On the zeros of the Riemann zeta function

Jorma Jormakka

Contact by: jorma.o.jormakka@gmail.com

Abstract. The paper proves the Riemann Hypothesis.

Key words: Riemann zeta function, Riemann Hypothesis, complex analysis.

1 Introduction

The idea of this proof is to construct a function of two complex variables, \( \xi(s, z) \), such that \( \xi(s, z) = \varphi(s, z)^{-1}\zeta(s) \). Then this function has all zeros \( s_0 \) of the Riemann zeta function \( \zeta(s) \). The function \( \varphi(s, z) \) is constructed to be finite in a point \((s, z)\) where \( s + z = s_0 \) for some zero \( s_0 \) of \( \zeta(s) \) where \( z \) is small but nonzero. Then \( \xi(s, z) \) does not have a zero at \((s, z)\). Some derivative of \( \zeta(s) \) must be nonzero at \( s_0 \) as \( \zeta \) is not the zero function. We can express

\[
\frac{d}{ds}\zeta(s) = h(s)\zeta(s) + g(s)\zeta(s)
\]

where \( g(s) \) is convergent at \( s_0 \). Then \( h(s) \) must have a pole at \( s_0 \). The function \( \varphi(s, z) \) is so constructed that \( \xi(s, z) \) has the expression

\[
\frac{\partial}{\partial z}\xi(s, z) = h(s + z)\xi(s, z) + u(s, z)\xi(s, z)
\]

where \( u(s, z) \) is convergent at \((s, z)\) where \( s + z = s_0 \). The pole of \( h(s) \) at \( s_0 \) means that \( h(s + z) = h(s_0) \) has a pole. As \( \xi(s, z) \) is finite and not zero and \( h(s, z) \) is finite, the partial derivative

\[
\frac{\partial}{\partial z}\xi(s, z)
\]
must be infinite. This is not possible as $\xi(s, z)$ is finite.

We construct $\varphi(s, z)$ in Lemma 1, show that $\xi(s, z)$ is finite and nonzero at $(s, z)$, $s + z = s_0$, in Lemma 2, and conclude that $h(s + z) = h(s_0)$ cannot have a pole at $(s, z)$ in Lemma 3. Lemma 1 is obvious, Lemma 3 is a simple calculation, Theorem 1 is simply a consequence of what Riemann proved, thus there remains only the proof of Lemma 2. This proof is given in the third section of the paper.

The idea of the proof of Lemma 2 is the following. Assuming that $\zeta(s)$ has an $r$-order zero at $s_0$, we can derive the expression (15) for the function $\xi(s, z)$. This expression is compared to two equations that are obtained from the definition (4) of $\xi(s, z)$ as an infinite product. The first is equation (17) which shows that if $r > 0$, then $r = 1$. The second equation (21) gives a first-order linear differential equation that gives $\zeta(s)$ in the whole half-plane $Re\{s\} > \frac{1}{2}$ in case $r = 1$. If the differential equation is correct, then $\zeta(s)$ does not have a pole at $s = 1$, but the Riemann zeta function has this pole. Therefore $r \neq 1$. The remaining possibility is that $r = 0$ and $s_0$ is not a zero of $\zeta(s)$.

2 Definitions

The Riemann zeta function is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

where $s = x + iy$, $x, y \in \mathbb{R}$, is a complex number. The zeta function can be continued analytically to the whole complex plane except for $s = 1$ where the function has a pole. The function has trivial zeros at even negative integers. It does not have zeros for $x \geq 1$ and the only zeros for $x \leq 0$ are the trivial ones. The nontrivial zeros lie in the strip $0 < x < 1$, see e.g. [1].

Let

$$P = \{p_1, p_2, \ldots | p_j \text{ is a prime, } p_{j+1} > p_j > 1, j \geq 1\}$$
be the set of all primes (larger than one). Let \( s = x + iy, x, y \in \mathbb{R} \) and \( x > \frac{1}{2} \).

The Riemann zeta function can be expressed as

\[
\zeta(s) = \prod_{j=1}^{\infty} (1 - p_j^{-s})^{-1},
\]

This infinite product converges absolutely if \( x = \text{Re}\{s\} > 1 \).

3 Lemmas and the theorem

**Lemma 1.** Let \( \text{Re}\{s\} > 1, \text{Re}\{s + z\} > 1 \) and \( x = \text{Re}\{z\} \). We define the absolutely convergent products

\[
\varphi(s, z) = \prod_{j=1}^{\infty} \left( 1 - p_j^{-s} + p_j^{-s-z} \right)^{-1},
\]

\[
\xi(s, z) = \prod_{j=1}^{\infty} \left( \frac{1 - p_j^{-s}}{1 - p_j^{-s} + p_j^{-s-z}} \right)^{-1}.
\]

Then

\[
\zeta(s) = \varphi(s, z) \xi(s, z)
\]

\( \varphi(s, 0) = 1, \xi(s, 0) = \zeta(s), \)

\[
\lim_{x \to \infty} \varphi(s, z) = \zeta(s) \ , \lim_{x \to \infty} \xi(s, z) = 1.
\]

**Proof.** Directly multiplying the absolutely convergent products shows (5). At \( z = 0 \) the term \( 1 - p_j^{-z} = 0 \) and if \( x \to \infty \) then \( p_j^{-z} \to 0 \). The limits follow.

**Lemma 2.** Let us assume there exists a zero \( s_0 \) of \( \zeta(s) \) with \( 1 > \text{Re}\{s_0\} > \frac{1}{2} \). Then there exists \( \epsilon > 0 \) such that for some \( (s, z) \) with if \( 0 < |z| < \epsilon \) and \( s + z = s_0 \) the functions \( \xi(s, z) \) and \( \varphi(s, z) \) are analytic at the point \( (s, z) \) as a function of \( z \) and \( s \). The function \( \xi(s, z) \) is finite and nonzero at such a point \( (s, z) \) and the partial derivative of \( \xi(s, z) \) with respect to \( z \) is finite at such a point.

The proof of this lemma is in the following section.
**Lemma 3.** Let $\zeta(s_0) = 0$ for $s_0$ such that $\text{Re}\{s_0\} > \frac{1}{2}$. Then for $n \geq 1$ and $D = \frac{d}{ds}$

$$D^{(n)} \zeta(s_0) = 0.$$ 

**Proof.** From (3)

$$\frac{\partial}{\partial z} \ln \xi(s, z) = \sum_{j=1}^{\infty} (-\ln p_j)p_j^{-s-z}(1 - p_j^{-s} + p_j^{-s-z})^{-1}$$

$$= -\sum_{j=1}^{\infty} \ln(p_j) p_j^{-s-z} - \sum_{j=1}^{\infty} \ln(p_j) p_j^{-2s-z}(1 - p_j^{-z})(1 - p_j^{-s} + p_j^{-s-z})^{-1}$$

$$= h(s + z) + u(s, z)$$ \hspace{1cm} (6)

where

$$u(s, z) = -\sum_{j=1}^{\infty} \ln(p_j) p_j^{-2s-z}(1 - p_j^{-z})(1 - p_j^{-s} + p_j^{-s-z})^{-1}$$ \hspace{1cm} (7)

and

$$h(s) = -\sum_{j=1}^{\infty} \ln(p_j) p_j^{-s}.$$ \hspace{1cm} (8)

From (8) follows that $h(z)$ converges absolutely if $\text{Re}\{s\} > 1$.

If $\alpha > 0$ is selected and $j$ is sufficiently large

$$|\ln(p_j) p_j^{-2s-z}(1 - p_j^{-z})(1 - p_j^{-s} + p_j^{-s-z})^{-1}| < 4|p_j^{-2s-z+\alpha}|.$$ 

Thus, $u(s, z)$ converges absolutely if $\text{Re}\{s\} > \frac{1}{2}$ and $\text{Re}\{s + z\} > \frac{1}{2}$. For $\text{Re}\{s + z\} > 1$, $\text{Re}\{s\} > 1$ holds

$$\frac{\partial}{\partial z} \xi(s, z) = h(s + z) \xi(s, z) + u(s, z) \xi(s, z).$$ \hspace{1cm} (9)

In a similar way

$$D \zeta(s) = h(s) \zeta(s) + g(s) \zeta(s)$$ \hspace{1cm} (10)
where

\[ g(s) = -\sum_{j=1}^{\infty} \ln(p_j) \frac{p_j^{-2s}}{1 - p_j^{-s}}. \]

The function \( g(s) \) converges absolutely if \( \text{Re}\{s\} > \frac{1}{2} \) because

\[ |\ln(p_j) p_j^{-2s}(1 - p_j^{-s})^{-1}| < 2|p_j^{-2s+\alpha}| \]

for any fixed \( \alpha > 0 \) if \( j \) is sufficiently large. The term \( g(s)\zeta(s) \) is finite because \( \zeta(s) \) is finite and \( g(s) \) converges and is therefore finite.

We can continue \( h(s) \) analytically to \( 1 > \text{Re}\{s\} > \frac{1}{2} \) by (10), except to points where \( \zeta(s) = 0 \). At those points \( h(s) \) has a pole. By Lemma 2 the function \( \xi(s, z) \)
can be continued analytically to a neighbourhood of \( s_0 \) and it is finite and nonzero,
and the function \( \frac{\partial}{\partial z} \xi(s, z) \) is finite at \( (s, z) \) where \( s + z = s_0 \) and \(|z| < \epsilon\). The term \( u(s, z)\xi(s, z) \) is finite because \( \xi(s, z) \) is finite and \( u(s, z) \) converges and is therefore finite.

If the derivative \( D\zeta(s_0) \) is nonzero, then \( h(s) \) must have a pole at \( s_0 \). If so, then \( h(s + z) \) has a pole at any point where \( s_0 = s + z \). It follows from (10) that
the partial derivative \( \frac{\partial}{\partial z} \xi(s, z) \) must be infinite at such a point because the term \( u(s, z)\xi(s, z) \) is finite and \( \xi(s, z) \) is finite and nonzero. This is not possible because \( \xi(s, z) \) is finite and therefore its partial derivative with respect to \( z \) is finite. It follows that \( h(s) \) does not have a pole at \( s = s_0 \) and therefore \( D\zeta(s_0) = 0 \).

Let us assume \( D^{(j)}\zeta(s_0) = 0 \) is shown by induction for \( j \leq n \). The initial step \( n = 1 \) is proven above. The convergent part \( g(s) \) and all its derivatives are
finite at \( s = s_0 \) and vanish when multiplied by a zero. As \( h(s_0) \) is finite, all its
derivatives must also be finite at \( s = s_0 \). At \( s = s_0 \) the function \( h(s) \) and all its
derivatives vanish when multiplied by a zero. Because of the induction assumption
\( D^{(j)}\zeta(s_0) = 0 \) for \( j \leq n \). All terms of the \((n+1)\)th derivative of \( \zeta(s) \) vanish when
the terms are expanded: \( D^{(j)}(g(s)\zeta(s))_{s=s_0} = 0 \) and \( D^{(j)}(h(s)\zeta(s))_{s=s_0} = 0 \) if
\( j \leq n \). Therefore \( D^{(n+1)}\zeta(s)_{s=s_0} = 0 \). The claim follows by induction on \( n \). \( \blacksquare \)
Theorem 1. If \( \zeta(s) = 0 \) and \(-1 < \Re\{s\} < 1\) then \( \Re\{s\} = \frac{1}{2} \).

Proof. Riemann showed that

\[
2^{1-s} \Gamma(s) \zeta(s) \cos \left( \frac{1}{2} s \pi \right) = \pi^s \zeta(1-s).
\]

Thus, if there exists a zero \( s_0 = x_0 + iy_0 \) of \( \zeta(s) \) with \( 0 < x_0 < \frac{1}{2} \) then there exists a zero of \( \zeta(s) \) at a symmetric point in \( \frac{1}{2} < x < 1 \). Therefore we only need to look at the strip \( \frac{1}{2} < \Re\{s\} < 1 \). If there exists \( s_0 \) with \( \zeta(s_0) = 0 \) and \( \frac{1}{2} < \Re\{s\} < 1 \), then by Lemma 3 all derivatives of \( \zeta(s) \) are zero at \( s_0 \). As \( \zeta(s) \) is analytic if \( s \neq 1 \), it has a converging Taylor series in some neighborhood of every point \( s \neq 1 \). Since \( \zeta(s_0) = 0 \) and all derivatives vanish at the point \( s_0 \), the function \( \zeta(s) = 0 \) in an open neighborhood and can be continued as zero everywhere, which is a contradiction. Therefore no such \( s_0 \) exists. \( \blacksquare \)

4 The proof of Lemma 2

We have to prove in Lemma 2 that \( \varphi(s, z) \) is finite and nonzero at some point \((s, z)\) where \( s + z = s_0 \) and \( s_0 \) is (a potential) zero of \( \zeta(s) \). It is clear from the definition that \( \varphi(s, 0) = 1 \), but it could be that \( \varphi(s, z) \) is infinite for any \( z \neq 0 \) in the chosen small environment of \( |z| < \epsilon, s + z = s_0 \). An example of such a function is

\[
\varphi(s, z) = (s - s_0)^r (s + z - s_0)^{-r} f_0(s, z), \quad r > 0
\]

where \( f_0(s, z) \) is some function that is finite and nonzero close to \( s_0 \). It turns out that if \( \zeta(s) \) has a zero at \( s_0 \) then this is the form we get to the function \( \varphi(s, z) \). The goal is to prove that \( r \) must be zero for \( \varphi(s, z) \) defined by the infinite product expression (3).

Firstly, we continue \( \varphi(s, z) \) and \( \xi(s, z) \) analytically to the strip \( \frac{1}{2} < \Re\{s\} < 1 \), \( \frac{1}{2} < \Re\{s + z\} < 1 \).
In Lemma 3 we defined $h(s)$ that has the power series representation (8). It is continued analytically to $\frac{1}{2} < \text{Re}\{s\}$ by (10) as

$$h(s) = \zeta(s)^{-i} \frac{d\zeta(s)}{ds} - g(s).$$

The function $g(s)$ is analytic (and therefore finite) when $\frac{1}{2} < \text{Re}\{s\}$. The function $\zeta(s)$ is analytic everywhere except for $s = 1$. The right side is defined and analytic for $\frac{1}{2} < \text{Re}\{s\}$ except for in points where $\zeta(s)$ has a zero or is infinite. In those points $h(s)$ has a pole. Consequently, all derivatives of $h(s)$ are analytic in $\frac{1}{2} < \text{Re}\{s\}$ with the exception of isolated poles (where $\zeta(s)$ is zero or infinite). We are interested only in a small environment $|z| < \epsilon$. In this area there is only one potential zero of $\zeta(s)$ at $s_0$ and $\zeta(s)$ is finite.

From (3) we get for $\text{Re}\{s\} > 1$ the equation

$$\frac{\partial}{\partial s} \ln \varphi(s, z) = -\sum_{j=1}^{\infty} \frac{\partial}{\partial s} \ln (1 - p_j^{-s} + p_j^{-s-z})$$

$$= -\sum_{j=1}^{\infty} \ln (p_j^{-s} - p_j^{-s-z})(1 - p_j^{-s} + p_j^{-s-z})^{-1}$$

$$= -\sum_{j=1}^{\infty} \ln (p_j^{-s} - p_j^{-s-z}) + f_2(s, z)$$

where

$$f_2(s, z) = -\sum_{j=1}^{\infty} \ln (p_j^{-s} - p_j^{-s-z})(1 - p_j^{-s}) \sum_{n=1}^{\infty} (p_j^{-s} - p_j^{-s-z})^n$$

converges absolutely and is an analytic (and therefore finite) function if $\text{Re}\{s\} > \frac{1}{2}, \text{Re}\{s+z\} > \frac{1}{2}$. Replacing the power series expressions of $h(s)$ with $h(s)$ we can continue the equation analytically to the strip $\frac{1}{2} < \text{Re}\{s\} < 1, \frac{1}{2} < \text{Re}\{s+z\} < 1$:

$$\frac{\partial}{\partial s} \ln \varphi(s, z) = h(s) - h(s + z) + f_2(s, z)$$

(12)
Equation (12) can be integrated over $s$ and it gives $\ln \varphi(s, z)$ in all points where $h(s)$ and $h(s + z)$ are analytic. The problem in Lemma 2 is that we are interested in points where $s + z = s_0$. Clearly, if $s + z = s_0$ is a zero of $\zeta(s)$, then $h(s + z)$ is infinite and $\varphi(s, z)$ is infinite if $z \neq 0$. We have to prove that this is not the case.

Equation (5) $\xi(s, z) = \varphi(s, z)^{-1} \zeta(s)$ gives an analytical continuation of $\xi(s, z)$ as a function of $s$ to the strip strip $\frac{1}{2} < \text{Re}\{s\} < 1$, $\frac{1}{2} < \text{Re}\{s + z\} < 1$. The function $\xi(s, z)$ is analytic in points where $\varphi(s, z)$ is nonzero, and $\xi(s, z)$ is nonzero when $\varphi(s, z)$ is finite. If $\xi(s, z)$ is finite and nonzero, then the partial derivative of $\xi(s, z)$ with respect to $z$ is finite at $(s, z)$ and Lemma 2 is proven. However, we have to show that $\varphi(s, z)$ is not a function of the type (11).

This particular form (11) of $\varphi(s, z)$ we get if $\zeta(s_0) = 0$. This is because then the Taylor series of zeta is

$$\zeta(s) = C(s - s_0)^r + (s - s_0)^{r+1}A(s)$$

(13)

where $A(s)$ has only nonzero powers of $(s - s_0)$ and $r > 0$. The function $h(s)$ contains the divergent part of $\zeta'(s)/\zeta(s)$, thus

$$h(s) = \frac{r}{s - s_0} + f_1(s)$$

(14)

where $f_1(s)$ is analytic in a small environment of $s_0$. Function $h(s)$ has always the form (14). It cannot have higher order poles. If $\zeta(s)$ has a pole or order $k$, then $r = -k$ in (14). The Riemann zeta function $\zeta(s)$ has only one pole, a simple pole at $s = 1$. Function $f_1(s)$ is defined by $h(s)$ in the half-plane $\text{Re}\{s\} > \frac{1}{2}$, but it does have at least one pole: because $\zeta(s)$ has a pole at $s = 1$, function $f_1(s)$ in (14) has a simple pole at $s = 1$.

Inserting (14) to (12) we can write

$$\frac{\partial}{\partial s} \ln \varphi(s, z) = -r(s + z - s_0)^{-1} + r(s - s_0)^{-1} + f_3(s, z)$$
where

\[ f_3(s, z) = f_1(s) - f_1(s + z) + f_2(s, z) \]

is defined in the strip \( \frac{1}{2} < \text{Re}\{s\} < 1, \frac{1}{2} < \text{Re}\{s\} < 1 \) and is analytic in a small neighborhood of \( s_0 \) but has poles, at least at \( s = 1 \). Then

\[
\ln \varphi(s, z) = -r \ln(s + z - s_0) + r \ln(s - s_0) + \int f_3(s, z) \, ds
\]

and we have the form (11):

\[
\varphi(s, z) = \left( \frac{s - s_0}{s + z - s_0} \right)^r e^{\int f_3(s, z) \, ds}.
\]

Because \( \varphi(s, 0) = 1 \) for every \( s \), holds \( f_3(s, 0) = 0 \).

By (5) the function \( \xi(s, z) \) has the form

\[
\xi(s, z) = (C(s + z - s_0)^r + (s - s_0)(s + z - s_0)^r A(s)) e^{-\int f_3(s, z) \, ds}.
\]

(15)

From (4) \( \xi(s, z) \) can be estimated in the area \( 1 < \text{Re}\{s\}, 1 < \text{Re}\{s + z\} \) and \( |z| \ll 1 \):

\[
\ln \xi(s, z) = \ln \zeta(s) + \sum_{j=1}^{\infty} \ln \left( 1 - p_j^{-s} + p_j^{-s-z} \right)
\]

\[
= \ln \zeta(s) + \sum_{j=1}^{\infty} \ln \left( 1 - z \ln(p_j) p_j^{-s} + O(z^2) \right)
\]

\[
= \ln \zeta(s) + \sum_{j=1}^{\infty} -z \ln(p_j) p_j^{-s} + O(z^2) = \ln \zeta(s) + zh(s) + O(z^2).
\]

Thus

\[
\xi(s, z) = \zeta(s) \exp(zh(s) + O(z^2)) = \zeta(s) + z\zeta(s)h(s) + O(z^2).
\]

(16)

Equation (16) can be continued to the strip \( \frac{1}{2} < \text{Re}\{s\} < 1, \frac{1}{2} < \text{Re}\{s + z\} < 1 \).
If $r > 0$, then in (15) the function $\xi(s, z) = 0$ if $s + z = s_0$. If $z = s_0 - s$ is very small in (16), then $\xi(s, z) = 0$ implies that

$$1 - zh(s) = O(z^2) \text{ i.e., } h(s) = z^{-1} + O(z^2).$$

(17)

Inserting $z = s_0 - s$ and $h(s)$ from (14) to (17) implies that $r = 1$. Thus, if $r > 0$, then $r = 1$.

Let us assume $r = 1$ and $|z| << 1$, then

$$f_3(s, z) = f_1(s) - f_1(s + z) + f_2(s, z)$$

$$= -zf_1'(s) + z \sum_{j=1}^{\infty} (\ln p_j)^2 p_j^{-2j} + O(z^2)$$

where the sum converges in the strip $\frac{1}{2} < \Re\{s\} < 1$. Let us integrate $f_2(s, z)$ over $s$:

$$\int f_2(s, z) ds = \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \int (-\ln p_j) (p_j^{-s} + p_j^{-s-z})(p_j^{-s} - p_j^{-s-z})^n ds$$

$$= \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n+1} (p_j^{-s} - p_j^{-s-z})^{n+1}$$

and derivate it with respect to $z$ and set $|z| << 1$:

$$\frac{\partial}{\partial z} \int f_2(s, z) ds = \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n+1} \frac{\partial}{\partial z} (p_j^{-s} - p_j^{-s-z})^{n+1}$$

$$= \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \ln(p_j) p_j^{-s-z} (p_j^{-s} - p_j^{-s-z})^n = O(z).$$

For $|z| << 1$ holds

$$f_1(s) - f_1(s + z) = zf_1'(s) + O(z^2).$$
Therefore if $|z| < c 1$

$$\frac{\partial}{\partial z} \int f_3(s, z) ds = \frac{\partial}{\partial z} \int zf_3'(s) ds + \frac{\partial}{\partial z} \int f_2(s, z) ds = f_1(s) + O(z)$$

and

$$\frac{\partial}{\partial z} \int f_3(s, z) ds|_{z=0} = f_1(s).$$

Also

$$e^{-\int f_3(s, z) ds} = 1 + O(z).$$

From (15)

$$\frac{\partial}{\partial z} \xi(s, z) = (C + (s - s_0)A(s))e^{-\int f_3(s, z) ds}$$

$$- \left(\frac{\partial}{\partial z} \int f_3(s, z) ds\right) e^{-\int f_3(s, z) ds} (C(s - s_0) + (s - s_0)(s + z - s_0)A(s)).$$

Thus

$$\frac{\partial}{\partial z} \xi(s, z)|_{z=0} = C + (s - s_0)A(s) - f_1(s)(C(s - s_0) + (s - s_0)^2A(s))$$

$$= (C + 2(s - s_0)A(s) + (s - s_0)^2A'(s)) - (s - s_0)A(s) - (s - s_0)^2A'(s)$$

$$- f_1(s)(C(s - s_0) + (s - s_0)^2A(s)).$$

$$= \zeta'(s) - ((A(s) + C f_1(s))(s - s_0) + (s - s_0)^2(A'(s) + f_1(s)A(s))).$$

(18)

Since $\xi(s, 0) = \zeta(s)$, equation (16) gives the limit

$$\lim_{z \to 0} \frac{\xi(s, z) - \xi(s, 0)}{z} = \zeta(s)h(s).$$

By (10) $h(s)\zeta(s) = \zeta'(s) - g(s)\zeta(s)$, thus

$$\frac{\partial \xi(s, z)}{\partial z}|_{z=0} = \zeta'(s) - g(s)\zeta(s).$$

(19)
Equations (18) and (19) imply that

\[ g(s)\zeta(s) = g(s)(C(s - s_0) + (s - s_0)^2A(s)) \]

\[ = ((A(s) + Cf_1(s))(s - s_0) + (s - s_0)^2(A'(s) + f_1(s)A(s))) . \]  

(20)

Simplifying (20) yields

\[ A'(s) = \frac{1}{s - s_0}(g(s)C - A(s) - f_1(s)C) + g(s)A(s) - f_1(s) . \]  

(21)

Since \( A(s) \) and \( A'(s) \) do not have a pole at \( s_0 \), we get \( g(s)C - A(s) - f_1(s)C = (s-s_0)f_0(s) \) for some \( f_0(s) \) that is analytic close to \( s_0 \), but our interest is not at the environment of \( s_0 \). It is at the environment of the pole of \( \zeta(s) \) at \( s = 1 \). Solving \( A(s) \) from the linear first order differential equation (21) yields \( \zeta(s) \) exactly in the complex half-plane \( \text{Re}\{s\} > \frac{1}{2} \) when \( A(s) \) is inserted in (13). The function we obtain this way to \( \zeta(s) \) is valid at an environment of \( s = 1 \), and here is the contradiction for \( r = 1 \): At \( s = 1 \) the Riemann zeta function \( \zeta(s) \) has a simple pole. Thus, \( A(s) \) solved from (21) must have a simple pole at \( s = 1 \) and \( A'(s) \) must have a double pole at \( s = 1 \). The function \( f_1(s) \) is a part of \( h(s) \) and it has a simple pole at \( s = 1 \). The function \( g(s) \) has no poles in \( \text{Re}\{s\} > \frac{1}{2} \). Equation (21) cannot hold for \( A(s) \) yielding the Riemann zeta function as the left side has a double pole at \( s = 1 \) and the right side has only simple poles at \( s = 1 \). Therefore \( r = 1 \) is not possible.

Let us still mention that we cannot get a pole to \( s = 1 \) by solving (20) with the trial

\[ \frac{C}{A(s)} = \frac{A(s) + Cf_1(s)}{A'(s) + f_1(s)A(s)} . \]  

(22)

This equation yields

\[ CA'(s) = A(s)^2 \quad \text{i.e.,} \quad A(s) = -(s - C_2)^{-1} \]
for some complex number $C_2$. Though this equation is not linear and allows a pole for $A(s)$ at $-C_2$, the pole cannot be set to $s = 1$. The solution yields an exact form for $\zeta(s)$ in the whole complex plane if $r = 1$:

\[
\zeta(s) = C(s - s_0) - C(s - s_0)^2(s - C_2)^{-1}. \tag{23}
\]

Clearly, (23) is not the Riemann zeta function, but additionally there is another problem: we get $g(s) = f_1(s) - (s - s_0)^{-1}$. Since $g(s)$ is analytic when $\text{Re}\{s\} > \frac{1}{2}$, the number $C_2$ cannot be $-1$. This method, or any other method, cannot give a simple pole to $\zeta(s)$ at $s = 1$. Therefore $r = 1$ is impossible.

If $r = 0$, then there is no contradiction, thus $r = 0$. The proof of Lemma 2 and the Riemann Hypothesis is complete. Notice that products that are convergent in the strip $\frac{1}{2} < \text{Re}\{s\} < 1$, $\frac{1}{2} < \text{Re}\{s + z\} < 1$ are not convergent if $\text{Re}\{s\} = \frac{1}{2}$, thus $\zeta(s)$ can (and does) have zeros on that line. 

References