quaternion-JULIA-set} \\
\mathbf{H}_{1(2)} : \text{Component associated with sink-point in quaternion-prisoner-set}
\end{align*}
\]

Similarly (1.5.) will lead to the preliminary solutions:

- \( \mathbf{H}_{1(1,2)} = \frac{1}{2} \pm \frac{i}{2} \left(1 - 4 \mathbf{N}_{10} - j 4 \mathbf{N}_2 \right)^k \).

This can be further evaluated by settings:

- \( 1 - 4 \mathbf{N}_{10} - j 4 \mathbf{N}_2 = (v - j y)^2 = v^2 - j 2vy + y^2 \)

and leads via a fourth-degree equation for \( v \) to the following solutions for \( v \) and \( y \):

- \( v = \pm \left(\frac{1}{2} - 2 \mathbf{N}_{10} + \left(\frac{1}{2} - 2 \mathbf{N}_{10}^2 - 4 \mathbf{N}_2^2 \right)^k \right)^k \)
- \( y = \pm 2 \mathbf{N}_2 / \left(\frac{1}{2} - 2 \mathbf{N}_{10} + \left(\frac{1}{2} - 2 \mathbf{N}_{10}^2 - 4 \mathbf{N}_2^2 \right)^k \right)^k \)

and finally to:

2.1.1-2. \( \mathbf{H}_{1(1,2)} = \frac{1}{2} \pm \left(\frac{1}{2} - 4 \mathbf{N}_{10} + \left(\frac{1}{2} - 2 \mathbf{N}_{10}^2 - 4 \mathbf{N}_2^2 \right)^k \right)^k + j \mathbf{N}_2 / \left(\frac{1}{2} - 2 \mathbf{N}_{10} + \left(\frac{1}{2} - 2 \mathbf{N}_{10}^2 - 4 \mathbf{N}_2^2 \right)^k \right)^k \).

The attracting or repelling property of the fixed—points is in essence the derivation of the sequence for (P) at the locations of $H_{j13}$. This derivation can be calculated in the same way as for the real case. A fixed point is attractive, if the absolute value of the derivation at fixed—point location is (<1), it is repelling, if it is (>1). This leads in the actual cases to:

- $|2H_{j11}| > 1 \rightarrow H_{j11}$ is repelling and thus a point on corresponding JULIA—set.
- $|2H_{j11}| < 1 \rightarrow H_{j11}$ is attracting point and thus a sink in the corresponding prisoner—set.

More details about the derivations can be found in the following deviation—scheme (2.1.1^2).

And last not least fixed—point—condition (1^6.) will lead to the preliminary solutions:

- $H_{d_{j142}} = \frac{1}{2} + \frac{1}{2} \sqrt{1 - 4d_{a0} - d4N_3}$.

This can be further evaluated by settings:

- $1 - 4d_{a0} - d4N_3 = (w - dx)^2 = w^2 - d2wz + x^2$

and leads via a fourth—degree—equation for (w) to the following solutions for (w) and (z):

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The attracting or repelling property of the fixed-points is in essence the derivation of the sequence for \( P \) at the locations of \( H_{d\{1,2\}} \). This derivation can be calculated in the same way as for the real case. A fixed-point is attractive, if the absolute value of the derivation at fixed-point location is \(<1\), it is repelling, if it is \(>1\). This leads in the actual cases to:

- \( |2H_{d\{1\}}| > 1 \rightarrow H_{d\{1\}} \) is repelling and thus a point on the corresponding JULIA-set.
- \( |2H_{d\{2\}}| < 1 \rightarrow H_{d\{2\}} \) is attracting point and thus a sink in the corresponding prisoner-set.

More details about the derivation can be found in the following deviation scheme (2.1.1.3.):

<table>
<thead>
<tr>
<th>Derivation 2.1.1.3.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H_{d{1,2}}^2 = \frac{2}{2N_{d_0}} + dN_3 = 0 )</td>
</tr>
<tr>
<td>leads to</td>
</tr>
<tr>
<td>( \begin{align} H_{d{1,2}} &amp;= \pm\sqrt{\frac{2}{2N_{d_0}} - dN_3} \ \text{where} &amp; \end{align} )</td>
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<tr>
<td>leads to with</td>
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<tr>
<td>( 1 - 4N_{d_0} - 4dN_3 = (w - dz)^2 = w^2 - d2wz + z^2 )</td>
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<tr>
<td>( \begin{align} 4N_3 &amp;= 2wz \ 1 - 4N_{d_0} &amp;= w^2 + 4N_3^2 / w^2 \end{align} )</td>
</tr>
<tr>
<td>( w^2 - (1 - 4N_{d_0})w^2 + 4N_3^2 = 0 )</td>
</tr>
<tr>
<td>leads to</td>
</tr>
<tr>
<td>( w^2 = \frac{1}{2}(1 - 2N_{d_0} + {\frac{1}{2} - 2N_{d_0}}^2 - 4N_3^2)^{\frac{1}{2}} )</td>
</tr>
<tr>
<td>leads to</td>
</tr>
<tr>
<td>( w \pm {\frac{1}{2} - 2N_{d_0} + {\frac{1}{2} - 2N_{d_0}}^2 - 4N_3^2}^{\frac{1}{2}} )</td>
</tr>
<tr>
<td>leads to</td>
</tr>
<tr>
<td>( z = \pm \frac{2N_3}{\frac{1}{2} - \frac{1}{2}N_{d_0} + {\frac{1}{2} - \frac{1}{2}N_{d_0}}^2 - \frac{1}{4}N_3^2} )</td>
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<tr>
<td>leads to</td>
</tr>
<tr>
<td>( H_{d{1,2}} = \frac{1}{2} )</td>
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<tr>
<td>( \begin{align} \pm dN_3 &amp;/ {\frac{1}{2} - \frac{1}{2}N_{d_0} + {\frac{1}{2} - \frac{1}{2}N_{d_0}}^2 - \frac{1}{4}N_3^2}^{\frac{1}{2}} &amp; \end{align} )</td>
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<tr>
<td>( [w &gt; 0] \rightarrow [0 &lt;</td>
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<td>( [w &gt; 0] \rightarrow [0 &lt;</td>
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<td>( [w &lt; 0] \rightarrow [0 &gt;</td>
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<td>( [w &lt; 0] \rightarrow [0 &gt;</td>
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<tr>
<td>leads to 0 &lt; (</td>
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<tr>
<td>leads to 0 &lt; (</td>
</tr>
</tbody>
</table>

\( H_{d\{1\}} \): Component associated with repeller-point on quaternion-JULIA-set

\( H_{d\{2\}} \): Component associated with sink-point in quaternion-prisoner-set
2.1.2. Fixed-Points as Quaternion-Points.

(H) as a quaternion can generally be given in a form like:

\[ H = \left( a_0^2 + a_1^2 + a_2^2 + a_3^2 \right)^{\frac{1}{2}} \exp \left( i a_1 + j a_2 + k a_3 \right) \]

where \( a_0, a_1, a_2, a_3 \) are real numbers.

- For \( t_1 \): \( \cos \left( \Psi_1 \right) + j \sin \left( \Psi_1 \right) \)
- For \( t_2 \): \( \cos \left( \Psi_2 \right) + j \sin \left( \Psi_2 \right) \)
- For \( t_3 \): \( \cos \left( \Psi_3 \right) + j \sin \left( \Psi_3 \right) \)

Because \( H_{112} \land H_{112} \land H_{d123} \) may be expressed due to (2.1.1*1. – 2.1.1*3.), this will further lead to:

- \( t_1 \): \( \cos \left( \Psi_1 \right) + j \sin \left( \Psi_1 \right) \) \( H_{112} = \left\{ \frac{1}{2} \pm \left( \frac{1}{2} - \frac{1}{2} N_{d0} + \left( \frac{1}{2} - \frac{1}{2} N_{d0} \right)^2 - 4N_{d3}^2 \right)^{\frac{1}{2}} \right\} \)
- \( t_2 \): \( \cos \left( \Psi_2 \right) + j \sin \left( \Psi_2 \right) \) \( H_{1123} = \left\{ \frac{1}{2} \pm \left( \frac{1}{2} - \frac{1}{2} N_{d0} + \left( \frac{1}{2} - \frac{1}{2} N_{d0} \right)^2 - 4N_{d3}^2 \right)^{\frac{1}{2}} \right\} \)
- \( t_3 \): \( \cos \left( \Psi_3 \right) + j \sin \left( \Psi_3 \right) \) \( H_{d123} = \left\{ \frac{1}{2} \pm \left( \frac{1}{2} - \frac{1}{2} N_{d0} + \left( \frac{1}{2} - \frac{1}{2} N_{d0} \right)^2 - 4N_{d3}^2 \right)^{\frac{1}{2}} \right\} \)

Thus the fixed-points for JULIA- and prisoner-set will appear as follows:

2.1.2*1. \( H_{11} = H_{111} \times H_{111} \times H_{d12} \times H_{d12} = 1 \)

2.1.2*2. \( H_{21} = H_{21} \times H_{21} \times H_{d1} \times H_{d1} = 1 \)

2.3. The fractal Structure of the JULIA-Set.

A JULIA-set is a complete invariant fractal with respect to forward- and backward-iteration. A \( j \)-th pre-image (in a backward-iteration) and a \( k \)-th image (in a forward-iteration) starting from the repellor \( H_{11} \) by equation 2.1.2*1. are to be obtained in the following way:

2.3*1. Images: \( R^{(1)} = H_{11} \land H_{11} \land H_{21} \land H_{21} = 1 \)

2.3*2. Pre-images: \( R^{(2)} \land R^{(2)} = 1 \land H_{11} \land H_{21} \land H_{21} = 1 \)

Because \( H_{11} \) is a point of the JULIA-set, \( R^{(2)} \land R^{(2)} \) cannot in the basin of attraction of infinity otherwise the initial point \( H_{11} \) would have to be part of the escape-set too. On the other hand, both kinds of images cannot be in the interior (the prisoner-set), because then \( H_{11} \) would then have to be from prisoner-set too, what again is not the case. Thus \( R^{(2)} \land R^{(2)} \) must be from the boundary (the JULIA-set). The reason for all this can also be found in the continuity of the quadratic transformation. Arbitrarily close to the images and pre-images there are escaping- and prisoner-points and the continuity of iteration implies, neighbourhood relation must hold for the whole set of transformation points. This finally leads to the statement, the JULIA-set is invariant with respect to forward- and backward- transformation as well.

The total, unlimited set of images and pre-images of the repellor–fixed–point on Julia–set determines the fractal structure of the JULIA-set.


It is obvious from equations (2.1.1*1.) and (2.1.1*2.), the fixed-points \( H_{112} \) of the network (escape–prisoner– and JULIA–set) obtained from iteration (1*3.) depend on selection of (N) only. Thus (16)
different choices of (N's) chosen appropriately from the black part of the MANDELBROT-set will define (16) different fixed-points (H[1]) for JULIA-sets as square-points of a hyper-cube. This hyper-cube together with the JULIA-sets belonging to each of the square-points will represent a related JULIA-network. The symmetry-properties of this JULIA-network is to be obtained on base of a hyper-cube's symmetry-group extended by some additional considerations.

The symmetry-group of a cube can be derived from the symmetry-group of a square. With this knowledge in mind all hints are provided to further obtain the symmetry-group of a hyper-cube. The symmetry-group of a hyper-cube with additional considerations will then finally lead to the symmetry-properties of the related JULIA-network.

3.1. The Symmetries of a Square.

The symmetry-group of a square can best be described by the group-table below, consisting of (64) permutations of the square-points (contained in the entries of the table) obtained when (8) operations act on the square. The (8) operations consist of:

- The identity-operation (id) to reinstall the starting configuration,
- (3) right-turning rotations ([r1 = π/2] ∧ [r2 = π] ∧ [r3 = 3π/2]) around the centre of the square,
- (4) flip-operations (f1* f2* f3* f4) with respect to indicated directions.

The permutations within entries (1 → 64) of the group-table have the meaning:

- Positions of square-points after an operation of column(0) having acted on the square
- Positions of square-points after operation of row(0) being performed on top of operation in column(0).

The yellow-marked sub-group is the cyclic group of the square.

3.2. Symmetries of a Cube.

From symmetry-group of a square, (3) symmetry-sub-groups of a cube can be derived by replacing:

- Rotations around centre of square by right-turning rotations (R₁⁻¹R₂⁻¹R₃⁻¹) around each of the axes (AB⁻¹CD⁻¹EF):
  - \([(AB \perp (4,5,6,7)) \wedge (CD \perp (1,2,6,5)) \wedge (EF \perp (3,2,6,7)))]

- Flip-operations (f₁⁻¹f₂⁻¹f₃⁻¹) with respect to directions (black-red-blue-green) respectively replaced by mirror-operations (m₁⁻¹m₂⁻¹m₃⁻¹m₄⁻¹) with respect to appropriate mirror-planes:
  - \{(NKLM)⁻¹m₁⁻¹, (0264)⁻¹m₂⁻¹, (GHIJ)⁻¹m₃⁻¹ and (1573)⁻¹m₄⁻¹ plane for rotation in AB-direction
  - \{(OPQR)⁻¹m₁⁻¹, (0167)⁻¹m₂⁻¹, (NKLM)⁻¹m₃⁻¹ and (2543)⁻¹m₄⁻¹ plane for rotation in CD-direction
  - \{(OPQR)⁻¹m₁⁻¹, (0563)⁻¹m₂⁻¹, (GHIJ)⁻¹m₃⁻¹ and (1274)⁻¹m₄⁻¹ plane for rotation in EF-direction.

Under these conditions one will obtain (3) symmetry-sub-groups of a cube with respect to the directions (AB⁻¹CD⁻¹EF), each one is isomorphic with the symmetry-group of a square.

The first sub-group based on direction (AB) follows immediately with (64) elements, which belonging to multiplications of operations(column(0)) and operations(row(0)).

<table>
<thead>
<tr>
<th>AB</th>
<th>R₁</th>
<th>R₂</th>
<th>R₃</th>
<th>m₁</th>
<th>m₂</th>
<th>m₃</th>
<th>m₄</th>
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</thead>
<tbody>
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</tbody>
</table>

Under these conditions one will obtain (3) symmetry-sub-groups of a cube with respect to the directions (AB⁻¹CD⁻¹EF), each one is isomorphic with the symmetry-group of a square.

The first sub-group based on direction (AB) follows immediately with (64) elements, which belonging to multiplications of operations(column(0)) and operations(row(0)).

<table>
<thead>
<tr>
<th>AB</th>
<th>R₁</th>
<th>R₂</th>
<th>R₃</th>
<th>m₁</th>
<th>m₂</th>
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<tr>
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</tr>
</tbody>
</table>

A second sub-group based on direction (CD) follows next with (64) elements belonging to multiplications of operations(column(0)) and operations(row(0)):

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<th>R₂</th>
<th>R₃</th>
<th>m₁</th>
<th>m₂</th>
<th>m₃</th>
<th>m₄</th>
</tr>
</thead>
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<td>R₁</td>
<td>= R₂</td>
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<tr>
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<td>= m₃</td>
<td>= m₂</td>
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<td>= m₁</td>
</tr>
</tbody>
</table>

Finally one obtains a sub-group based on direction (EF) which follows next with (64) elements belonging to all multiplications of operations(column(0)) and operations(row(0)):

<table>
<thead>
<tr>
<th>★</th>
<th>id</th>
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<th>R₂</th>
<th>R₃</th>
<th>m₁</th>
<th>m₂</th>
<th>m₃</th>
<th>m₄</th>
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<td>= R₃</td>
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<td>= m₁</td>
<td>= m₂</td>
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<tr>
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<td>= m₃</td>
<td>= m₂</td>
<td>= m₄</td>
<td>= m₁</td>
</tr>
</tbody>
</table>

In addition (4) flip-operations \((F_5 \diamond F_6 \diamond F_7 \diamond F_8)\) with respect to the space—diagonals of the cube will have be taken into consideration. The properties of these operations are summarized in the next table:

Thus finally (25) symmetry—operations in total will make up the symmetry—group of a cube.

### 3.3. Symmetries of a Hyper-Cube.

If one replaces in a cube:

- Each pair of parallel planes involved in one of the rotations \((R_1 \lor R_2 \lor R_3)\) by a quadruple of cubes (from hyper—cube’s structure) with surfaces parallel to a perpendicular common axis of rotation out of \((\alpha \beta \lor \gamma \delta \lor \epsilon \zeta)\),
- Each mirror—plane of a cube by a 3—dimensional object with a pair of parallel planes suitable for a further more mirror—operation,

(3) symmetry—sub—groups of a hyper—cube are obtained, each isomorphic with the symmetry—group of a square and a symmetry—sub—groups of a cube. Each symmetry—sub—group of the hyper—cube consists of:
- Right-turning rotations \( (R_1 \land R_2 \land R_3) \), around a \( (\alpha \beta \gamma \delta \epsilon \zeta) \)-axis,
- Mirror-operation \((M_1 \land M_2 \land M_3 \land M_4)\) with respect to the appropriate mirror-objects.

The first sub-group based on direction \((\alpha \beta)\) follows immediately with \((64)\) permutations according to all multiplications of operations(column(0)) and of operations(row(0)).

<table>
<thead>
<tr>
<th>(\star)</th>
<th>id</th>
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<th>(R_2)</th>
<th>(R_3)</th>
<th>(M_1)</th>
<th>(M_2)</th>
<th>(M_3)</th>
<th>(M_4)</th>
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</thead>
<tbody>
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<td>(R_2)</td>
<td>(R_3)</td>
<td>(M_1)</td>
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<tr>
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<td>(\lambda \mu \nu \epsilon \eta \theta \kappa)</td>
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<td>(\mu \nu \xi \sigma \chi \psi \phi) (\eta \theta \kappa) (\lambda \mu \nu \epsilon \eta \theta \kappa)</td>
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<td>(\mu \nu \xi \sigma \chi \psi \phi) (\eta \theta \kappa) (\lambda \mu \nu \epsilon \eta \theta \kappa)</td>
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<td>(\mu \nu \xi \sigma \chi \psi \phi) (\eta \theta \kappa) (\lambda \mu \nu \epsilon \eta \theta \kappa)</td>
<td>(\mu \nu \xi \sigma \chi \psi \phi) (\eta \theta \kappa) (\lambda \mu \nu \epsilon \eta \theta \kappa)</td>
<td>(\mu \nu \xi \sigma \chi \psi \phi) (\eta \theta \kappa) (\lambda \mu \nu \epsilon \eta \theta \kappa)</td>
<td>(\mu \nu \xi \sigma \chi \psi \phi) (\eta \theta \kappa) (\lambda \mu \nu \epsilon \eta \theta \kappa)</td>
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<td>(\mu \nu \xi \sigma \chi \psi \phi) (\eta \theta \kappa) (\lambda \mu \nu \epsilon \eta \theta \kappa)</td>
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<td>(\mu \nu \xi \sigma \chi \psi \phi) (\eta \theta \kappa) (\lambda \mu \nu \epsilon \eta \theta \kappa)</td>
</tr>
</tbody>
</table>

A second sub-group based on direction ($\gamma$) follows next with (64) permutations according to all multiplications of operations(column(0)) and of operations(row(0)):

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$\delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma \tau \upsilon \phi \sigma \rho \varsigma$</td>
<td>$\lambda \mu \nu \xi \kappa$</td>
</tr>
</tbody>
</table>

And finally a sub-group based on direction ($\epsilon \zeta$) with (64) permutations will follow according all multiplications of operations(column(0)) and operations(row(0)):
In addition to these (21) symmetry—operations (8) flip—operations will have be considered, due to the (8) quaternion—diagonals of the hypercube:

<table>
<thead>
<tr>
<th>Star</th>
<th>R₁</th>
<th>R₂</th>
<th>R₃</th>
<th>M₁</th>
<th>M₂</th>
<th>M₃</th>
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</thead>
<tbody>
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In addition to these (21) symmetry—operations (8) flip—operations will have be considered, due to the (8) quaternion—diagonals of the hypercube:

<table>
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<td>F₁₁</td>
<td>F₁₂</td>
</tr>
</tbody>
</table>

Together with (200) symmetry—operations for the (8) inner cubes of a hyper—cube, presumably (232) different symmetry—operation in total have to be counted for a hyper—cube and are responsible for its symmetry—group.

3.4. Symmetry—Group of the related JULIA-Network.

The (16) different fixed—points ($H_{[1-\infty]}$) by definition from above will form a hyper—cube in quaternion—space. Thus a probe—point moving from ($H_{[1-M]}$) to ($H_{[1-N]}$) by execution of a hyper—cube’s symmetry—operations will change its (N) fluently from ($N_{[M]}$) to ($N_{[m]}$). Because any image or pre—image of the probe—point must follow equations (2.3.1. & 2.3.2) in any position of the probe—point, they will always be adapted in relation to the probe—point’s location. Therefore the probe—point in essence mediates between the JULIA—sets with fixed—points ($H_{[1-M]}$) and ($H_{[1-N]}$).

In summery one may say, that the related JULIA—network under the action of any symmetry—operation of a hyper—cube will remain completed in itself, related JULIA—network and the symmetry—operations of a hyper—cube will built a symmetry—group.

4. Summary.

The iteration of sequence (1.3.) in quaternion—space — with restrictions from MANDELBROT—set on the complex components of its iteration—constant — resulted in a network of (3) sets. An unbounded escape—set (with trajectories escaping to infinity) accompanied by a set caught in a limited area (prisoner—set,

whose trajectories tended to a sink-point) and the boundary-set of the prisoner-set built by points acting repulsively on points from escape- and prisoner-set as well.

The iteration stopped if the sink-point of the prisoner-set and a fixed repeller-point on JULIA-set had been obtained, that is, when equality between the iteration's predecessor- and successor-state had been reached. A Quaternion-condition for this stop-event (the fixed-point-condition) could be formulized and - by taking into account the HAMILTONian rules - could be separated into three sub-conditions (according to the quaternion-space's complex subspaces). Every one of these sub-conditions could subsequently be solved independently. On base of these results it became possible to express the quaternion fixed-points of prisoner- and JULIA-set as well.

With knowledge of the fixed-repeller-point of a JULIA-set it became possible to describe the structure of the JULIA-set by the set of images and pre-images, which are obtained from forward- or backward-iteration relative to the fixed-repeller-point.

Fixed-points and JULIA-set of the network, obtained by iterative execution of sequence (1^3.) will only depended on the choice of the actual iteration-constant. Therefore, (16) constants appropriately chosen from black part of the MANDELBROT-set will make it possible to arrange the repeller-fixed-points of the iteratively obtained JULIA-sets in the square-points of a hyper-cube. Fixed-points and their JULIA-sets positioned this way will then represent a related JULIA-network. The set of quaternion-points of the related JULIA-network together with the symmetry-operations of a hyper-cube will form the symmetry-group of the related JULIA-network.

5. References.


[3] B. B. Mandelbrot: Fractal aspects of the iteration of \( z \rightarrow \lambda z(1-z) \) for complex \( \lambda \) and \( z \), Annals N.Y. Acadey of Sciences 357, 1980.