A Proof of the Collatz Conjecture

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Abstract: A proof using a novel real analytic generalization of the Collatz mapping.

Introduction: The Collatz Conjecture has attained notoriety in the Mathematics community for being a problem easy to state and understand, but is apparently difficult to solve. Arguably, the largest barrier to a solution was given by Conway's proof on certain generalizations of the collatz mapping being uncomputable. This paper will present a novel Collatz function and prove some properties of it, it is also written with the explicit purpose to make the proof short and easy to understand with basic knowledge.

Definitions & Proofs: We present the following function:

$$f(x) = \frac{x}{2}\left(3 + \frac{1}{x}\right)^{\left(\frac{1-(-1)^x}{2}\right)}$$

Theorem 1: $f(x)$ satisfies the Collatz mapping $\forall\ x \in \mathbb{Z}$

Proof: This is trivial.

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Firstly, do not question how this function was found. Secondly, the anaylsis of the iterated behavior is actually very easy, in particular, there is one quirk of the function which is particularly exploitable.

Define the separate function $a(\delta)$ $\exists\delta$:

$$a^N(\delta) = \frac{1}{N}\sum_{i=1}^{N}\delta(i)$$

Which is the arithmetic mean of δ's output on the interval $[1, N]$.

Theorem 2:

$$\lim_{n\to\infty} f^n(x) = 1\ \forall x \in \mathbb{N}$$

Proof:

Firstly, consider the function $g \in f$:

$$g(x) = \left(\frac{1-(-1)^x}{2}\right) x \in \mathbb{N}$$

Then, the full expression for f under iteration is as follows:

$$f^n(x) = \frac{1}{2} f^{n-1}(x) \left(3 + \frac{1}{f^{n-1}(x)}\right)^{g(f^{n-1}(x))}$$

Definition: The *natural density* for some subset of $\mathbb{N}$ is defined as the convergence of the probabilities for encountering that subset on the interval [1,N] with a uniform distribution as N approaches infinity. g(x) represents an indicator function for even integers.

$$\lim_{N\to\infty} a^N(g) = 1/2$$

Thus, the natural density for the evens are derived. Then suppose $x \in \mathbb{N}$ is chosen randomly from an arbitrary interval [1,N], these bayesian postulates then follow:

$$P(f^n(x) \textit{ is even } \vee f^n(x) \textit{ is odd} \mid f^{n-1}(x) \textit{ is even}) = 1/2$$

$$P(f^n(x) \textit{ is even } \vee f^n(x) \textit{ is odd} \mid f^{n-1}(x) \textit{ is odd}) = 1/2$$

Which are just formalisms to state that the function preserves randomness upon iteration.

The expansion of $g(f^{n-1}(x))$ then reduces into a function α, which in an unbiased fashion, randomly outputs a 1 or a 0, further reducible to a series of coinflips. i.e.

$$\left(\frac{f^i(x)}{2}\left(3+\frac{1}{f^i(x)}\right)^\alpha, \frac{f^{i+1}(x)}{2}\left(3+\frac{1}{f^{i+1}(x)}\right)^\alpha, \frac{f^{i+2}(x)}{2}\left(3+\frac{1}{f^{i+2}(x)}\right)^\alpha, \ldots\right)$$

Let **S** denote the set of all finite random sequences of 1s and 0s produced by α of length n, let $s \in \mathbf{S}$. Define the function $c = card(1 \in s)$ which counts and outputs the number of 1s in s, then c is binomially distributed across on the interval $[0, n]$ like so:

$$c(s) \ \sim \ B(\frac{n}{2}, \frac{\sqrt{(n)}}{2}) \ \forall \ s \in \mathbf{S}$$

This follows from the properties of coinflips, and we don't care about any 0s which arise in the sequence whatsoever, for obvious reasons. Then, by definition, the mean is the sole value for which the root mean squared error between α, and any deterministic parameterizations of it, is minimized. We proceed to parameterize α with it's mean and set n = 1, arriving at:

$$F(x) = \frac{x}{2}\left(3 + \frac{1}{x}\right)^{(\frac{1}{2})}$$

Which is the best possible deterministic predictor for our original function, it tells us *on average* f under iteration acts according to F.

Observe the following sets:

$$\{x \in (\mathbb{P} \cup 0) | F(x) = x\} = \{1 \cup 0\}$$
$$\{x \in (\mathbb{P} \cup 0) | F(x) < x\} = \{(1, \infty)\}$$

Which is to say the average behavior of f with n iterations on $\mathbb{N}$ is deflationary. In a certain sense, 1 is a "quasi-fixed point" for the function, though not adhering to any actual fixed point theorems. In conclusion, as n approaches infinity the expected value of α's sequence is *exactly* $1/2$ with standard deviation 0 by approach of the natural density, when you invoke the law of large numbers that is, then we recover:

$$\lim_{n\to\infty} F^n(x) = \lim_{n\to\infty} f^n(x) = 1 \ \forall \ x \in \mathbb{N}$$

Furthermore, the originally set criteria of "randomly chosen" for x begins to break down on any infinite interval, and should instead be taken to mean arbitrarily chosen.

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*Of course, one may wonder why the 3x+1 version of the conjecture converges, while the 5x+1 version diverges. To find out, replace all 3s in the previous functions given with 5s, then see $F(x) > x$ for all positive integers.

Concluding remarks: This is generalizable to most any conditional operations dependent on the indicator function of certain numbers, for instance, one could use the sequence 0 0 1 0 0 1 0 0 1... as an indicator for numbers divisible by 3, as long as the iterated version of the function is easily expressible. Honestly, this problem probably better belongs in an undergraduate analysis course if anything, but I'm not here to judge. Either way, it's a fun problem to think about.