Schwarzschild Metric Total Energy Inconsistency

Karl De Paepe

Abstract

We determine the total energy of the Schwarzschild metric. We show use of divergence theorem leads to a total energy inconsistency.

1 Schwarzschild metric

Units are chosen so that $c = G = 1$. The Schwarzschild metric [1] is

$$ds^2 = -\left[1 - \frac{2M}{r}\right]dt^2 + \left[1 - \frac{2M}{r}\right]^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2 \tag{1}$$

Let

$$x^1 = r\sin\theta\cos\varphi \qquad x^2 = r\sin\theta\sin\varphi \qquad x^3 = r\cos\theta \tag{2}$$

hence

$$\begin{aligned} ds^2 &= -\left[1 - \frac{2M}{r}\right]dt^2 + \left\{\left[1 - \frac{2M}{r}\right]^{-1} - 1\right\}\frac{(\mathbf{x}\cdot d\mathbf{x})^2}{r^2} + d\mathbf{x}^2 \\ &= (-1 + h_{00})dt^2 + (\delta_{kl} + h_{kl})dx^k dx^l \end{aligned} \tag{3}$$

where

$$h_{00} = \frac{2M}{r} \qquad h_{0k} = 0 \qquad h_{kl} = \frac{2Mx^k x^l}{r^2(r - 2M)} \tag{4}$$

2 Energy-momentum tensor of gravitational field

The Einstein field equations are [1]

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -8\pi T_{\mu\nu} \tag{5}$$

hence

$$R^{(1)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R^{(1)\lambda}{}_{\lambda} = -8\pi(T_{\mu\nu} + t_{\mu\nu}) \tag{6}$$

where

$$16\pi t_{\mu\nu} = 2R_{\mu\nu} - g_{\mu\nu}R - 2R^{(1)}_{\mu\nu} + \eta_{\mu\nu}R^{(1)\lambda}{}_{\lambda} \tag{7}$$

and

$$2R^{(1)}_{\mu\nu} = \frac{\partial^2 h^\lambda{}_\lambda}{\partial x^\mu \partial x^\nu} - \frac{\partial^2 h^\lambda{}_\mu}{\partial x^\lambda \partial x^\nu} - \frac{\partial^2 h^\lambda{}_\nu}{\partial x^\lambda \partial x^\mu} + \frac{\partial^2 h_{\mu\nu}}{\partial x^\lambda \partial x_\lambda} \tag{8}$$

Indices on $h_{\mu\nu}$, $R^{(1)}_{\mu\nu}$, and $\partial/\partial x^\lambda$ are raised and lowered with η's. For example $h^\lambda{}_\lambda = \eta^{\lambda\nu} h_{\lambda\nu}$ and $\partial/\partial x_\lambda = \eta^{\lambda\nu}\partial/\partial x^\nu$. We interpret $t^{\mu\nu}$ as the energy- momentum tensor of the gravitational field [1]. We have by (4) and (8) that $R^{(1)}_{00} = 0$. Now for the Schwarzschild metric $R_{\mu\nu} = 0$ so by (4), (7), (8), and $R_{\mu\nu} = R^{(1)}_{\mu\nu} = 0$ we have

$$\begin{aligned}
-32\pi t_{00} &= 2R^{(1)}_{11} + 2R^{(1)}_{22} + 2R^{(1)}_{33} \\
&= -\ \frac{\partial^2 h_{00}}{\partial x^2} + \frac{\partial^2 h_{11}}{\partial y^2} + \frac{\partial^2 h_{11}}{\partial z^2} + \frac{\partial^2 h_{22}}{\partial x^2} + \frac{\partial^2 h_{33}}{\partial x^2} - 2\frac{\partial^2 h_{12}}{\partial x \partial y} - 2\frac{\partial^2 h_{13}}{\partial x \partial z} \\
&\quad -\ \frac{\partial^2 h_{00}}{\partial y^2} + \frac{\partial^2 h_{22}}{\partial x^2} + \frac{\partial^2 h_{22}}{\partial z^2} + \frac{\partial^2 h_{11}}{\partial y^2} + \frac{\partial^2 h_{33}}{\partial y^2} - 2\frac{\partial^2 h_{12}}{\partial x \partial y} - 2\frac{\partial^2 h_{23}}{\partial y \partial z} \\
&\quad -\ \frac{\partial^2 h_{00}}{\partial z^2} + \frac{\partial^2 h_{33}}{\partial y^2} + \frac{\partial^2 h_{33}}{\partial x^2} + \frac{\partial^2 h_{22}}{\partial z^2} + \frac{\partial^2 h_{11}}{\partial z^2} - 2\frac{\partial^2 h_{23}}{\partial y \partial z} - 2\frac{\partial^2 h_{13}}{\partial x \partial z}
\end{aligned} \tag{9}$$

At a point $(t, x, 0, 0)$ with $x > 0$

$$\begin{gathered}
\frac{\partial^2 h_{00}}{\partial x^2} = \frac{4M}{x^3} \qquad \frac{\partial^2 h_{11}}{\partial y^2} = \frac{\partial^2 h_{11}}{\partial z^2} = \frac{2M(4M-3x)}{x^2(x-2M)^2} \qquad \frac{\partial^2 h_{12}}{\partial x \partial y} = \frac{\partial^2 h_{13}}{\partial x \partial z} = \frac{4M(M-x)}{x^2(x^2-2M)^2} \\
\frac{\partial^2 h_{00}}{\partial y^2} = \frac{\partial^2 h_{00}}{\partial z^2} = \frac{-2M}{x^3} \qquad \frac{\partial^2 h_{23}}{\partial y \partial z} = \frac{2M(x-2M)}{x^2(x-2M)^2} \qquad \frac{\partial^2 h_{22}}{\partial x^2} = \frac{\partial^2 h_{33}}{\partial x^2} = \frac{\partial^2 h_{33}}{\partial y^2} = \frac{\partial^2 h_{22}}{\partial z^2} = 0
\end{gathered} \tag{10}$$

Using (9) and (10) we can calculate t_{00} at $(t, x, 0, 0)$. Now t_{00} is spherically symmetric so we can replace x by r in t_{00} calculated at $(t, x, 0, 0)$ giving

$$t_{00}(r) = \frac{-M^2}{2\pi r^2 (r-2M)^2} \tag{11}$$

3 Total energy and divergence theorem

Since $T_{\mu\nu} = 0$ the total energy [1] of the Schwarzschild metric is then using (11)

$$P^0 = \int \eta^{0\mu}\eta^{0\nu}(T_{\mu\nu} + t_{\mu\nu}) d^3x = \int t_{00} d^3x = -M \int_0^\infty \frac{du}{(1-u)^2} \tag{12}$$

which is not finite. Alternatively let us calculate the total energy using the divergence theorem. We have [1]

$$R^{(1)\mu\nu} - \frac{1}{2}\eta^{\mu\nu} R^{(1)\lambda}{}_\lambda = \frac{\partial Q^{\rho\mu\nu}}{\partial x^\rho} \tag{13}$$

where

$$2Q^{\rho\mu\nu} = \frac{\partial h^\lambda{}_\lambda}{\partial x_\mu}\eta^{\rho\nu} - \frac{\partial h^\lambda{}_\lambda}{\partial x_\rho}\eta^{\mu\nu} - \frac{\partial h^{\lambda\mu}}{\partial x^\lambda}\eta^{\rho\nu} + \frac{\partial h^{\lambda\rho}}{\partial x^\lambda}\eta^{\mu\nu} + \frac{\partial h^{\mu\nu}}{\partial x_\rho} - \frac{\partial h^{\rho\nu}}{\partial x_\mu} \tag{14}$$

The total energy using (6) and (13) is then

$$\begin{aligned}
P^0 &= \int \eta^{0\mu}\eta^{0\nu}(T_{\mu\nu} + t_{\mu\nu}) d^3x = -\frac{1}{8\pi}\int \left(R^{(1)00} - \frac{1}{2}\eta^{00} R^{(1)\lambda}{}_\lambda\right) d^3x \\
&= -\frac{1}{8\pi}\int \frac{\partial Q^{\rho 00}}{\partial x^\rho} d^3x = -\frac{1}{8\pi}\int \frac{\partial Q^{k00}}{\partial x^k} d^3x
\end{aligned} \tag{15}$$

where repeated Latin indices are summed over 1,2,3. We now assume we can apply the divergence theorem to (15) giving

$$P^0 = -\frac{1}{8\pi}\int Q^{k00} n_k r^2 d\Omega = -\frac{1}{16\pi}\int \left\{\frac{\partial h_{jj}}{\partial x^i} - \frac{\partial h_{ij}}{\partial x^j}\right\} n_i r^2 d\Omega \tag{16}$$

where $n_k = x^k/r$, $d\Omega = \sin\theta d\theta d\varphi$, and the integral is over a large sphere of radius r. Calculating this for the metric (3) gives $P^0 = M$ which is finite. Without using the divergence theorem P^0 is not finite but using the divergence theorem $P^0 = M$. This is an inconsistency.

References

[1] S. Weinberg, *Gravitation and Cosmology*

[2] K. De Paepe, Physics Essays, September 2012

k.depaepe@utoronto.ca