MOTZKIN ISLANDS: A 3-DIMENSIONAL EMBEDDING OF MOTZKIN PATHS

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Abstract. A Motzkin Path is a walk left-to-right starting at the horizontal axis, consisting of up, down or horizontal steps, never descending below the horizontal axis, and finishing at the horizontal axis. Interpret Motzkin Paths as vertical geologic cuts through mountain ranges with limited slopes. The natural embedding of these paths defines Motzkin Islands as sets of graphs labeled on vertices by non-negative integers (altitudes), a graph cycle defining a shoreline at zero altitude, and altitude differences along edges never larger than one. We address some of these islands with simple shapes on triangular and quadratic meshes.

1. Motzkin Islands

Let a Motzkin Island be defined as a 3-dimensional extension of Motzkin Paths \cite{5, A001006}: Given a connected finite graph with single, undirected edges take one cycle and call it shoreline. Vertices along a path along this cycle are labeled with zeros; labels represent geographic altitudes; the shoreline is at sea level. All other vertices are labeled by non-negative integers such that the labels of adjacent vertices differ at most by one.

In consequence, each path from shore to shore along graph edges is a Motzkin Path of up, horizontal and down steps.

The concept is likely restricted to planar graphs, because otherwise crossing edges are mapped to “Motzkin bridges and tunnels” in the landscape of the islands.

2. Regular Triangular Shoreline

Motzkin Islands can be defined on any type of finite graphs. One particular class are Motzkin Regular Triangular Islands: a triangular grid with a triangular shoreline. They consist of a graph of an isosceles triangle, each side split into \( n \) edges of unit length, the \( 3n \) edges defining the shoreline, and the interior edges defined by connecting these vertices of the shoreline by straight lines parallel to the 3 edges. Each vertex not on the shoreline has 6 adjacent vertices. For \( n = 5 \) this looks like
The edges are not shown from here on, only the integer labels at the vertices.

An example of such an island with $n = 9$ and highest peak at 2 is:

```
 0 0 0 0 0
 0 0 0 0 0 0 0 0 0
 0 0 0 0 0 0 0 0
 0 0 1 1 0
 0 1 1 2 1 0
 0 1 2 2 1 1 0
 0 1 2 1 2 1 0
 0 1 1 1 1 1 0 1 0
 0 0 0 0 0 0 0 0 0 0
```

The same island converted to a STL file and rendered with meshlab illustrates why these geometric objects may be called islands:

The full set of islands with edge lengths $n = 0, 1, 2$ and 3 is

```
0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 1 0
0 0 0 0 0 0 0 1 0
0 0 0 0 0 0 0 1 0
```

Let $M^\circ(n)$ count “fixed” islands where rotations by multiples of 120 degrees around a vertical axis or mirrors along one of the three vertical planes through
the center may generate other islands. [This is in tune with the standard count of Motzkin paths which are considered different walking left-to-right or right-to-left. By analogy with the “free” polyominoes we could also count “free” islands where symmetry-related copies are counted only once.]

A lower bound is

\[ M^S(n) \geq 2^{T(n-2)} \]

where \( T(n) \equiv n(n+1)/2 \) are the triangular numbers.

\textbf{Proof.} \( T(n-2) \) is the number of vertices that are not on the shoreline [7]. Any set of labels of ones and zeros at these internal vertices creates a Motzkin Island, and there are \( 2^{T(n-2)} \) possible multisets of zeros and ones. So there are \( 2^{T(n-2)} \) islands with maximum peak height 1, and more if peak heights \( \geq 2 \) are achievable, i.e., at sufficiently large islands such that the distance to the shore is larger. \( \square \)

The number of Motzkin Reg-Triangular Islands is

\[ M^S(n) = 1, 1, 1, 2, 8, 64, 1032, 33640, 2221952, 297891576, 81173202920, 45006474922560, 50821273347381064, \ldots \quad (n \geq 0). \]

One can refine these enumerations by counting Reg-Triangular Islands with maximum altitude \( h \): Table 1.

The first case where the lower bound (1) is surpassed is \( M^S(6) = 1032 = 8 + 2^{10} = 8 + 2^{T(4)} \) where a peak height of 2 may be reached in the center of the triangle:

\[
\begin{array}{cccccc}
0 & 0 & 0 & a & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 2 & 1 & 0 \\
0 & b & 1 & 1 & c & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
\]

\begin{verbatim}
Table 1. Reg-Triangular Islands \( M^S(n) \) with maximum altitude \( h \). The sum of the columns \( h = 0 \) and \( h = 1 \) is (1). Row sums are (2).
\end{verbatim}
The $8 = 2^3$ islands of that shape are those where any combination of zeros and ones appears at the three vertices $a$, $b$ and $c$: Entry $n = 6, h = 2$ in Table 1.

3. Regular Rectangular Shoreline

3.1. Overview. On the simple $m \times n$ square grid with 4 neighbors adjacent to each internal vertex and an area of $mn$ unit squares, the prototypical shore line is a rectangle.

\[
\begin{array}{cccccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

The $3 \times 4$ Motzkin Islands have the labels arranged like this,

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & a & b & c \\
0 & d & e & f \\
0 & 0 & 0 & 0 \\
\end{array}
\]

where $|b-a| \leq 1$, $|b-c| \leq 1$, $|b-e| \leq 1$, $|c-0| \leq 1$, $|c-f| \leq 1$, $|d-a| \leq 1$ and so on are the 4 requirements at each of the vertices $a, \ldots, f$.

**Definition 1.** (Reg-Rect Islands) $M_m^\square \times n$ is the number of “fixed” Motzkin islands of this class with paths on a (projected) grid of squares and a rectangular shoreline of length $2(m+n)$.

Because there are $(m-1)(n-1)$ internal vertices, the number of islands with maximum peak height 1 is given by distributing ones and zeros on all nodes in all possible ways, and, similar to (1), the associated lower bound is

\[
M_m^\square \times n \geq 2^{(m-1)(n-1)}.
\]

For small widths of the islands, all points are close to shores, so no peaks of height $>1$ are possible, and

\[
M_0^\square \times n = 1;
\]

\[
M_m^\square \times n = 2^{(m-1)(n-1)}, \quad 1 \leq m \leq 3, \, n \geq 1.
\]

Table 2 shows numeric results for $M_m^\square \times n$.

Glueing a Motzkin Path of length $m$ at the right of a Motzkin Island to get an array of $n+1$ columns with a finite set of paths that are compatible with the altitude constraint shows that the Transfer Matrix method [6] is applicable to relate $M_m^\square \times n$ for fixed $m$, so linear recurrences with constant coefficients arise along each row and each column of Table 2.

The 4th column obeys a 3rd order linear recurrence obtained by inverting a $9 \times 9$ transfer matrix:

\[
M_4^\square \times n = 9M_{4 \times (n-1)}^\square - 4M_{4 \times (n-2)}^\square - 16M_{4 \times (n-3)}^\square, \quad n \geq 4,
\]

with generating function

\[
\sum_{n \geq 0} M_4^\square \times n x^n = 1 + x \frac{1 - x - 4x^2}{1 - 9x + 4x^2 + 16x^3}.
\]
The order of the transfer matrix for the $m \times n$ shape is the $m$-th Motzkin number [5, A001006]. Because the counts are symmetric, $M_{m \times n} = M_{n \times m}$, so only one triangular part of it needs to be shown. The columns $m = 2, 3$ are entirely characterized by (5) and cut short to save space.

**Definition 2.** $M_{m \times n} (x) = \sum_{n \geq 0} M_{m \times n} x^n$ are the (rational) generating functions for Reg-Rect Islands of fixed width $m$.

The Motzkin path of $n$ steps through the centre axis of the $4 \times n$ island is a word of length $n - 1$ of the alphabet \{0, 1, 2\} avoiding the patterns 02 and 20.

The growth of the sequences can commonly be estimated with the Binet formulas from the smallest roots (in absolute value) of the denominator polynomials of the generating functions:[8, §5.2]

\begin{equation}
1 + x \frac{1 - x - 4x^2}{1 - 9x + 4x^2 + 16x^3} \approx 3 \frac{0.04303 + 0.0565}{4} + \frac{0.55768}{x + 0.928} + \frac{0.12071}{1 - x}
\end{equation}

**Table 2.** The number of Motzkin Rectangular Islands $M_{m \times n}$.

The array is symmetric, $M_{m \times n} = M_{n \times m}$, so only one triangular part of it needs to be shown. The columns $m = 2, 3$ are entirely characterized by (5) and cut short to save space.
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\[ (9) \quad M_{4 \times n} \propto \frac{0.0135}{0.12071^{1+n}} \approx 0.0135 \times 8.284^{n+1}. \]

The 5th column obeys a 5th order linear recurrence obtained by inverting a
21 \times 21 transfer matrix:
\[ M_{5 \times n}^{\square} = 21M_{5 \times (n-1)}^{\square} - 52M_{5 \times (n-2)}^{\square} - 184M_{5 \times (n-3)}^{\square} + 32M_{5 \times (n-4)}^{\square} + 128M_{5 \times (n-5)}^{\square}, \quad n \geq 6, \]
with generating function
\[ \sum_{n \geq 0} M_{5 \times n}^{\square} x^n = \frac{1-x}{1-5x-28x^2+8x^3+32x^4}. \]

\[ (10) \quad M_{5 \times n}^{\square} \propto \frac{0.0027}{0.05741^{1+n}} \approx 0.0027 \times 17.415^{n+1}. \]

The 6th column obeys a 13th order linear recurrence obtained by inverting a
51 \times 51 transfer matrix The generating function is
\[ \sum_{n \geq 0} M_{6 \times n}^{\square} x^n = 1 + x \frac{p_{6,3}(x)}{q_{6,3}(x)} = 1 + x + 32x^2 + 1024x^3 + 36176x^4 + 1312656x^5 + 48185392x^6 + \cdots \]
where
\[ p_{6,3} \equiv 1 - 17x - 185x^2 + 1339x^3 - 7130x^4 - 32536x^5 - 61584x^6 + 186080x^7 + 97536x^8 - 298496x^9 + 43008x^{10} + 98304x^{11} - 32768x^{12}; \]
\[ q_{6,3} \equiv 1 - 49x + 359x^2 + 3851x^3 - 23750x^4 - 68392x^5 + 321168x^6 + 352480x^7 - 1284352x^8 - 401408x^9 + 1615872x^{10} - 393216x^{11} + 131072x^{12}. \]

The 7th column obeys a 25 order linear recurrence obtained by inverting a 127 \times 127 transfer matrix. The generating function is
\[ \sum_{n \geq 0} M_{7 \times n}^{\square} x^n = 1 + x \frac{p_{7,3}(x)}{q_{7,3}(x)} \]
where
\[ p_{7,3} \equiv 1 - 57x - 434x^2 + 30928x^3 + 52707x^4 - 5195215x^5 - 2295186x^6 + 33047942x^7 - 86356656x^8 - 953114448x^9 + 7586615232x^{10} + 131598114560x^{11} - 146226275584x^{12} - 901632193024x^{13} + 1173476425728x^{14} + 3008236953600x^{15} - 4337494769664x^{16} - 4485232918528x^{17} + 7302221398016x^{18} + 2352974135296x^{19} - 5122050490368x^{20} - 7225914880x^{21} + 1221716869120x^{22} - 30870077440x^{23} - 90194313216x^{24}; \]
$q_{7,3} \equiv 1 - 121x + 3214x^2 - 21184x^3 - 1052701x^4 - 188559x^5 + 111757902x^6 - 91039024x^7$
\[+ 5161897360x^8 + 5669119488x^9 + 115099761600x^{10} - 14600847232x^{11} - 1294977514752x^{12} + 20396717817856x^{16} + 31750174605312x^{17} + 25819486879744x^{18} - 44558677180416x^{19} - 11796274479104x^{20} + 27229555785728x^{21} + 110528299008x^{22} - 5667209347072x^{23} + 123480309760x^{24} + 360777252864x^{25}.

3.2. Reduction to Island without straight canals. A Motzkin Path may return multiple times to the horizontal line, and the Motzking Ballot numbers [5, A091836] count how Motzking Paths can be described as left-to-right compositions of “atomic” Motzkin Paths without intermediate returns to the horizontal.

In the same style, a list of Rectangular Motzkin Islands of size $m \times n$, $m \times n'$, $m \times n''$ etc can be chained to form an island of size $m \times (n + n' + n'' + \cdots)$ by mergers of eastern and western shorelines. The composite island has straight “canals” of length $m$ at sea level at the places of these mergers.

```
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 a b . . . 0 0 A B . 0 0 a b . . . 0 A B . 0
0 . . . . . 0 <-> 0 . . . 0 => 0 . . . . . 0 . . . 0
0 . . . . . 0 0 . . . 0 0 . . . . . 0 . . . 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
```

^canal

Definition 3. $M_{m \times n}^{\square(i)}$ is the number of Reg-Rect Islands of shape $m \times n$ which are composed of $i$ atomic Reg-Rect Islands (i.e., islands without straight canals of length $m$). The associated (rational) generating function is $M_{m \times n}^{\square(i)}(x) \equiv \sum_{n \geq 0} M_{m \times n}^{\square(i)}(x)^n$. The inverse INVERT transformation [1][4, Th. I.1] decomposes an integer sequence into the number of atomic parts, and the $i$th power of the generating function of the atomic parts provides the number of composite islands:

\[
M_{m \times n}^{\square(1)}(x) = 1 - \frac{1}{M_{m \times n}^{\square}(x)};
\]

\[
M_{m \times n}^{\square(i)}(x) = (M_{m \times n}^{\square(1)}(x))^i.
\]

A consequence of these formulas: because the $M_{m \times n}^{\square}$ have rational generating functions, the $M_{m \times n}^{\square(i)}$ also have rational generating functions.

Example 1. The case $m = 3$, where $M_{m}^{\square}$ are powers of 4, has an inverse INVERT transformation which are the powers of 3, so we arrive that the table of $m \times n$ Reg-Rect islands composed of $i$ smaller islands [5, A027465]: Table 3.

Example 2. The $4 \times n$ Reg-Rect Motzkin Islands which are mergers of smaller $4 \times n'$ Islands without canals are counted in Table 4.

3.3. Statistics for Maximum Peak altitudes. Refining the counts of Table 2 by the maximum altitude $h$ in the islands is simply a matter of leaving out the Motzkin Paths from the transfer matrices which climb too high. The reduction of the order of the transfer matrix by deleting entries with a given height from the table is tabulated in the Online Encyclopedia of Integer Sequences (OEIS) [5,
Table 3. The number of $M_{3\times n}^{(i)}$ islands that consist of $i$ atomic islands that have no straight canals of length 3.

<table>
<thead>
<tr>
<th>$n\setminus i$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>$\sum_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
<td>6</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td>16</td>
</tr>
<tr>
<td>4</td>
<td>27</td>
<td>27</td>
<td>9</td>
<td>1</td>
<td></td>
<td></td>
<td>64</td>
</tr>
<tr>
<td>5</td>
<td>81</td>
<td>108</td>
<td>54</td>
<td>12</td>
<td>1</td>
<td></td>
<td>256</td>
</tr>
<tr>
<td>6</td>
<td>243</td>
<td>405</td>
<td>270</td>
<td>90</td>
<td>15</td>
<td>1</td>
<td>1024</td>
</tr>
</tbody>
</table>

Table 4. The number of $M_{4\times n}^{(i)}$ islands that consist of $i$ atomic $M_{4\times n'}$ islands.

<table>
<thead>
<tr>
<th>$n\setminus i$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>$\sum_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>7</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>49</td>
<td>14</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>64</td>
</tr>
<tr>
<td>4</td>
<td>359</td>
<td>147</td>
<td>21</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>528</td>
</tr>
<tr>
<td>5</td>
<td>2641</td>
<td>1404</td>
<td>294</td>
<td>28</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td>4368</td>
</tr>
<tr>
<td>6</td>
<td>19463</td>
<td>12709</td>
<td>3478</td>
<td>490</td>
<td>35</td>
<td>1</td>
<td></td>
<td></td>
<td>36176</td>
</tr>
<tr>
<td>7</td>
<td>143473</td>
<td>111082</td>
<td>37407</td>
<td>6924</td>
<td>735</td>
<td>42</td>
<td>1</td>
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<td>299664</td>
</tr>
<tr>
<td>8</td>
<td>1057703</td>
<td>947127</td>
<td>378051</td>
<td>86339</td>
<td>1029</td>
<td>49</td>
<td>1</td>
<td></td>
<td>2482384</td>
</tr>
</tbody>
</table>

A097862]. The generic result is that the $M_{m\times n}$ that reach at most an altitude $h$ have rational generating functions. Looking at a straight Motzkin Path of $m$ steps reveals that the maximum altitude is $h \leq \lceil m/2 \rceil$, $h \leq \lfloor n/2 \rfloor$.

For maximum altitude $h = 1$ the outcome is given by Eq. (5). The generating function of these Reg-Rect Islands is

\begin{equation}
M_{m\times h\leq 1}^{(i)}(x) = 1 + \sum_{n \geq 1} 2^{(m-1)(n-1)} x^n = 1 + 2^{m-1} \frac{1}{1 - 2x}, \quad m > 0.
\end{equation}

Subtracting this from Eq. (7) gives the $4 \times n$ Reg-Rect Islands with maximum altitude 2; subtracting this from Eq. (11) gives the $5 \times n$ Reg-Rect Islands with maximum altitude 2.

The first slightly more complicated result occurs at $m = 6$ and $m = 7$, where altitudes of $h = 3$ are possible. Inverting a $50 \times 50$ transfer matrix we obtain

\begin{equation}
M_{6\times h\leq 2}^{(i)}(x) = 1 + x \frac{p_{6,2}(x)}{q_{6,2}(x)} = 1 + x + 32x^2 + 1024x^3 + 36176x^4 + 1312656x^5 + 48175392x^6 + \cdots
\end{equation}

where

\begin{align*}
p_{6,2} &= 1 - 16x - 192x^2 + 1084x^3 + 7144x^4 - 22816x^5 - 64704x^6 + 95616x^7 + 139264x^8 - 114688x^9 - 49152x^{10} + 32768x^{11},
\end{align*}
\[ q_{6,2} = 1 - 48x + 320x^2 + 3820x^3 - 18984x^4 - 71232x^5 + 214592x^6 - 505056x^7 - 655872x^8 - 759808x^9 + 606208x^{10} + 196608x^{11} - 131072x^{12}. \]

The results for maximum altitudes exactly equal to some \( h \) are derived from these intermediate results with the inclusion-exclusion principle.

**Example 3.** Eq. (18) determines \( M_{6 \times h \leq 2}^\square(x) \) and (13) determines \( M_{6 \times h \leq 3}^\square(x) \):

\[ (19) \quad M_{6 \times h=3}(x) = M_{6 \times h \leq 3}(x) - M_{6 \times h \leq 2}(x) = 10'000x^6 + 103'64'000x^7 + 69'5'42'224x^8 + 38'5'34'6'9'7'1'2x^9 + \cdots \]

**Example 4.** Inverting a 120 \( \times \) 120 transfer matrix yields \( M_{7 \times h \leq 2}^\square(x) \) and

\[ (20) \quad M_{7 \times h=3}(x) = M_{7 \times h \leq 3}(x) - M_{7 \times h \leq 2}(x) = 103'64'000x^6 + 24'8'19'7'6'8'o x^7 + 37'6'5'5'8'9'5'5'6'8x^8 + 46'4'3'1'6'2'7'3'8'2'5'6x^9 + \cdots \]

The order 120 of the transfer matrix is the order 127 without constraint that governs Eq. (14) minus the 7 Motzkin paths of length 7 and height 3 [5, A097862].

One may also mix the results of this chapter and the previous one, looking at the Reg-Rect Islands with some maximum \( h \) that have no straight canals of length \( m \). The basic ansatz is that islands which have altitudes not larger than \( H \) can be decomposed into atomic islands of altitudes not larger than \( H \):

\[ (21) \quad M_{m \times n, h \leq H}^\square = \sum_i M_{m \times n, h \leq H}^{\square(i)}. \]

Equations (15)–(16) for the associated generating functions remain valid if the constraint \( h \leq H \) is added (and leads to another family of rational generating functions).

4. **Regular Parallelogram Shoreline**

If the underlying grid is the triangular grid with 6 neighbors adjacent to each internal vertex, the shoreline may be a parallelogram with 2 sides of length \( m \) and 2 sides of length \( n \). The example of a \( M_{5 \times 5}^\Diamond \) distribution of vertices looks like this:

```
.---.---.---.---.---.
/ \ / \ / \ / \ / \ / 
.---.---.---.---.---.
/ \ / \ / \ / \ / \ / 
.---.---.---.---.---.
/ \ / \ / \ / \ / \ / 
.---.---.---.---.---.
```

The number of internal vertices is \( (n - 1)(m - 1) \) as in the case of the square lattice, and basically every aspect related to the availability of transfer matrix algorithms and canals (they are slanted now) remains intact.

The lower bound obtained for islands with maximum height of 1 is again 2 to the power of the number of internal vertices,

\[ M_{m \times n}^\Diamond \leq 2^{(m-1)(n-1)}, \quad n \geq 1. \]
The basic counting numbers are in Table 5. The columns \( m \leq 3 \) are the regular powers of 2. Further generating functions are

\[
\sum_{n \geq 0} M_{4 \times n}^\Diamond x^n = 1 + x \frac{1 - x - x^2}{1 - 9x + 7x^2 + 4x^3},
\]

\[
\sum_{n \geq 0} M_{5 \times n}^\Diamond x^n = 1 + x \frac{(1 - x)(1 - 4x - 8x^2 + 3x^3)}{1 - 21x + 76x^2 + 7x^3 - 99x^4 - 44x^5 + 16x^6},
\]

\[
\sum_{n \geq 0} M_{6 \times n}^\Diamond x^n = 1 + x \frac{r_{6,3}(x)}{s_{6,3}(x)},
\]

with

\[
r_{6,3} = 1 - 19x + 53x^2 + 286x^3 - 1266x^4 - 200x^5 - 348x^6 + 12112x^7 - 6896x^8 - 6752x^9 + 384x^{10} + 512x^{11};
\]

\[
s_{6,3} = (1 - 47x + 481x^2 - 702x^3 - 1866x^4 + 1268x^5 + 7228x^6 - 3616x^7 - 4064x^8 - 64x^9 + 256x^{10})
\times (1 - 4x - 8x^2).
\]

\[
\sum_{n \geq 0} M_{7 \times n}^\Diamond x^n = 1 + x \frac{r_{7,3}(x)}{s_{7,3}(x)},
\]

with

\[
r_{7,3} = -1 + 63x - 1143x^2 + 364x^3 + 35019x^4 - 768871x^5 + 960584x^6 + 327136x^7
\]
\[+ 61740672x^8 - 441671236x^9 + 1577898892x^{10} - 1632794220x^{11} - 6633776796x^{12}
\]
\[+ 11128738368x^{13} + 17012212988x^{14} - 16432634428x^{15} - 27719932816x^{16} + 551175824x^{17}
\]
\[+ 23451095152x^{18} + 12993764672x^{19} - 7682412560x^{20} - 7410479680x^{21}
\]
\[- 22526464x^{22} + 1183185920x^{23} + 245771264x^{24} - 18113536x^{25}
\]
\[- 7204864x^{26} + 32768x^{27} + 65536x^{28};
\]

\[
s_{7,3} = -1 + 127x - 5175x^2 + 85916x^3 - 503349x^4 - 1111931x^5 + 17547656x^6 + 18640380x^7
\]
\[- 390222952x^8 - 288723512x^9 + 5057701996x^{10} + 3601279348x^{11}
\]
\[- 3624480044x^{12} - 26797173296x^{13} + 116223283500x^{14} + 137289708452x^{15}
\]
\[- 129459816960x^{16} - 288061565440x^{17} - 64508507216x^{18} + 197636554032x^{19}
\]
\[+ 162786560752x^{20} - 7310842880x^{21} - 49814909440x^{22} - 8841032192x^{23}
\]
\[+ 4239736832x^{24} + 1174548480x^{25} - 50548736x^{26}
\]
\[- 28753920x^{27} + 131072x^{28} + 262144x^{29}.
\]

Remark 1. The heuristics \( M_{m \times n}^\Diamond \leq M_{m \times n}^\Box \) is a reduction of counts caused by the additional constraint of having 2 more neighbors with restricted slope in the triangular grid than in the square grid.
Table 5. The number of Motzkin Parallelogram Islands $M^\diamondsuit_{m \times n}$.

The array is symmetric, $M^\diamondsuit_{m \times n} = M^\diamondsuit_{n \times m}$, so only one triangular part of it needs to be shown. The columns $m = 2, 3$ are entirely characterized by (22) and cut short to save space.

### 5. Regular Rhomboid Shoreline

#### 5.1. Subclassing Parallelograms

The shoreline with the lozenge/rhomboid shape of $n$ edges on each of the 4 sides defines Reg-Rhomb Motzkin Islands $M^\diamondsuit_{n \times n}$. They are a special case of the Parallelogram Islands for equilateral sides: There are $(n+1)^2$ vertices, of which $4n$ are on the shoreline and $(n-1)^2$ inside the island. The islands for $n = 0, 1, 2$ and 4 of the 16 islands for $n = 3$ have altitudes like this:

```
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
```

From the diagonal of Table 5 we extract

\begin{equation}
M^\diamondsuit_{n \times n} = 1, 1, 2, 16, 516, 67972, 36817540, \ldots \quad (n \geq 0).
\end{equation}
Glueing two Reg-Triangular islands such that the two shorelines overlap creates a sub-class of Reg-Rhomb islands with a canal across the short diagonal of the rhomboid, so these yield the lower bound

\[ M_{n \times n}^\diamond \geq [M^q(n)]^2. \]

5.2. **Hand-counted Examples.** The first case where the lower bound (22) is surpassed are the 4 islands with \( n = 4 \) and maximum height 2, \( M_{4 \times 4}^\diamond = 2^{(4-1)^2} + 4 = 516 \), which have the altitude map

\[
\begin{array}{cccccccccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

where \( a \) and \( b \) are one of the \( 2^2 = 4 \) combinations of zeros and ones.

For \( n = 5 \) the islands with maximum height 2 have one of the following maps:

- **A single 2 at one of the 4 inner vertices:**
  \[
  \begin{array}{cccccccccccccccccccc}
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 1 & 1 & a & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 1 & 1 & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 1 & 1 & e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & f & g & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  \end{array}
  \]
  where \( a-i \) are 9 vertices supporting \( 2^9 = 512 \) combinations of zeros and ones. The 4 places of the peak altitude give \( 4 \times 512 = 2048 \) islands.

- **Two 2’s parallel to sides:**
  \[
  \begin{array}{cccccccccccccccccccc}
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 1 & 1 & a & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 1 & 1 & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 1 & 1 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & e & f & g & h & i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  \end{array}
  \]
  where \( a-f \) are 6 vertices supporting \( 2^6 = 64 \) combinations of zeros and ones. The 4 places of the peak altitude specify \( 4 \times 64 = 256 \) islands.

- **Two 2’s on the short or long diagonal:**
  \[
  \begin{array}{cccccccccccccccccccc}
  0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & a & 1 & 1 & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 1 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & c & d & e & f & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  \end{array}
  \]
  where \( a-f \) are vertices with combinations of zeros and ones. This specifies \( 2^4 + 2^6 = 80 \) islands.

- **Three 2’s:**
Table 6. Reg-Rhomb Islands $M_{n \times n}^{h}$ with maximum altitude $h$. 
The sum of the columns $h = 0$ and $h = 1$ is (22). Row sums are (27).

<table>
<thead>
<tr>
<th>$n \backslash h$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>$\Sigma h \geq 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>15</td>
<td></td>
<td></td>
<td>16</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>511</td>
<td>4</td>
<td></td>
<td>516</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>65535</td>
<td>2436</td>
<td></td>
<td>67972</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>33554431</td>
<td>3263008</td>
<td>100</td>
<td>36817540</td>
</tr>
</tbody>
</table>

The total $M_{5 \times 5}^{h}$ with maximum height 2 is $2048 + 256 + 80 + 48 + 4 = 2436$, entry $(n = 5, h = 2)$ in Table 6.

The $M_{6 \times 6}^{h}$ with maximum height 3 have the altitude maps

\[
\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & a & 0 & 0 \\
0 & 1 & 2 & 2 & 1 & 0 & 0 \\
0 & 1 & 2 & 1 & 1 & 0 & 0 \\
0 & b & 1 & 1 & c & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
\]

where $a$–$d$ are vertices with combinations of zeros and ones. This specifies $2^3 + 2^4 + 2^3 + 2^4 = 48$ islands.

- Four 2’s
  \[
  \begin{array}{ccccccc}
  0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 1 & 1 & 1 & a & 0 & 0 \\
  0 & 1 & 2 & 2 & 1 & 0 & 0 \\
  0 & 1 & 2 & 1 & 1 & 0 & 0 \\
  0 & b & 1 & 1 & c & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & 0
  \end{array}
  \]
  where $a$–$b$ are vertices with combinations of zeros and ones. This specifies $2^2 = 4$ islands.

This leads to $4 + 64 + 2 \times 16 = 100$ islands, entry $(n = 6, h = 3)$ in Table 6.
6. Summary

We have introduced Motzkin Islands as finite vertex-labeled graphs defined on meshes with a perimeter describing a coastline, considered in particular the triangular and simple-square meshes and pointed out that the generating functions for simple shapes of the coast can be calculated with the transfer matrix method.

References


URL: https://www.mpia.de/~mathar

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