New Principles of Differential Equations II

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Abstract

This is the second part of the total paper. Three kinds of Z Transformations are used to get many laws for general solutions of \( m \)-th order linear partial differential equations with \( n \) variables in the present thesis. Some general solutions of first-order linear partial differential equations, which cannot be obtained by using the characteristic equation method, can be solved by the Z Transformations. By comparing, we find that general solutions of some first-order partial differential equations got by the characteristic equation method are not complete.

Keywords: Z Transformations; banal PDEs; non-banal PDEs; general solutions; the characteristic equation method.

Introduction

In recent years, many numerical methods [1-4] and analytical methods [5-7] have been developed to solve linear partial differential equations (PDEs), the existence [8], uniqueness [9, 10] and stability [11] of their solutions are also the focus of research.

In [12], we used new analytical methods to preliminarily study some laws of general solutions of linear PDEs. In present paper, we will use three kinds of Z Transforms to further study the \( m \)-th order linear PDEs with \( n \) variables.

1. New principles and methods

In order to obtain the general solutions of PDEs in various orthogonal coordinate systems, we proposed the concepts and laws of the independent variable transformational equations (IVTEs) and the dependent variable transformational equations (DVTEs) in [12]. About the IVTEs, there is an important new theorem:

**Theorem 1.** In the domain \( D \), \((D \subset \mathbb{R}^n)\), if \( G(y_1, y_2, \ldots, y_n, u, u_{y_1}, u_{y_2}, \ldots, u_{y_n}, u_{y_1y_2}, u_{y_1y_3}, \ldots) = 0 \) is an arbitrary independent variable transformational equation of a \( m \)-th order PDE \( F(x_1, x_2, \ldots, x_n, u, u_{x_1}, u_{x_2}, \ldots, u_{x_n}, u_{x_1x_2}, u_{x_1x_3}, \ldots) = 0, \) so

1. If \( F=0 \) is a linear PDE, then \( G=0 \) is an \( m \)-th order linear PDE.
2. If \( F=0 \) is a nonlinear PDE, then \( G=0 \) is an \( m \)-th order nonlinear PDE.

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Proof. Since

$$u_{x_i} = \sum_{p=1}^{n} u_{y_p} \frac{\partial y_p}{\partial x_i},$$

(1)

$$u_{x_i x_j} = \sum_{p=1}^{n} u_{y_p} \frac{\partial^2 y_p}{\partial x_i \partial x_j} + \sum_{p=1 \leq q, r}^{n} u_{y_p y_q} \frac{\partial y_p}{\partial x_i} \frac{\partial y_q}{\partial x_j},$$

(2)

$$u_{x_i x_j x_k} = \sum_{p=1}^{n} u_{y_q} \frac{\partial^3 y_p}{\partial x_i \partial x_j \partial x_k} + \sum_{p=1 \leq q, r}^{n} u_{y_p y_q y_r} \frac{\partial y_p}{\partial x_i} \frac{\partial y_q}{\partial x_j} \frac{\partial y_r}{\partial x_k},$$

(3)

where \(i, j, k, p, q, r \in \{1, 2, \ldots, n\}\), namely \(u_{x_i}, u_{x_i x_j}, u_{x_i x_j x_k} \ldots\) are linear relationship with \(u_{y_p}, u_{y_p y_q}, u_{y_p y_q y_r}, \ldots\), and the highest order of the partial derivatives on both sides of the equations are equal. Therefore each linear term in \(F = 0\) is transformed into a new linear term; every non-linear term in \(F = 0\) is transformed into a new nonlinear term; and the highest order of the partial derivative of the dependent variable of each term is constant. So the independent variable transformation not only does not change the linearity or non-linearity of the original PDEs, but also does not change their order. Then Theorem 1 is proved. \(\square\)

For the DVTEs, the specific \(l^{\text{th}}\)-order transformation \(v = h(x_1, \ldots, x_n, u, u_{x_1}, \ldots, u_{x_n}, u_{x_1 x_2}, \ldots)\) may be linear or non-linear. For the linear transformation, each linear term of an \(m^{\text{th}}\)-order linear PDE is usually transformed into a new linear term, so a \((l + m)^{\text{th}}\)-order linear PDE will be transformed. For an \(m^{\text{th}}\)-order nonlinear PDE, each nonlinear term will transform a new nonlinear term, normally it will be transformed a \((l + m)^{\text{th}}\)-order nonlinear PDE.

For the nonlinear transformation, an \(m^{\text{th}}\)-order linear PDE is usually transformed into a \((l + m)^{\text{th}}\)-order nonlinear PDE, an \(m^{\text{th}}\)-order nonlinear PDE may be transformed into a \((l + m)^{\text{th}}\)-order nonlinear PDE or a \((l + m)^{\text{th}}\)-order linear PDE.

In \(\mathbb{R}^n\) whose independent variables are \(x_1, x_2, \ldots, x_n\), if the solution of a PDE contains an arbitrary function, the number of independent variables of the arbitrary function is generally not equal to \(n\). Unless for any a smooth function, the essence of a PDE is \(0 = 0\), such as

$$u_{x_i} - u_{x_i} = 0,$$

(4)

It is obvious that the general solution of Eq. (4) can be an arbitrary first differentiable function with \(n\) variables, but this PDE has no practical meaning. We call a PDE whose essence is \(0 = 0\) a banal PDE and a PDE whose nature is not \(0 = 0\) a non-banal PDE.

For non-banal PDEs, we propose a conjecture:

Conjecture 1. In \(\mathbb{R}^n\), \((n \geq 2)\), if the solution of a non-banal PDE contains an arbitrary function, the number \(l\) of independent variables of the arbitrary function satisfies \(1 \leq l \leq n - 1\).

In [13], we presented \(Z_1-Z_3\) transformations with the following contents.

**Z1 Transformation.** In the domain \(D, (D \subset \mathbb{R}^n)\), any established \(m^{\text{th}}\)-order PDE with \(n\) space variables \(F(x_1, \ldots, x_n, u, u_{x_1}, \ldots, u_{x_n}, u_{x_1 x_2}, \ldots) = 0\), set \(y_i = y_i(x_1, \ldots, x_n)\) and \(u = f(y_1, \ldots, y_l)\) are both undetermined \(m^{\text{th}}\)-differentiable functions \((u, y_i \in C^m(D), 1 \leq i \leq l \leq n)\), \(y_1, y_2, \ldots, y_l\) are independent of each other, then substitute \(u = f(y_1, \ldots, y_l)\) and its partial derivatives into
$F = 0$

1. In case of working out $y_i = y_i(x_1, \ldots x_n)$ and $f(y_1, \ldots y_l)$ then $u = f(y_1, \ldots y_l)$ is the solution of $F = 0$.

2. In case of dividing out $u = f(y_1, \ldots y_l)$ and its partial derivative, also working out $y_i = y_i(x_1, \ldots x_n)$ then $u = f(y_1, \ldots y_l)$ is the solution of $F = 0$, and $f$ is an arbitrary $m$th-differentiable function.

3. In case of dividing out $u = f(y_1, \ldots y_l)$ and its partial derivative, also getting $k = 0$, but in fact $k \neq 0$, then $u = f(y_1, \ldots y_l)$ is not the solution of $F = 0$, and $f$ is an arbitrary $m$th-differentiable function.

**Z$_2$ Transformation.** In the domain $D$, ($D \subset \mathbb{R}^n$), any established $m$th-order PDE with $n$ space variables $F(x_1, \ldots x_n, u, u_{x_1}, \ldots u_{x_n}, u_{x_1x_2}, \ldots) = 0$, set $y_i = y_i(x_1, \ldots x_n)$ known and $u = f(y_1, \ldots y_l)$ undetermined ($u, y_i \in C^m(D), 1 \leq i \leq l \leq n$), $y_1, y_2, \ldots y_l$ are independent of each other, then substitute $u = f(y_1, \ldots y_l)$ and its partial derivatives into $F = 0$

1. In case of working out $f(y_1, \ldots y_l)$, then $u = f(y_1, \ldots y_l)$ is the solution of $F = 0$.

2. In case of dividing out $u = f(y_1, \ldots y_l)$ and its partial derivative, also getting $0 = 0$, then $u = f(y_1, \ldots y_l)$ is the solution of $F = 0$, and $f$ is an arbitrary $m$th-differentiable function.

3. In case of dividing out $u = f(y_1, \ldots y_l)$ and its partial derivative, also getting $k = 0$, but in fact $k \neq 0$, then $u = f(y_1, \ldots y_l)$ is not the solution of $F = 0$, and $f$ is an arbitrary $m$th-differentiable function.

**Z$_3$ Transformation.** In the domain $D$, ($D \subset \mathbb{R}^n$), any established $m$th-order PDE with $n$ space variables $F(x_1, \ldots x_n, u, u_{x_1}, \ldots u_{x_n}, u_{x_1x_2}, \ldots) = 0$, setting $g(x_1, \ldots x_n)$, $h(y_1, \ldots y_l)$ and $y_i = y_i(x_1, \ldots x_n)$ are all undetermined function, $y_1, y_2, \ldots y_l$ are independent of each other, $(g, h, y_i \in C^m(D), 1 \leq i \leq l \leq n)$, then substitute $u = gh(y_1, \ldots y_l)$ and its partial derivatives into $F = 0$

1. In case of working out $g, h$ and $y_i$, then $u = gh(y_1, \ldots y_l)$ is the solution of $F = 0$.

2. In case of dividing out $h$ and its partial derivative, also working out $g$ and $y_i$, then $u = gh(y_1, \ldots y_l)$ is the solution of $F = 0$, and $h$ is an arbitrary $m$th-differentiable function.

3. In case of getting $k = 0$, but in fact $k \neq 0$, then $u = gh(y_1, \ldots y_l)$ is not the solution of $F = 0$.

In this paper, we upgrade the Z$_3$ transformation:
2. General solutions’ laws of linear partial differential equations with variable coefficients

In this section, if there is no special interpretation, the acquiescent independent variables of \( \mathbb{R}^n \) are \( x_1, x_2, \ldots, x_n \). \( a_i = a_i (x_1, \ldots, x_k) \), \( a_{ij} = a_{ij} (x_1, \ldots, x_k) \), \( a_{ij\ldots i_k} = a_{ij\ldots i_k} (x_1, \ldots, x_k) \), \( b_i = b_i (x_1, \ldots, x_k) \), and \( b_{ij} = b_{ij} (x_1, \ldots, x_k) \) are arbitrary known functions. \( y_i = y_i (x_1, \ldots, x_k) \).

Theorem 1. In \( \mathbb{R}^n \), if the exact solutions \( u = y_i (x_1, \ldots, x_k) \) of Eq. (5), which are independent of each other, are known, \( 1 \leq i \leq l \leq k - 1 \),

\[
a_1 u_{x_1} + a_2 u_{x_2} + \cdots + a_k u_{x_k} = 0, \tag{5}
\]

then the general solution of Eq. (5) is

\[
u = f (y_1, y_2, \ldots, y_l, x_{k+1}, x_{k+2}, \ldots, x_n). \tag{6}
\]

Prove. By \( Z_1 \) Transformation, set \( u (x_1, \ldots, x_n) = f (y_1, y_2, \ldots, y_l, x_{k+1}, x_{k+2}, \ldots, x_n) \), then

\[
a_1 u_{x_1} + a_2 u_{x_2} + \cdots + a_k u_{x_k} = a_1 (f_{y_1} y_{x_1} + f_{y_2} y_{x_1} + \cdots + f_{y_l} y_{x_1}) + a_2 (f_{y_1} y_{x_2} + f_{y_2} y_{x_2} + \cdots + f_{y_l} y_{x_2}) + \cdots + a_k (f_{y_1} y_{x_k} + f_{y_2} y_{x_k} + \cdots + f_{y_l} y_{x_k})
\]

\[
= f_{y_1} (a_1 y_{x_1} + a_2 y_{x_1} + \cdots + a_k y_{x_1}) + f_{y_2} (a_1 y_{x_2} + a_2 y_{x_2} + \cdots + a_k y_{x_2}) + \cdots + f_{y_l} (a_1 y_{x_l} + a_2 y_{x_l} + \cdots + a_k y_{x_l}) = 0.
\]

Since \( a_1 y_{x_1} + a_2 y_{x_2} + \cdots + a_k y_{x_k} = 0, (1 \leq i \leq l \leq k - 1) \), the above equation is correct externally. So Theorem 1 is proved. According to the characteristic equation method and Conjecture 1 we can know: Usually \( l = k - 1 \).

According to Theorem 1, if the exact solutions \( u = y_i (x_1, \ldots, x_n) \) of \( a_1 u_{x_1} + a_2 u_{x_2} + \cdots + a_n u_{x_n} = 0 \), which are independent of each other, are known, \( 1 \leq i \leq l \leq n - 1 \), then its general solution is \( u = f (y_1, y_2, \ldots, y_l) \).

Theorem 2. In \( \mathbb{R}^n \), if the general solution \( u = f (y_1, y_2, \ldots, y_l, x_{k+1}, x_{k+2}, \ldots, x_n) \) of

\[
(b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k}) u = 0 \]

is known, then

\[
(b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k})^2 u = 0, \tag{7}
\]

the general solution of Eq. (7) is

\[
u = f_1 (y_1, \ldots, y_l, x_{k+1}, \ldots, x_n) + (\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k) f_2 (y_1, \ldots, y_l, x_{k+1}, \ldots, x_n). \tag{8}
\]

Prove. Because the general solution \( u = f (y_1, y_2, \ldots, y_l, x_{k+1}, x_{k+2}, \ldots, x_n) \) of

\[
(b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k}) u = 0 \]

is also the solution of \( (b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k})^2 u = 0 \), setting \( u = g (x_1, \ldots, x_k) \), \( f = \lambda_s x_s \) is the solution of \( (b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k})^2 u = 0 \) too, that is \( g (x_1, \ldots, x_k) = \lambda_s x_s \), and
\[ \lambda_s \text{ are arbitrary constants, } (s = 1, 2, \ldots, k), \]

\[ (b_1D_{x_1} + b_2D_{x_2} + \cdots + b_kD_{x_k})^2u = \left( \sum_{i=1}^{k} b_i^2D_{x_i}^2 + 2 \sum_{i<j} b_ib_jD_{x_i}D_{x_j} \right)gf \]

\[ = \sum_{i=1}^{k} b_i^2 \left( g_{x_i}f + 2g_{x_i}f_{x_i} + g_{x_i}f_{x_i} \right) + 2 \sum_{i<j} b_ib_j \left( g_{x_i}f + g_{x_i}f_{x_i} + g_{x_i}f_{x_i} + g_{x_i}f_{x_i} \right) \]

\[ = g \left( \sum_{i=1}^{k} b_i^2f_{x_i} + 2 \sum_{i<j} b_ib_jf_{x_i} \right) + f \left( \sum_{i=1}^{k} b_i^2g_{x_i} + 2 \sum_{i<j} b_ib_jg_{x_i} \right) + 2 \sum_{i<j} b_i^2g_{x_i}f_{x_i} \]

\[ + 2 \sum_{i<j} b_ib_j \left( g_{x_i}f_{x_i} + g_{x_i}f_{x_i} \right) = 2 \sum_{i=1}^{k} b_i^2g_{x_i} + 2 \sum_{i<j} b_ib_jg_{x_i}f_{x_i} + 2 \sum_{i<j} b_i^2g_{x_i}f_{x_i} = 2b_s^2\lambda_s f_{x_s} + 2\lambda_s b_s \sum_{j<s} b_jf_{x_j} + 2\lambda_s b_s \sum_{i<s} b_if_{x_i} = 2\lambda_s b_s \sum_{i=1}^{k} b_if_{x_i} = 0. \]

That \( u = \lambda_s x_s f \) is the solution of \((b_1D_{x_1} + b_2D_{x_2} + \cdots + b_kD_{x_k})^2u \) is proved. So its general solution is (8). □

According to Theorem 2, we present Theorem 3.

**Theorem 3.** In \( \mathbb{R}^n \), if the general solution \( u = f(y_1, y_2, \ldots, y_l, x_{k+1}, \ldots x_n) \) of \((b_1D_{x_1} + b_2D_{x_2} + \cdots + b_kD_{x_k})u = 0 \) is known, then the general solution of

\[ (b_1D_{x_1} + b_2D_{x_2} + \cdots + b_kD_{x_k})^mu = 0, \quad (9) \]

is

\[ u = \sum_{j=1}^{m} \left( \lambda_1x_1 + \lambda_2x_2 + \cdots + \lambda_kx_k \right)^{j-1} f_j(y_1, \ldots, y_l, x_{k+1}, \ldots x_n). \quad (10) \]

**Prove.** Two mathematical formulas are required to prove the theorem:

\[ (a_1 + a_2 + \cdots + a_k)^m = \sum_{l_1+l_2+\ldots+l_k=m} \frac{m!}{l_1!l_2!\ldots l_k!} a_1^{l_1}a_2^{l_2}a_k^{l_k}, \quad (11) \]

\[ D^n_{x_i} (uv) = \sum_{k=0}^{n} C_n^k u^{(n-k)}v^{(k)}_{x_i}. \quad (12) \]

Then

\[ (b_1D_{x_1} + b_2D_{x_2} + \cdots + b_kD_{x_k})^mu = \sum_{l_1+l_2+\ldots+l_k=m} \frac{m!}{l_1!l_2!\ldots l_k!} b_1^{l_1}b_2^{l_2}\ldots b_k^{l_k}D_{x_1}^{l_1}D_{x_2}^{l_2}\ldots D_{x_k}^{l_k}u. \quad (13) \]

According to Theorem 2, we know that \( u = f_1(y_1, \ldots, y_l, x_{k+1}, \ldots x_n) + (\lambda_1x_1 + \lambda_2x_2 + \cdots + \lambda_kx_k) \)

\( f_2(y_1, y_2, \ldots, y_l, x_{k+1}, \ldots x_n) \) is the general solution of \((b_1D_{x_1} + b_2D_{x_2} + \cdots + b_kD_{x_k})^2u = 0 \). Using mathematical induction, we set

\[ u = \Gamma = \sum_{j=1}^{m} \left( \lambda_1x_1 + \lambda_2x_2 + \cdots + \lambda_kx_k \right)^{j-1} f_j(y_1, \ldots, y_l, x_{k+1}, \ldots x_n), \quad (14) \]

is the general solution of \((b_1D_{x_1} + b_2D_{x_2} + \cdots + b_kD_{x_k})^mu = D^{(m)}u = 0, \) namely \( D^{(m)}\Gamma = 0 \).

\[ (b_1D_{x_1} + b_2D_{x_2} + \cdots + b_kD_{x_k})^{m+1}u = D^{(m+1)}u = 0. \quad (15) \]
Note \( u = \Gamma \) is also the solution of Eq. (15). Setting \( u = g(x_1, \ldots, x_k) \Gamma = \lambda_s x_s \Gamma \) is the solution of Eq. (15) too, that is \( g(x_1, \ldots, x_k) = \lambda_s x_s \), and \( \lambda_s \) are arbitrary constants, \( (s = 1, 2, \ldots, k) \), then

\[
D^n_{x_s} (g \Gamma) = \sum_{k=0}^{n} C_{n}^{k} g^{(n-k)} \Gamma_{x_s}^{(k)} x_s = \lambda_s x_s \Gamma_{x_s}^{(n)} + n \lambda_s \Gamma_{x_s}^{(n-1)}
\]

\[ D^l_{x_s} (g \Gamma) = \sum_{k=0}^{l} C_{l}^{k} g^{(l-k)} \Gamma_{x_s}^{(k)} = \lambda_s x_s \Gamma_{x_s}^{(l)} \]

\[ D^l_{x_s} D^n_{x_s} (g \Gamma) = D^n_{x_s} D^l_{x_s} (g \Gamma) = \lambda_s x_s \Gamma_{x_s}^{(l,n)} + n \lambda_s \Gamma_{x_s}^{(l,n-1)} , \ (s \not= i). \]

The same can be obtained:

\[
D^l_{x_1} D^l_{x_2} (g \Gamma) = D^l_{x_2} D^l_{x_1} (g \Gamma) = \lambda_s x_s \Gamma_{x_1,x_2,...,x_k}^{(l,l)} + l \lambda_s \Gamma_{x_1,x_2,...,x_k}^{(l,l-1)}, \ (s \not= i, j)
\]

\[
D^l_{x_1} D^l_{x_2} \ldots D^l_{x_k} (g \Gamma) = \lambda_s x_s \Gamma_{x_1,x_2,...,x_k}^{(l,l,...,l)} + l \lambda_s \Gamma_{x_1,x_2,...,x_k}^{(l,l,...,l-1)}. \]

Then

\[(b_1 D_{x_1} + b_2 D_{x_2} + \ldots + b_k D_{x_k})^{m+1} \]

\[= (b_1 D_{x_1} + b_2 D_{x_2} + \ldots + b_k D_{x_k}) \left( \sum_{l_1+l_2+\ldots+l_k=m} \frac{m!}{l_1! l_2! \ldots l_k!} b_1^{l_1} b_2^{l_2} \ldots b_k^{l_k} D^{l_1}_{x_1} D^{l_2}_{x_2} \ldots D^{l_k}_{x_k} \right) \]

\[= \sum_{l_1+l_2+\ldots+l_k=m} \frac{m!}{l_1! l_2! \ldots l_k!} b_1^{l_1} b_2^{l_2} \ldots b_k^{l_k} D^{l_1}_{x_1} D^{l_2}_{x_2} \ldots D^{l_k}_{x_k} + \ldots + b_1^{l_1} b_2^{l_2} \ldots b_s^{l_s} D^{l_1}_{x_1} D^{l_2}_{x_2} \ldots D^{l_s}_{x_s} \ldots + b_1^{l_1} b_2^{l_2} \ldots b_k^{l_k} D^{l_1}_{x_1} D^{l_2}_{x_2} \ldots D^{l_k}_{x_k}) \]

Because of \( (b_1 D_{x_1} + b_2 D_{x_2} + \ldots + b_k D_{x_k})^{m+1} \Gamma = 0 \).

So

\[(b_1 D_{x_1} + b_2 D_{x_2} + \ldots + b_k D_{x_k})^{m+1} \Gamma = \sum_{l_1+l_2+\ldots+l_k=m} \frac{m!}{l_1! l_2! \ldots l_k!} b_1^{l_1} b_2^{l_2} \ldots b_k^{l_k} (x_s \Gamma_{x_1,x_2,...,x_k}^{(l_1,l_2,\ldots,l_s)}) + \ldots + b_1^{l_1} b_2^{l_2} \ldots b_k^{l_k} (x_s \Gamma_{x_1,x_2,...,x_k}^{(l_1,l_2,\ldots,l_s)}) \]

Notice:

\[
\lambda_s x_s \sum_{l_1+l_2+\ldots+l_k=m} \frac{m!}{l_1! l_2! \ldots l_k!} (b_1^{l_1} b_2^{l_2} \ldots b_k^{l_k} (x_s \Gamma_{x_1,x_2,...,x_k}^{(l_1,l_2,\ldots,l_s)})) + b_1^{l_1} b_2^{l_2} \ldots b_k^{l_k} (x_s \Gamma_{x_1,x_2,...,x_k}^{(l_1,l_2,\ldots,l_s)}) \]

\[= \lambda_s x_s (b_1 D_{x_1} + b_2 D_{x_2} + \ldots + b_k D_{x_k})^{m+1} \Gamma = 0. \]
Then
\[
(b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k})^{m+1}(g \Gamma) = \lambda_s \sum_{l_1+l_2+\cdots+I_k=m} \frac{m!}{l_1!l_2!\cdots l_k!} (b_1^{l_1+1} b_2^{l_2} \cdots b_k^{l_k} \Gamma^{(1+1,l_2,\ldots,l_k-1,\ldots,l_k)} + b_1^{l_1} b_2^{l_2+1} \cdots b_k^{l_k} \Gamma^{(1+1,l_2,\ldots,l_k-1,\ldots,l_k)} + \cdots + b_1^{l_1} b_2^{l_2} \cdots b_k^{l_k+1} \Gamma^{(1+1,l_2,\ldots,l_k-1,\ldots,l_k)}).
\]

Since
\[
(b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k})^{m+1} \Gamma = (b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k}) (b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k})^{m-1} \Gamma = (b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k}) \sum_{l_1+l_2+\cdots+(l_k-1)+\cdots+l_k=m-1} \frac{m!}{l_1!l_2!\cdots l_k!} b_1^{l_1} b_2^{l_2} \cdots b_k^{l_k} (b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k})
\]
\[
\cdots b_k^{l_k} \Gamma^{(l_1,l_2,\ldots,l_k-1,\ldots,l_k)} + b_1^{l_1} b_2^{l_2+1} \cdots b_k^{l_k+1} (b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k}) \Gamma^{(1+1,l_2,\ldots,l_k-1,\ldots,l_k)} + \cdots + b_1^{l_1} b_2^{l_2} \cdots b_k^{l_k+1} \Gamma^{(l_1,l_2,\ldots,l_k-1,\ldots,l_k) \cdots + b_1^{l_1} b_2^{l_2+1} \cdots b_k^{l_k+1} \Gamma^{(1+1,l_2,\ldots,l_k-1,\ldots,l_k)} + \cdots + b_1^{l_1} b_2^{l_2} \cdots b_k^{l_k+1} \Gamma^{(l_1,l_2,\ldots,l_k-1,\ldots,l_k)+1} = 0.
\]

That is
\[
\sum_{l_1+l_2+\cdots+(l_k-1)+\cdots+l_k=m-1} \frac{m!}{l_1!l_2!\cdots l_k!} (b_1^{l_1+1} b_2^{l_2} \cdots b_k^{l_k} \Gamma^{(1+1,l_2,\ldots,l_k-1,\ldots,l_k)} + b_1^{l_1} b_2^{l_2+1} \cdots b_k^{l_k+1} \Gamma^{(1+1,l_2,\ldots,l_k-1,\ldots,l_k)} + \cdots + b_1^{l_1} b_2^{l_2} \cdots b_k^{l_k+1} \Gamma^{(l_1,l_2,\ldots,l_k-1,\ldots,l_k)+1}) = 0.
\]

So
\[
\lambda_s \sum_{l_1+l_2+\cdots+I_k=m} \frac{m!}{l_1!l_2!\cdots l_k!} (b_1^{l_1+1} b_2^{l_2} \cdots b_k^{l_k} \Gamma^{(1+1,l_2,\ldots,l_k-1,\ldots,l_k)} + b_1^{l_1} b_2^{l_2+1} \cdots b_k^{l_k+1} \Gamma^{(1+1,l_2,\ldots,l_k-1,\ldots,l_k)} + \cdots + b_1^{l_1} b_2^{l_2} \cdots b_k^{l_k+1} \Gamma^{(l_1,l_2,\ldots,l_k-1,\ldots,l_k)+1}) = \lambda_s m \sum_{l_1+l_2+\cdots+(l_k-1)+\cdots+l_k=m-1} \frac{m!}{l_1!l_2!\cdots l_k!} (b_1^{l_1+1} b_2^{l_2} \cdots b_k^{l_k} \Gamma^{(1+1,l_2,\ldots,l_k-1,\ldots,l_k)} + b_1^{l_1} b_2^{l_2+1} \cdots b_k^{l_k+1} \Gamma^{(1+1,l_2,\ldots,l_k-1,\ldots,l_k)} + \cdots + b_1^{l_1} b_2^{l_2} \cdots b_k^{l_k+1} \Gamma^{(l_1,l_2,\ldots,l_k-1,\ldots,l_k)+1}) = 0.
\]

Namely
\[
(b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k})^{m+1}(\lambda_s x_s \Gamma) = 0. \tag{18}
\]

Since \(s = 1, 2, \ldots k\), according to the principle of linear superposition, the general solution of Eq. (9) is (10), so the theorem is proved. \(\Box\)

In \(\mathbb{R}^n\), for the \(m\)-th order linear PDE with variable coefficients
\[
\sum_{i_1+i_2+\cdots+i_k=m} a_{i_1i_2\cdots i_k} x^{(i_1i_2\cdots i_k)} = 0, \tag{19}
\]
where \(i_j\) are natural number, \(1 \leq j \leq k \leq n\). If Eq. (19) can be translated into
\[
(b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k}) \cdots (b_{m_1} D_{x_1} + b_{m_2} D_{x_2} + \cdots + b_{m_k} D_{x_k}) u = 0. \tag{20}
\]
For
\[(b_{j_1} D_{x_1} + b_{j_2} D_{x_2} + \cdots + b_{j_k} D_{x_k}) u = b_{j_1} u_{x_1} + b_{j_2} u_{x_2} + \cdots + b_{j_k} u_{x_k} = 0, \quad (j = 1, 2, \ldots, m). \quad (21)\]

If the particular solutions \(u = y_{j_1}(x_1, \ldots, x_k)\) of Eq. (21), which are independent of each other, are all known, \((1 \leq s \leq l_j)\), by Theorem 1 the general solution of Eq. (19) is
\[u = \sum_{j=1}^{m} f_j \left( y_{j_1}, y_{j_2}, \ldots, y_{j_k}, x_{k+1}, x_{k+2}, \ldots, x_n \right). \quad (22)\]

If Eq. (19) can be translated into:
\[\prod_{j=1}^{q} (b_{j_1} D_{x_1} + b_{j_2} D_{x_2} + \cdots + b_{j_k} D_{x_k})^{p_j} u = 0, \quad (23)\]

where \(\sum_{j=1}^{q} p_j = m\), its general solution can be written by Theorem 3.

In \(\mathbb{R}^n\), for the \(m\)th-order linear PDE with variable coefficients
\[\sum_{0 \leq i_1 + i_2 + \cdots + i_k \leq m} a_{i_1 i_2 \ldots i_k} (x_1, x_2, \ldots, x_n) = A(x_1, x_2, \ldots, x_n), \quad (24)\]

where \(A(x_1, x_2, \ldots, x_n)\) is an arbitrary known function. In general, we need to solve the particular solution of (24) first, and then use the general solution of its homogeneous equation to obtain its general solution. Such as

**Theorem 4.** In \(\mathbb{R}^n\), if particular solutions \(u = y_i(x_1, \ldots, x_k)\) of \(a_1 u_{x_1} + a_2 u_{x_2} + \cdots + a_k u_{x_k} = 0\) are all known, \((1 \leq i \leq k - 1)\), then
\[a_1 u_{x_1} + a_2 u_{x_2} + \cdots + a_k u_{x_k} = A(x_1, x_2, \ldots, x_n), \quad (25)\]

the general solution of Eq. (25) is
\[u = f(y_1, \ldots, y_{k-1}, x_{k+1}, \ldots, x_n) + \int \frac{A(y_1, \ldots, y_k, x_{k+1}, \ldots, x_n) dy_k}{a_1 y_{x_1} + a_2 y_{x_2} + \cdots + a_k y_{x_k}}, \quad (26)\]

where \(y_1, y_2, \ldots, y_k\) are independent of each other, and \(x_j = x_j(y_1, y_2, \ldots, y_k)\) should be got, \((1 \leq j \leq k)\).

**Proof.** By \(Z_1\) Transformation, set \(u(x_1, x_2, \ldots, x_n) = u(y_1, y_2, \ldots, y_k, x_{k+1}, x_{k+2}, \ldots, x_n)\) and
\[a_i(x_1, x_2, \ldots, x_n) = a_i(y_1, y_2, \ldots, y_k, x_{k+1}, x_{k+2}, \ldots, x_n), \quad (s = 1, 2, \ldots, k),\]

and \(y_1, y_2, \ldots, y_k\) are independent of each other, then
\[a_1 u_{x_1} + a_2 u_{x_2} + \cdots + a_k u_{x_k}\]
\[= a_1 \sum_{s=1}^{k} y_{s_{x_1}} u_{y_s} + a_2 \sum_{s=1}^{k} y_{s_{x_2}} u_{y_s} + \cdots + a_k \sum_{s=1}^{k} y_{s_{x_k}} u_{y_s}\]
\[= \left( a_1 y_{x_1} + a_2 y_{x_2} + \cdots + a_k y_{x_k} \right) u_{y_1} + \left( a_1 y_{x_1} + a_2 y_{x_2} + \cdots + a_k y_{x_k} \right) u_{y_2} + \cdots\]
\[+ \left( a_1 y_{x_1} + a_2 y_{x_2} + \cdots + a_k y_{x_k} \right) u_{y_k} = A(x_1, x_2, \ldots, x_n). \quad (27)\]
If the particular solutions $u = y_i(x_1, \ldots, x_k)$ of $a_1u_{x_1} + a_2u_{x_2} + \cdots + a_ku_{x_k} = 0$ are all known ($1 \leq i \leq k-1$), then Eq. (27) can be translated into

$$\left( a_1y_{k_1} + a_2y_{k_2} + \cdots + a_ky_{k_k} \right) u_{yk} = A(x_1, x_2, \ldots, x_n).$$

(28)

A particular solution of Eq. (28) is

$$u = \int \frac{A(y_1, \ldots, y_k, x_{k+1}, \ldots, x_n) \, dy_k}{a_1y_{k_1} + a_2y_{k_2} + \cdots + a_ky_{k_k}}.$$  

(29)

To compute (29) we need to find the first differentiable $y_k = y_k(x_1, x_2, \ldots, x_k)$ and $y_1, y_2, \ldots, y_k$ independent of each other, and can get $x_j = x_j(y_1, y_2, \ldots, y_k)$, then the general solution of Eq. (25) is (26). So Theorem 4 is proved. □

**Theorem 5.** In $\mathbb{R}^n$, if particular solutions $u = y_i(x_1, \ldots, x_k)$ of $b_1u_{x_1} + b_2u_{x_2} + \cdots + b_ku_{x_k} = 0$ are all known, ($1 \leq i \leq k-1$), then

$$(b_1D_{x_1} + b_2D_{x_2} + \cdots + b_kD_{x_k})^m u = A(x_1, \ldots, x_n),$$

(30)

the general solution of Eq. (30) is

$$u = \sum_{j=1}^{m} \left( \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k \right)^{j-1} f_j(y_1, \ldots, y_{k-1}, x_{k+1}, \ldots, x_n)$$

$$+ \frac{\int \cdots \int A(y_1, \ldots, y_k, x_{k+1}, \ldots, x_n) \, dy_k}{(b_1c_1 + b_2c_2 + \cdots + b_kc_k)^m},$$

(31)

where $y_1, y_2, \ldots, y_k$ are independent of each other, $y_k = c_1 x_1 + c_2 x_2 + \cdots + c_k x_k$ and $x_j = x_j(y_1, y_2, \ldots, y_k)$ can be got, ($1 \leq j \leq k$).

**Prove.** By $Z_1$ Transformation, set $u(x_1, x_2, \ldots, x_n) = u(y_1, y_2, \ldots, y_k, x_{k+1}, x_{k+2}, \ldots, x_n)$ and $a(x_1, x_2, \ldots, x_n) = a_i(y_1, y_2, \ldots, y_k, x_{k+1}, x_{k+2}, \ldots, x_n)$, where $y_s = y_s(x_1, x_2, \ldots, x_k), (s = 1, 2, \ldots, k)$, $y_1, y_2, \ldots, y_k$ are independent of each other, and

$$y_k = c_1 x_1 + c_2 x_2 + \cdots + c_k x_k.$$

Then

$$(b_1D_{x_1} + b_2D_{x_2} + \cdots + b_kD_{x_k})^m u$$

$$= (b_1D_{x_1} + b_2D_{x_2} + \cdots + b_kD_{x_k})^{m-1} \left( b_1 \sum_{s=1}^{k} y_{s_1} u_{y_s} + b_2 \sum_{s=1}^{k} y_{s_2} u_{y_s} + \cdots + b_k \sum_{s=1}^{k} y_{s_k} u_{y_s} \right)$$

$$= (b_1D_{x_1} + b_2D_{x_2} + \cdots + b_kD_{x_k})^{m-1} \left( b_1 y_{1x_1} + b_2 y_{1x_2} + \cdots + b_k y_{1x_k} \right) u_{y_1}$$

$$+ \left( b_1 y_{2x_1} + b_2 y_{2x_2} + \cdots + b_k y_{2x_k} \right) u_{y_2} + \cdots + \left( b_1 y_{kx_1} + b_2 y_{kx_2} + \cdots + b_k y_{kx_k} \right) u_{y_k}$$

$$= (b_1D_{x_1} + b_2D_{x_2} + \cdots + b_kD_{x_k})^{m-1} \left( b_1 y_{kx_1} + b_2 y_{kx_2} + \cdots + b_k y_{kx_k} \right) u_{y_k}$$

$$= (b_1c_1 + b_2c_2 + \cdots + b_kc_k) (b_1D_{x_1} + b_2D_{x_2} + \cdots + b_kD_{x_k})^{m-1} u_{y_k}$$

$$= (b_1c_1 + b_2c_2 + \cdots + b_kc_k) (b_1D_{x_1} + b_2D_{x_2} + \cdots + b_kD_{x_k})^{m-2} (b_1 \sum_{s=1}^{k} y_{s_1} u_{y_s} + \cdots$$

$$+ b_2 \sum_{s=1}^{k} y_{s_2} u_{y_s} + \cdots + b_k \sum_{s=1}^{k} y_{s_k} u_{y_s})$$

$$= (b_1c_1 + b_2c_2 + \cdots + b_kc_k) (b_1D_{x_1} + b_2D_{x_2} + \cdots + b_kD_{x_k})^{m-2}$$

$$\left( b_1 y_{1x_1} + b_2 y_{1x_2} + \cdots + b_k y_{1x_k} \right) u_{y_1} + \left( b_1 y_{2x_1} + b_2 y_{2x_2} + \cdots + b_k y_{2x_k} \right) u_{y_2} + \cdots + \left( b_1 y_{kx_1} + b_2 y_{kx_2} + \cdots + b_k y_{kx_k} \right) u_{y_k}$$

$$= (b_1c_1 + b_2c_2 + \cdots + b_kc_k) (b_1D_{x_1} + b_2D_{x_2} + \cdots + b_kD_{x_k})^{m-2} u_{y_k}$$

$$= \cdots = (b_1c_1 + b_2c_2 + \cdots + b_kc_k)^m u^{(m)}_{y_k}.$$
Namely
\[(b_1D_{x_1} + b_2D_{x_2} + \cdots + b_kD_{x_k})^m u = (b_1c_1 + b_2c_2 + \cdots + b_kc_k)^m u_{y_k} = A(y_1, \ldots, y_k, x_{k+1}, \ldots, x_n). \tag{32}\]

The particular solution of Eq. (32) is
\[u = \int \cdots \int A(y_1, \ldots, y_k, x_{k+1}, \ldots, x_n) \, dm \, y_k. \]

According to Theorem 3, the general solution of Eq. (30) is (31), so the theorem is proved. \(\square\)

**Theorem 6.** In \(\mathbb{R}^n\), if the particular solution \(g = g(x_1, \ldots, x_k)\) of \(a_1g_{x_1} + a_2g_{x_2} + \cdots + a_kg_{x_k} + a_{k+1}g = 0\) and exact solutions \(u = y_i(x_1, \ldots, x_k)\) of \(a_1u_{x_1} + a_2u_{x_2} + \cdots + a_ku_{x_k} = 0\), which are independent of each other, are all known, \(1 \leq i \leq l \leq k - 1\), then
\[a_1u_{x_1} + a_2u_{x_2} + \cdots + a_ku_{x_k} + a_{k+1}u = 0, \tag{33}\]

the general solution of Eq. (33) is
\[u = g(x_1, \ldots, x_k) f(y_1, \ldots, y_l, x_{k+1}, \ldots, x_n). \tag{34}\]

**Prove.** By \(Z_3\) Transformation, set \(u = g(x_1, \ldots, x_k)f(y_1, \ldots, y_l, x_{k+1}, \ldots, x_n)\), then
\[a_1u_{x_1} + a_2u_{x_2} + \cdots + a_ku_{x_k} + a_{k+1}u\\= a_1 \left( f_{x_1} + g \sum_{i=1}^l f_{y_i}y_{x_1} \right) + a_2 \left( f_{x_2} + g \sum_{i=1}^l f_{y_i}y_{x_2} \right) + \cdots + a_k \left( f_{x_k} + g \sum_{i=1}^l f_{y_i}y_{x_k} \right) + a_{k+1}gf\\= f \left( a_1g_{x_1} + a_2g_{x_2} + \cdots + a_kg_{x_k} + a_{k+1}g \right) + g \sum_{i=1}^l f_{y_i} \left( a_1y_{ix_1} + a_2y_{ix_2} + \cdots + a_ky_{ix_k} \right)\\= 0.\]

Since \(a_1g_{x_1} + a_2g_{x_2} + \cdots + a_kg_{x_k} + a_{k+1}g = 0\) and \(a_1y_{ix_1} + a_2y_{ix_2} + \cdots + a_ky_{ix_k} = 0\). The above equation is correct eternally, so Theorem 6 is proved. \(\square\)

**Theorem 7.** In \(\mathbb{R}^n\), if the general solution \(u = g(x_1, \ldots, x_k)f(y_1, \ldots, y_l, x_{k+1}, \ldots, x_n)\) of \((b_1D_{x_1} + b_2D_{x_2} + \cdots + b_kD_{x_k} + b_{k+1})u = 0\) is known, then
\[(b_1D_{x_1} + b_2D_{x_2} + \cdots + b_kD_{x_k} + b_{k+1})^2 u = 0, \tag{35}\]

the general solution of Eq. (35) is
\[u = g(x_1, \ldots, x_k) \left( f_1(y_1, \ldots, y_l, x_{k+1}, \ldots, x_n) + (\lambda_1x_1 + \lambda_2x_2 + \cdots + \lambda_kx_k) f_2(y_1, \ldots, y_l, x_{k+1}, \ldots, x_n) \right). \tag{36}\]

**Prove.** Because the general solution \(u = g(x_1, \ldots, x_k)f(y_1, \ldots, y_l, x_{k+1}, \ldots, x_n)\) of \((b_1D_{x_1} + b_2D_{x_2} + \cdots + b_kD_{x_k} + b_{k+1})u = 0\) is known, apparently \(u = g(x_1, \ldots, x_k)f(y_1, \ldots, y_l, x_{k+1}, \ldots, x_n)\) is also the solution of \((b_1D_{x_1} + b_2D_{x_2} + \cdots + b_kD_{x_k} + b_{k+1})^2 u = 0\), setting
\[u = hA = \lambda_s x_s g(x_1, \ldots, x_k) f(y_1, \ldots, y_l, x_{k+1}, \ldots, x_n), \quad (1 \leq s \leq k), \tag{37}\]
where $h = \lambda x_s, \Lambda = g(x_1, \ldots, x_k)f(y_1, \ldots, y_l, x_{k+1}, \ldots, x_n)$, and $\lambda_s$ are arbitrary constants. Assuming (37) is a solution of (35), then

\[
(b_1D_{x_1} + b_2D_{x_2} + \cdots + b_kD_{x_k} + b_{k+1})^2 u \\
= \left( \sum_{i=1}^{k} b_i^2 D_{x_i}^2 + b_{k+1}^2 + 2 \sum_{1 \leq i < j \leq k} b_ib_j D_{x_i} D_{x_j} + 2b_{k+1} \sum_{i=1}^{k} b_i D_{x_i} \right) (h\Lambda) \\
= \sum_{i=1}^{k} b_i^2 (h_{x_i} x_i, h + \Lambda_{x_i}) + b_{k+1}^2 h\Lambda \\
+ 2 \sum_{1 \leq i < j \leq k} b_ib_j (h_{x_i} x_j + h_{x_j} x_i + h\Lambda_{x_i x_j}) + 2b_{k+1} \sum_{i=1}^{k} b_i (h_{x_i} x_i + h\Lambda_{x_i}) \\
= h \left( \sum_{i=1}^{k} b_i^2 x_i \Lambda + b_{k+1}^2 + 2 \sum_{1 \leq i < j \leq k} b_ib_j x_{i j} + 2b_{k+1} \sum_{i=1}^{k} b_i x_i \right) + \Lambda \sum_{i=1}^{k} b_i h_{x_i x_i} \\
+ 2 \sum_{i=1}^{k} b_i^2 x_i \Lambda + 2 \Lambda \sum_{1 \leq i < j \leq k} b_ib_j x_{i j} + 2 \sum_{1 \leq i < j \leq k} b_ib_j x_{i j} + 2 \sum_{1 \leq i < j \leq k} b_ib_j x_{i j} \\
+ 2b_{k+1} \Lambda \sum_{i=1}^{k} b_i h_{x_i} \\
= 2 \sum_{i=1}^{k} b_i^2 x_i \Lambda + 2 \sum_{1 \leq i < j \leq k} b_ib_j x_{i j} + 2 \sum_{1 \leq i < j \leq k} b_ib_j x_{i j} + 2b_{k+1} \Lambda \sum_{i=1}^{k} b_i h_{x_i} \\
= 2b_s h_{x_s} + 2b_{s+1} h_{x_s} + 2b_{s+1} h_{x_s} + 2b_{k+1} \Lambda \sum_{i=1}^{k} b_i h_{x_i} \\
= 2b_s h_{x_s} \sum_{i=1}^{k} b_i x_i + 2b_{k+1} b_s h_{x_s} \Lambda = 2b_s h_{x_s} \left( \sum_{i=1}^{k} b_i x_i + b_{k+1} \Lambda \right) = 0.
\]

That (37) is a solution of (35) is proved. So its general solution is (36). $\square$

According to Theorem 7, we present Theorem 8.

**Theorem 8.** In $\mathbb{R}^n$, if the general solution $u = g(x_1, \ldots, x_k)f(y_1, \ldots, y_l, x_{k+1}, \ldots, x_n)$ of $(b_1D_{x_1} + b_2D_{x_2} + \cdots + b_kD_{x_k} + b_{k+1})u = 0$ is known, then the general solution of

\[
(b_1D_{x_1} + b_2D_{x_2} + \cdots + b_kD_{x_k} + b_{k+1})^m u = 0, \quad (m \geq 2),
\]

is

\[
u = (\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k)^{j-1} f_j (y_1, \ldots, y_l, x_{k+1}, \ldots, x_n).
\]

**Prove.** Using Eq. (11), we get

\[
(b_1D_{x_1} + b_2D_{x_2} + \cdots + b_kD_{x_k} + b_{k+1})^m u \\
= \sum_{l_1 + l_2 + \cdots + l_{k+1} = m} m! l_1! l_2! \cdots l_{k+1}! b_1^{l_1} b_2^{l_2} \cdots b_k^{l_k} b_{k+1}^{l_{k+1}} D_{x_1}^{l_1} D_{x_2}^{l_2} \cdots D_{x_k}^{l_k} u.
\]

According to Theorem 7, we know that $u = g(x_1, \ldots, x_k)(f_1(y_1, \ldots, y_l, x_{k+1}, \ldots, x_n) + (\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k) f_2(y_1, \ldots, y_l, x_{k+1}, \ldots, x_n))$ is the general solution of $(b_1D_{x_1} + b_2D_{x_2} + \cdots + b_kD_{x_k} + b_{k+1})^2 u = 0$. Using mathematical induction, we set

\[
u = \Gamma = g(x_1, \ldots, x_k)^m (\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k)^{j-1} f_j (y_1, \ldots, y_l, x_{k+1}, \ldots, x_n).
\]
is the general solution of \((b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k} + b_{k+1})^m u = D^{(m)} u = 0\), namely \(D^{(m)} \Gamma = 0\), so \(u = \Gamma\) is also the solution of \((b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k} + b_{k+1})^{m+1} u = D^{(m+1)} u = 0\).

Setting \(u = h(x_1, \ldots, x_k)\) is the solution of \(D^{(m+1)} u = 0\) yet, that is \(h(x_1, \ldots, x_k) = \lambda x_s \Gamma, \lambda \in \mathbb{R}\) and \(\lambda\) is arbitrary constants, \((s = 1, 2, \ldots, k)\), by Eq. (17) we obtain

\[
\sum D_{x_s} x_{x_s} (h \Gamma) = \lambda x_s \Gamma (l_1 x_{x_1 x_2, \ldots, x_s, x_k} + l_s x_s \Gamma (l_{1,2,\ldots,(s-1),l_s})).
\]

Then

\[
(b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k} + b_{k+1})^{m+1} \Gamma = 0.
\]

So

\[
(b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k} + b_{k+1})^{m+1} (h \Gamma)
\]

\[
= \sum l_{1,2,\ldots, l_{k+1}=m} (l_{1,2,\ldots, l_{k+1}} \Gamma (l_{1,2,\ldots,(s-1),l_s} + l_s x_s \Gamma (l_{1,2,\ldots,(s-1),l_s})).
\]
Since
\[ (b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k} + b_{k+1})^m \Gamma \]
\[ = \left( b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k} + b_{k+1} \right) (b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k} + b_{k+1})^{m-1} \Gamma . \]
\[ = (b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k} + b_{k+1}) \left( \sum_{l_1 + l_2 + \cdots + (l_{m-1}) + l_{k+1} = m-1} \frac{(m-1)!}{l_1!l_2! \cdots l_{m-1}!l_{k+1}!} b_1^{l_1} b_2^{l_2} \cdots b_{k+1}^{l_{k+1}} R_{x_1 \cdots x_k} \right) \]
\[ = 0. \]

We get
\[ \sum_{l_1 + l_2 + \cdots + (l_{m-1}) + l_{k+1} = m-1} \frac{(m-1)!}{l_1!l_2! \cdots l_{m-1}!l_{k+1}!} (b_1^{l_1} b_2^{l_2} \cdots b_{k+1}^{l_{k+1}} R_{x_1 \cdots x_k}) = 0, \]
\[ \text{for the } \theta_i, \text{ according to the principle of linear superposition, the general solution of} \]
\[ \text{Eq.} (38) \text{ is} \]
\[ \text{Eq.} (39), \text{ so the theorem is proved.} \]

Then
\[ \sum_{l_1 + l_2 + \cdots + (l_{m-1}) + l_{k+1} = m} \frac{m! \lambda}{l_1!l_2! \cdots l_{m-1}!l_{k+1}!} (b_1^{l_1} b_2^{l_2} \cdots b_{k+1}^{l_{k+1}} \Gamma_{x_1 \cdots x_k}) \]
\[ = \lambda \sum_{l_1 + l_2 + \cdots + (l_{m-1}) + l_{k+1} = m} \frac{(m-1)!}{l_1!l_2! \cdots l_{m-1}!l_{k+1}!} \Gamma_{x_1 \cdots x_k} \]
\[ \Gamma_{x_1 \cdots x_k} = 0. \]

Namely
\[ (b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k} + b_{k+1})^{m+1} (\lambda x_\theta \Gamma) = 0. \]

Since \( s = 1, 2, \ldots, k \), according to the principle of linear superposition, the general solution of Eq. (38) is (39), so the theorem is proved. \( \square \)

In \( \mathbb{R}^n \), for the \( m \)-th order linear PDE with variable coefficients
\[ \sum_{0 \leq i_1 + i_2 + \cdots + i_k \leq m} a_{i_1i_2 \cdots i_k} u_{i_1i_2 \cdots i_k} = 0. \]

where \( i_r \) are natural number, \( 1 \leq r \leq k \leq n \). If Eq. (45) can be translated into
\[ (b_1 D_{x_1} + b_1 D_{x_2} + \cdots + b_k D_{x_k} + b_{k+1}) (b_2 D_{x_1} + b_2 D_{x_2} + \cdots + b_2 D_{x_k} + b_{k+1}) \]
\[ \cdots \]
\[ = 0. \]

If the particular solution \( u = g_j(x_1, \ldots, x_k) \) of \( (b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k} + b_{k+1}) u = 0 \) and exact solutions \( u = y_j, (x_1, \ldots, x_k) \) of \( (b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k}) u = 0 \), which are independent of each other, are all known, \( 1 \leq j \leq m, 1 \leq s \leq l_j \leq k - 1 \), by Theorem 6 the general solution of Eq. (45) is
\[ u(x_1, \ldots, x_n) = \sum_{j=1}^{m} \left( g_j(x_1, \ldots, x_k) f_j(y_{j_1}, y_{j_2}, \ldots, y_{j_l}, x_{k+1}, x_{k+2}, \ldots, x_n) \right). \]

If Eq. (45) can be translated into:
\[ \prod_{j=1}^{q} (b_{j_1} D_{x_1} + b_{j_2} D_{x_2} + \cdots + b_{j_m} D_{x_n} + b_{j_{m+1}})^{p_j} u = 0, \]
where $\sum_{j=1}^{q} p_j = m$, its general solution can be written by Theorem 8.

Next we propose Theorem 9.

**Theorem 9.** In $\mathbb{R}^n$, if the particular solution $g = g(x_1, \ldots, x_k)$ of $a_1 g_{x_1} + a_2 g_{x_2} + \cdots + a_k g_{x_k} + a_{k+1} g = 0$ and the exact solutions $u = y_i(x_1, \ldots, x_k)$ of $a_1 u_{x_1} + a_2 u_{x_2} + \cdots + a_k u_{x_k} = 0$, which are independent of each other, are all known, $(1 \leq i \leq k-1)$, then

$$a_1 u_{x_1} + a_2 u_{x_2} + \cdots + a_k u_{x_k} + a_{k+1} u = A(x_1, \ldots, x_n),$$

the general solution of Eq. (49) is

$$u = g(x_1, \ldots, x_k) f(y_1, \ldots, y_{k+1}, \ldots, y_n) + \int \frac{A(y_1, \ldots, y_{k+1}, \ldots, y_n) dy_k}{(a_1 y_{k+1} + a_2 y_{k+2} + \cdots + a_k y_{k+1}) g},$$

where $y_1, y_2, \ldots, y_k$ are independent of each other.

**Prove.** By $Z_3$ Transformation, set $u = g(x_1, \ldots, x_k) f(y_1, \ldots, y_{k+1}, \ldots, y_n)$, and $a_s(x_1, \ldots, x_n) = a_s(y_1, \ldots, y_k, x_{k+1}, \ldots, x_n)$, $(1 \leq s \leq k+1)$, where $y_t = y_t(x_1, \ldots, x_k)$, $(t = 1, 2, \ldots, k)$, $y_1, y_2, \ldots, y_k$ are independent of each other, and $x_j = x_j(y_1, \ldots, y_k)$, $(1 \leq j \leq k)$, then $u = y_k(x_1, \ldots, x_k)$ is not the exact solutions of $a_1 u_{x_1} + a_2 u_{x_2} + \cdots + a_k u_{x_k} = 0$.

$$a_1 u_{x_1} + a_2 u_{x_2} + \cdots + a_k u_{x_k} + a_{k+1} u = 0,$$

$$= a_1 \left( f g_{x_1} + g \sum_{t=1}^{k} f y_t y_{x_1} \right) + a_2 \left( f g_{x_2} + g \sum_{t=1}^{k} f y_t y_{x_2} \right) + \cdots$$

$$+ a_k \left( f g_{x_k} + g \sum_{t=1}^{k} f y_t y_{x_k} \right) + a_{k+1} g f$$

$$= f \left( a_1 g_{x_1} + \cdots + a_k g_{x_k} + a_{k+1} g \right) + g \sum_{t=1}^{k} f y_t \left( a_1 y_{x_1} + a_2 y_{x_2} + \cdots + a_k y_{x_k} \right).$$

Since $a_1 g_{x_1} + \cdots + a_k g_{x_k} + a_{k+1} g = 0$ and $a_1 y_{x_1} + a_2 y_{x_2} + \cdots + a_k y_{x_k} = 0$, $(1 \leq i \leq k-1)$, so

$$a_1 u_{x_1} + a_2 u_{x_2} + \cdots + a_k u_{x_k} + a_{k+1} u = g f y_k \left( a_1 y_{x_1} + a_2 y_{x_2} + \cdots + a_k y_{x_k} \right)$$

$$= A(y_1, \ldots, y_k, x_{k+1}, \ldots, x_n).$$

We get

$$f = \int \frac{A(y_1, \ldots, y_k, x_{k+1}, \ldots, x_n) dy_k}{(a_1 y_{x_1} + a_2 y_{x_2} + \cdots + a_k y_{x_k}) g}.$$ 

A particular solution of Eq. (34) is

$$u = g \int \frac{A(y_1, \ldots, y_k, x_{k+1}, \ldots, x_n) dy_k}{(a_1 y_{x_1} + a_2 y_{x_2} + \cdots + a_k y_{x_k}) g}.$$  

(51)

To compute (51) we need to find the first differentiable $y_k = y_k(x_1, x_2, \ldots, x_k)$ and $y_1, y_2, \ldots, y_k$ independent of each other, and can get $x_j = x_j(y_1, y_2, \ldots, y_k)$. According to Theorem 6, the general solution of Eq. (49) is (50), so Theorem 9 is proved. □

Note that Theorem 9 obtains the general solution of the variable coefficient transport equation.

We propose Theorem 10 as follows.

**Theorem 10.** In $\mathbb{R}^n$, if the particular solution $g = g(x_1, \ldots, x_k)$ of $b_1 g_{x_1} + b_2 g_{x_2} + \cdots + b_k g_{x_k} +$
\( b_{k+1}g = 0 \) and exact solutions \( u = y_i(x_1, \ldots, x_k) \) of \( b_1u_{x_1} + b_2u_{x_2} + \cdots + b_ku_{x_k} = 0 \), which are independent of each other, are all known, \( (1 \leq i \leq k - 1) \), then

\[
(b_1D_{x_1} + b_2D_{x_2} + \cdots + b_kD_{x_k} + b_{k+1})^m u = A(x_1, \ldots, x_n),
\]

the general solution of Eq. (52) is

\[
u = g(x_1, \ldots, x_k) \sum_{j=1}^{m} (\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k)^{j-1} f_j (y_1, \ldots, y_{k-1}, x_{k+1}, \ldots, x_n) + g \int \cdots \int g^{-1} A(y_1, \ldots, y_k, x_{k+1}, \ldots, x_n) d^m y_k,\]

where \( y_1, y_2, \ldots, y_k \) are independent of each other, \( y_k = c_1 x_1 + c_2 x_2 + \cdots + c_k x_k \), and \( x_j = x_j(y_1, y_2, \ldots, y_k) \) can be got, \( (1 \leq j \leq k) \).

**Proof.** By \( Z_3 \) Transformation, set \( u = g(x_1, \ldots, x_k)f(y_1, \ldots, y_k, x_{k+1}, \ldots, x_n) \) and \( a_i(x_1, x_2, \ldots, x_n) = a_i(y_1, y_2, \ldots, y_k, x_{k+1}, \ldots, x_n). y_1, y_2, \ldots, y_k \) are independent of each other, \( y_s = y_s(x_1, x_2, \ldots, x_k), \) \( (s = 1, 2, \ldots, k) \), and

\[
y_k = c_1 x_1 + c_2 x_2 + \cdots + c_k x_k.
\]

Note that \( u = y_k(x_1, \ldots, x_k) \) is not the exact solutions of \( b_1u_{x_1} + b_2u_{x_2} + \cdots + b_ku_{x_k} = 0 \), then

\[
(b_1D_{x_1} + b_2D_{x_2} + \cdots + b_kD_{x_k} + b_{k+1})^m u = (b_1D_{x_1} + b_2D_{x_2} + \cdots + b_kD_{x_k} + b_{k+1})^m a_i(x_1, x_2, \ldots, x_n) + g \int \cdots \int g^{-1} A(y_1, \ldots, y_k, x_{k+1}, \ldots, x_n) d^m y_k.
\]

Then

\[
(b_1D_{x_1} + b_2D_{x_2} + \cdots + b_kD_{x_k} + b_{k+1})^m u = (b_1D_{x_1} + b_2D_{x_2} + \cdots + b_kD_{x_k} + b_{k+1})^m g f^{(m)}(y_k) = A(y_1, \ldots, y_k, x_{k+1}, \ldots, x_n).
\]
We get
\[ f = \int \cdots \int g^{-1} A(y_1, \ldots, y_k, x_{k+1}, \ldots, x_n) \, d^m y_k. \]
So the particular solution of Eq. (52) is
\[ u = g \int \cdots \int g^{-1} A(y_1, \ldots, y_k, x_{k+1}, \ldots, x_n) \, d^m y_k. \]  
(56)

By Theorem 8, the general solution of Eq. (52) is (53). So Theorem 10 is proved. □

2. General solutions’ laws of linear partial differential equations with constant coefficients

Here we will research the general solutions’ laws of linear PDEs with constant coefficients, which are the special cases of linear PDEs with variable coefficients. In this section, if there is no special interpretation, the acquiescent independent variables of \( \mathbb{R}^n \) are \( x_1, x_2, \ldots, x_n \); \( D_{x_i} \equiv \frac{\partial}{\partial x_i} \), \( 2 \leq k \leq n \); \( a_i, b_i, b_j \) and \( a_{i_1 i_2 \ldots i_p} \) are arbitrary known constants, \( c_i, c_{ij}, c_{i_1j}, l_i, l_j, \lambda_i, C_i \) and \( C \) are arbitrary constants, \( f \) and \( f_i \) are arbitrary smooth functions \( (i, j, s = 1, 2, \ldots) \).

Theorem 11. In \( \mathbb{R}^n \),
\[ a_1 u_{x_1} + a_2 u_{x_2} + \cdots + a_k u_{x_k} = A(x_1, x_2, \ldots, x_n), \quad (2 \leq k \leq n), \]  
(57)
the general solution of Eq. (57) is
\[ u = f(y_1, y_2, \ldots, y_{k-1}, x_{k+1}, x_{k+2}, \ldots, x_n) + \frac{\int A(y_1, y_2, \ldots, y_k, x_{k+1}, x_{k+2}, \ldots, x_n) \, dy_k}{a_1 c_{k1} + a_2 c_{k2} + \cdots + a_k c_{kk}}, \]  
(58)
where
\[ y_i = c_{i1} x_1 + c_{i2} x_2 + \cdots + c_{ik} x_k, \quad (1 \leq i \leq k), \]  
(59)
\[ \frac{\partial}{\partial (x_1, x_2, \ldots, x_n)} (y_1, y_2, \ldots, y_k, x_{k+1}, x_{k+2}, \ldots, x_n) \neq 0, \]  
(60)
\[ c_{11} = -a_2 c_{i2} - a_3 c_{i3} - \cdots - a_k c_{ik}, \quad (1 \leq i \leq k - 1). \]  
(61)

Prove. According to \( Z_1 \) Transformation, set \( u(x_1, x_2, \ldots, x_n) = u(y_1, y_2, \ldots, y_k, x_{k+1}, x_{k+2}, \ldots, x_n) \), \( A(x_1, x_2, \ldots, x_n) = A(y_1, y_2, \ldots, y_k, x_{k+1}, x_{k+2}, \ldots, x_n) \) and
\[ \begin{cases} 
  y_1 = c_{11} x_1 + c_{12} x_2 + \cdots + c_{1k} x_k \\
  y_2 = c_{21} x_1 + c_{22} x_2 + \cdots + c_{2k} x_k \\
  \vdots \\
  y_k = c_{k1} x_1 + c_{k2} x_2 + \cdots + c_{kk} x_k,
\end{cases} \]  
(62)
and
\[ \frac{\partial (y_1, y_2, \ldots, y_k)}{\partial (x_1, x_2, \ldots, x_n)} \neq 0, \]  
(60)
where \( c_{ij} \) are undetermined constants. According to (60, 62), \( x_i = x_i(y_1, y_2, \ldots y_k) \) always has a unique solution \((1 \leq i, j \leq k)\). So

\[
a_1 u_{x_1} + a_2 u_{x_2} + \cdots + a_k u_{x_k} \\
= a_1 \sum_{i=1}^{k} c_{1i} y_i + a_2 \sum_{i=1}^{k} c_{2i} y_i + \cdots + a_k \sum_{i=1}^{k} c_{ki} y_i \\
= (a_1 c_{11} + a_2 c_{12} + \cdots + a_k c_{1k}) u_{y_1} + (a_1 c_{21} + a_2 c_{22} + \cdots + a_k c_{2k}) u_{y_2} + \cdots \\
+ (a_1 c_{k1} + a_2 c_{k2} + \cdots + a_k c_{kk}) u_{y_k}.
\]

Set

\[
a_1 c_{11} + a_2 c_{12} + \cdots + a_k c_{1k} = a_1 c_{21} + a_2 c_{22} + \cdots + a_k c_{2k} \\
= \cdots = a_1 c_{(k-1)1} + a_2 c_{(k-1)2} + \cdots + a_k c_{(k-1)k} = 0.
\]

We obtain

\[
c_{i1} = \frac{-a_2 c_{i2} - a_3 c_{i3} - \cdots - a_k c_{ik}}{a_1}, (1 \leq i \leq k - 1).
\]

Then

\[
a_1 u_{x_1} + a_2 u_{x_2} + \cdots + a_k u_{x_k} = (a_1 c_{k1} + a_2 c_{k2} + \cdots + a_k c_{kk}) u_{y_k} \\
= A(y_1, y_2, \ldots y_k, x_{k+1}, x_{k+2}, \ldots x_n).
\]

The general solution of (63) is

\[
u = f(y_1, y_2, \ldots y_{k-1}, x_{k+1}, x_{k+2}, \ldots x_n) + \frac{\int A(y_1, y_2, \ldots y_k, x_{k+1}, x_{k+2}, \ldots x_n) \, dy_k}{a_1 c_{k1} + a_2 c_{k2} + \cdots + a_k c_{kk}}.
\]

So Theorem 11 is proved. □

According to Theorem 11, in \( \mathbb{R}^n \),

\[
a_1 u_{x_1} + a_2 u_{x_2} + \cdots + a_n u_{x_n} = A(x_1, x_2, \ldots x_n), \quad (64)
\]

the general solution of Eq. (64) is

\[
u = f(y_1, y_2, \ldots y_{n-1}) + \frac{\int A(y_1, y_2, \ldots y_n) \, dy_n}{a_1 c_{n1} + a_2 c_{n2} + \cdots + a_n c_{nn}},
\]

where

\[
y_i = c_{i1} x_1 + c_{i2} x_2 + \cdots + c_{in} x_n,
\]

\[
c_{i1} = \frac{-a_2 c_{i2} - a_3 c_{i3} - \cdots - a_n c_{in}}{a_1}, (1 \leq i \leq n - 1).
\]

According to Theorem 11 we can get Theorem 12:

**Theorem 12.** In \( \mathbb{R}^n \),

\[
a_1 u_{x_1} + a_2 u_{x_2} + \cdots + a_k u_{x_k} = 0,
\]

the general solution of Eq. (68) is

\[
u = f(y_1, y_2, \ldots y_{k-1}, x_{k+1}, x_{k+2}, \ldots x_n),
\]

where \( y_i \) satisfy (59 — 61).

The proof of Theorem 12 is not complicated, the readers may try it.
According to Theorem 2 and 12, we can get directly Theorem 13:

**Theorem 13.** In $\mathbb{R}^n$,
\[
(b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k})^2 u = 0,
\]
the general solution of Eq. (70) is
\[
u = f_1(y_1, \ldots, y_{k-1}, x_{k+1}, \ldots, x_n) + (\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k) f_2(y_1, \ldots, y_{k-1}, x_{k+1}, \ldots, x_n).
\]
where $y_i$ satisfy (59, 60) and
\[
c_i = \frac{-b_2 c_i - b_3 c_{i3} - \cdots - b_k c_{ik}}{a_1}, (1 \leq i \leq k - 1).
\]

According to Theorem 3, we can present Theorem 14:

**Theorem 14.** In $\mathbb{R}^n$, the general solution of
\[
(b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k})^m u = 0,
\]
is
\[
u = \sum_{i=1}^{m} (\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k)^{i-1} f_i(y_1, \ldots, y_{k-1}, x_{k+1}, \ldots, x_n).
\]
where $y_i$ satisfy (59, 60) and (72).

In $\mathbb{R}^n$, for the $m$th-order linear PDE with constant coefficients
\[
\sum_{i_1+i_2+\cdots+i_k=m} a_{i_1i_2\ldots i_k} u^{(i_1i_2\ldots i_k)}_{x_1x_2\ldots x_k} = 0,
\]
where $i_j$ are natural number, $1 \leq j \leq k \leq n$. If Eq. (75) can be translated into
\[
(b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k})(b_2 D_{x_1} + b_2 D_{x_2} + \cdots + b_3 D_{x_k})\cdots(b_m D_{x_1} + b_m D_{x_2} + \cdots + b_m D_{x_k})u = 0.
\]
According to Theorem 12, we can get the general solution of Eq. (75) is
\[
u = \sum_{r=1}^{m} f_r(y_{r_1}, y_{r_2}, \ldots, y_{r_{k-1}}, x_{k+1}, \ldots, x_n), (1 \leq r \leq m),
\]
where
\[
y_{r_s} = c_{r_s} x_1 + c_{r_s} x_2 + \cdots + c_{r_s} x_k,
\]
\[
c_{r_s} = \frac{-b_{s_2} c_{r_s} - b_{s_3} c_{r_s} - \cdots - b_{s_k} c_{r_s}}{b_{s_1}}, (1 \leq s \leq k - 1).
\]
If Eq. (75) can be translated into:
\[
\prod_{j=1}^{q} (b_{j_1} D_{x_1} + b_{j_2} D_{x_2} + \cdots + b_{j_k} D_{x_k})^{p_j} u = 0,
\]
where $\sum_{j=1}^{q} p_j = m$, its general solution can be written by Theorem 14.
According to Theorem 5 and 12, we can present Theorem 15.

**Theorem 15.** In $\mathbb{R}^n$, 
\[
(b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k})^m u = A(x_1, \ldots, x_n),
\]
the general solution of Eq. (81) is
\[
u = \frac{\int \ldots \int A(y_1, \ldots, y_k, x_{k+1}, \ldots, x_n) \, d^m y}{(b_1 c_{k1} + b_2 c_{k2} + \cdots + b_k c_{kk})^m}
+ \sum_{j=1}^{m} (\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k)^{j-1} f_j (y_1, \ldots, y_{k-1}, x_{k+1}, \ldots, x_n),
\]
where $y_i$ satisfy (59, 60) and (72).

Next we propose Theorem 16.

**Theorem 16.** In $\mathbb{R}^n$,
\[
a_1 u_{x_1} + a_2 u_{x_2} + \cdots + a_k u_{x_k} + a_{k+1} u = 0,
\]
the general solution of Eq. (83) is
\[
u = \sum_{i=1}^{k} l_i c \frac{\partial (y_1, y_2, \ldots, y_k)}{\partial (x_1, x_2, \ldots, x_k)}
\frac{e^{-a_{k+1} x_i}}{u_i},
\]
where $l_i$ are arbitrary constants and $y_i$ satisfy (59 – 61).

**Prove.** According to $Z_3$ Transformation, set $u(x_1, \ldots, x_n) = g(x_1, \ldots, x_k) h(y_1, \ldots, y_k, x_{k+1}, \ldots, x_n)$, $y_i = c_{i1} x_1 + c_{i2} x_2 + \cdots + c_{ik} x_k$, and
\[
\frac{\partial (y_1, y_2, \ldots, y_k)}{\partial (x_1, x_2, \ldots, x_k)} \neq 0.
\]

So
\[
\begin{align*}
a_1 u_{x_1} + a_2 u_{x_2} + \cdots + a_k u_{x_k} + a_{k+1} u \\
= a_1 g_{x_1} h + a_1 g \sum_{i=1}^{k} c_{i1} h_{y_i} + a_2 g_{x_2} h + a_2 g \sum_{i=1}^{k} c_{i2} h_{y_i} + \cdots + a_k g_{x_k} h + a_k g \sum_{i=1}^{k} c_{ik} h_{y_i} \\
+ a_{k+1} g h \\
= (a_1 c_{i1} + a_2 c_{i2} + \cdots + a_k c_{ik}) g h_{y_i} + (a_1 c_{i1} + a_2 c_{i2} + \cdots + a_k c_{ik}) g h_{y_2} + \cdots \\
+ (a_1 c_{i1} + a_2 c_{i2} + \cdots + a_k c_{ik}) g h_{y_k} + (a_1 g_{x_1} + a_2 g_{x_2} + \cdots + a_k g_{x_k} + a_{k+1} g) h = 0.
\end{align*}
\]

Set
\[
c_{i1} = \frac{-a_2 c_{i2} - a_3 c_{i3} - \cdots - a_k c_{ik}}{a_1}, (1 \leq i \leq k - 1),
\]
\[
a_1 g_{x_1} + a_2 g_{x_2} + \cdots + a_k g_{x_k} + a_{k+1} g = 0.
\]
And set $g(x_1, \ldots, x_k) = g(x_i), (i = 1, 2, \ldots, k)$, Then
\[
\begin{align*}
a_1 g_{x_1} + a_2 g_{x_2} + \cdots + a_k g_{x_k} + a_{k+1} g = a_1 g_{x_1} + a_{k+1} g = 0 \\
\Rightarrow g(x_i) = l_i e^{-a_{k+1} x_i}. 
\end{align*}
\]
where \( l_i \) are arbitrary constants. Namely \( g(x_1, \ldots, x_k) = \sum_{i=1}^{k} l_i e^{-a_{k+1} x_i} \) is a particular solution of 
\[ a_1 g_{x_1} + a_2 g_{x_2} + \cdots + a_k g_{x_k} + a_{k+1} g = 0, \]
thus
\[ a_1 u_{x_1} + a_2 u_{x_2} + \cdots + a_k u_{x_k} + a_{k+1} u = (a_1 c_{k1} + a_2 c_{k2} + \cdots + a_k c_{kk}) g h_{y_k} = 0 \Rightarrow h_{y_k} = 0. \]
We get
\[ h = f\left(y_1, y_2, \ldots, y_{k-1}, x_{k+1}, x_{k+2}, \ldots, x_n\right), \]
where \( f \) is an arbitrary first differentiable function, so the general solution of Eq. (83) is
\[ u = gh = f\left(y_1, y_2, \ldots, y_{k-1}, x_{k+1}, x_{k+2}, \ldots, x_n\right) \sum_{i=1}^{k} l_i e^{-a_{k+1} x_i}. \]
\( \square \)

According to Theorem 8 and 16, we can present Theorem 17:

**Theorem 17.** In \( \mathbb{R}^n \),
\[ (b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k} + b_{k+1})^m u = 0, \]  
the general solution of Eq. (87) is
\[ u = \left( \sum_{j=1}^{m} \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k \right)^{j-1} f_j \left(y_1, \ldots, y_{k-1}, x_{k+1}, \ldots, x_n\right) \sum_{i=1}^{k} l_i e^{-a_{k+1} x_i}, \]
where \( \lambda_j \) and \( \beta_i \) are arbitrary constants, \( y_j \) satisfy (59, 60) and (72).

In \( \mathbb{R}^n \), for the \( m \)th-order linear PDE with constant coefficients
\[ \sum_{0 \leq i_1 + i_2 + \cdots + i_k \leq m} a_{i_1 i_2 \ldots i_k} u^{(i_1 \ldots i_k)} = 0, \]
where \( i_j \) are natural number, \( 1 \leq j \leq k \leq n \), If Eq. (89) can be translated into
\[ (b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k} + b_{k+1}) \left( b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k} + b_{k+1} \right) \]
\[ \cdots \left( b_{m_1} D_{x_1} + b_{m_2} D_{x_2} + \cdots + b_{m_k} D_{x_k} + b_{m_{k+1}} \right) \]
\[ u = 0. \]
By Theorem 16, the general solution of Eq. (89) is
\[ u = \sum_{r=1}^{m} \left( f_{r} (y_{r_1}, y_{r_2}, \ldots, y_{r_{k-1}}, x_{k+1}, \ldots, x_n) \sum_{i=1}^{k} l_i e^{-a_{k+1} x_i} \right), \]
where \( y_{r_s} \) satisfies (78, 79), \( 1 \leq s \leq k - 1 \).

If Eq. (89) can be translated into:
\[ \prod_{r=1}^{q} \left( b_{r_1} D_{x_1} + b_{r_2} D_{x_2} + \cdots + b_{r_k} D_{x_k} + b_{r_{k+1}} \right)^{p_r} u = 0, \]
where \( \sum_{r=1}^{q} p_r = m \), its general solution can be written by Theorem 17.
Using Theorem 9 and 16, we can get Theorem 18.

**Theorem 18.** In \( \mathbb{R}^n \),
\[
a_1u_{x_1} + a_2u_{x_2} + \cdots + a_ku_{x_k} + a_{k+1}u = A(x_1, x_2, \ldots, x_n), \quad (2 \leq k \leq n),
\]
the general solution of Eq. (93) is
\[
u = e^{-\frac{a_{k+1}x_1}{a_1}} \left( f(y_1, \ldots, y_{k-1}, x_{k+1}, \ldots, x_n) + \int e^{\frac{a_{k+1}y_1}{a_1}} A(y_1, \ldots, y_k, x_{k+1}, \ldots, x_n) dy_k \right),
\]
where \( y_i \) satisfy (59 - 61).

According to Theorem 9 and 16,
\[
u = \sum_{i=1}^{k} l_i e^{-\frac{a_{k+1}x_i}{a_i}} \left( f(y_1, \ldots, y_{k-1}, x_{k+1}, \ldots, x_n) + \int \frac{A(y_1, \ldots, y_k, x_{k+1}, \ldots, x_n)}{(a_1c_{k1} + \cdots + a_kc_{kk}) \sum_{i=1}^{k} l_i e^{-\frac{a_{k+1}x_i}{a_i}}} dy_k \right),
\]
is the general solution of Eq. (93) too, however, in the specific calculation, (95) may be more complicated. In order to simplify the calculation, in Theorem 18 we choose
\[
g(x_1, \ldots, x_k) = e^{-\frac{a_{k+1}x_1}{a_1}}.
\]
So (95) is simplified to (94).

Below we use a new method to calculate the general solution of Eq. (93) and propose Theorem 19.

**Theorem 19.** In \( \mathbb{R}^n \),
\[
a_1u_{x_1} + a_2u_{x_2} + \cdots + a_ku_{x_k} + a_{k+1}u = A(x_1, x_2, \ldots, x_n), \quad (2 \leq k \leq n),
\]
the general solution of Eq. (93) is
\[
u = f(y_1, \ldots, y_{k-1}, x_{k+1}, \ldots, x_n) e^{-\frac{a_{k+1}x_1}{a_1}}
+ e^{-a_{k+1}B y_k} \left( C - B \int e^{a_{k+1}B y_k} A(y_1, \ldots, y_k, x_{k+1}, \ldots, x_n) dy_k \right),
\]
where \( y_i \) satisfy (59 - 61), and
\[
B = (a_1c_{k1} + a_2c_{k2} + \cdots + a_kc_{kk})^{-1}.
\]

**Prove.** According to Z1 Transformation, set \( u(x_1, x_2, \ldots, x_n) = u(y_1, y_2, \ldots, y_k, x_{k+1}, x_{k+2}, \ldots, x_n) \), \( A(x_1, x_2, \ldots, x_n) = A(y_1, y_2, \ldots, y_k, x_{k+1}, x_{k+2}, \ldots, x_n) \) and
\[
\begin{align*}
y_1 &= c_{11}x_1 + c_{12}x_2 + \cdots + c_{1k}x_k \\
y_2 &= c_{21}x_1 + c_{22}x_2 + \cdots + c_{2k}x_k \\
\vdots \\
y_k &= c_{k1}x_1 + c_{k2}x_2 + \cdots + c_{kk}x_k.
\end{align*}
\]
and
\[ \frac{\partial (y_1, y_2, \ldots, y_k)}{\partial (x_1, x_2, \ldots, x_k)} \neq 0, \]
where \(c_{ij}\) are undetermined constants. According to (60, 62), \(x_i = x_i(y_1, y_2, \ldots, y_k)\) always has a unique solution \((1 \leq i, j \leq k)\). So
\[
\begin{align*}
& a_1 u_{x_1} + a_2 u_{x_2} + \cdots + a_k u_{x_k} + a_{k+1} u \\
& = a_1 \sum_{i=1}^{k} c_{1i} u_{y_i} + a_2 \sum_{i=1}^{k} c_{2i} u_{y_i} + \cdots + a_k \sum_{i=1}^{k} c_{ki} u_{y_i} + a_{k+1} u \\
& = (a_1 c_{11} + a_2 c_{12} + \cdots + a_k c_{1k}) u_{y_1} + (a_1 c_{21} + a_2 c_{22} + \cdots + a_k c_{2k}) u_{y_2} + \cdots \\
& + (a_1 c_{k1} + a_2 c_{k2} + \cdots + a_k c_{kk}) u_{y_k} + a_{k+1} u.
\end{align*}
\]
Set
\[
\begin{align*}
& a_1 c_{11} + a_2 c_{12} + \cdots + a_k c_{1k} = a_1 c_{21} + a_2 c_{22} + \cdots + a_k c_{2k} \\
& = a_1 c_{(k-1)1} + a_2 c_{(k-1)2} + \cdots + a_k c_{(k-1)k} = 0.
\end{align*}
\]
We obtain
\[ c_{i1} = -\frac{a_2 c_{i2} - a_3 c_{i3} - \cdots - a_k c_{ik}}{a_1}, (1 \leq i \leq k - 1). \]

Then
\[
\begin{align*}
& a_1 u_{x_1} + a_2 u_{x_2} + \cdots + a_k u_{x_k} + a_{k+1} u = (a_1 c_{k1} + a_2 c_{k2} + \cdots + a_k c_{kk}) u_{y_k} + a_{k+1} u \\
& = A(y_1, y_2, \ldots, y_k, x_{k+1}, x_{k+2}, \ldots, x_n).
\end{align*}
\]

Namely
\[
(a_1 c_{k1} + a_2 c_{k2} + \cdots + a_k c_{kk}) u_{y_k} + a_{k+1} u = A(y_1, y_2, \ldots, y_k, x_{k+1}, x_{k+2}, \ldots, x_n).
\]

According to the general solution of the first-order linear ordinary differential equation, it is not difficult to find the solution of Eq. (98) is
\[
\begin{align*}
& u = e^{-a_1 x_1 + a_2 x_2 + \cdots + a_k x_k} \\
& \quad \times C + \int e^{-a_1 x_1 + a_2 x_2 + \cdots + a_k x_k} A(y_1, y_2, \ldots, y_k, x_{k+1}, x_{k+2}, \ldots, x_n) \\
& \quad \times a_1 c_{k1} + a_2 c_{k2} + \cdots + a_k c_{kk} dy_k.
\end{align*}
\]

Combined with Theorem 16, the general solution of Eq. (93) is (96), so the theorem is proved. □

Theorems 9, 18, and 19 have special significance because they reveal the laws of the general solution of the transport equation. Since the transport equation has important applications in many fields, it is one of the most active linear PDEs being studied at present. Current research methods mainly use numerical methods to solve definite solution problems, such as Petrov-Galerkin methods [14], Discontinuous Galerkin Method [15, 16], the finite difference method [17,18], etc., the existence [19, 20] and uniqueness [21, 22] of solutions are also important areas of research.

According to Theorem 10 and 18, we can obtain Theorem 20.

**Theorem 20.** In \(\mathbb{R}^n\),
\[
(b_1 D_{x_1} + b_2 D_{x_2} + \cdots + b_k D_{x_k} + b_{k+1})^m u = A(x_1, \ldots, x_n),
\]
the general solution of Eq. (99) is
\[
\begin{align*}
u &= e^{-a_1 x_1 + a_2 x_2 + \cdots + a_k x_k} \\
& \quad \times \sum_{j=1}^{m} (\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k)^{j-1} f_j(y_1, \ldots, y_{k-1}, x_{k+1}, \ldots, x_n) \\
& + e^{-a_1 x_1 + a_2 x_2 + \cdots + a_k x_k} \\
& \quad \times \int \int \int e^{-a_1 x_1 + a_2 x_2 + \cdots + a_k x_k} A(y_1, \ldots, y_k, x_{k+1}, \ldots, x_n) \\
& \quad \times b_1 c_{k1} + b_2 c_{k2} + \cdots + b_k c_{kk})^m dy_k,
\end{align*}
\]

where \( y_i \) satisfy (59–61).

If we choose \( \sum_{i=1}^{k} l_i e^{-\frac{a_{k+1}x_i}{a_i}} \) as \( g(x_1, \ldots, x_k) \), the general solution form of Eq. (99) will be more complicated.

Next we propose Theorem 21.

**Theorem 21.** In \( \mathbb{R}^2 \),

\[
a_1u_{xx} + a_2u_{yy} + a_3u_{xy} + a_4u_x + a_5u_y + a_6u = 0,
\]

(101)

the general solution of Eq. (101) is,

\[
u = \left( C_1 e^{\lambda_1 v_1} + C_2 e^{\lambda_2 v_1} \right) h_1 \left( l_1 x + l_2 y + l_3 \right) + \left( C_3 e^{\lambda_3 v_2} + C_4 e^{\lambda_4 v_2} \right) h_2 \left( l_4 x + l_5 y + l_6 \right),
\]

(102)

where \( a_i \) are arbitrary known constants, \( h_1 \) and \( h_2 \) are arbitrary second differentiable functions, and

\[
\lambda_1 = \frac{-a_4k_1 - a_5k_2 + \sqrt{(a_4k_1 + a_5k_2)^2 - 4a_6 (a_1k_1^2 + a_2k_2^2 + a_3k_1k_2)}}{2 (a_1k_1^2 + a_2k_2^2 + a_3k_1k_2)},
\]

(103)

\[
\lambda_2 = \frac{-a_4k_1 - a_5k_2 - \sqrt{(a_4k_1 + a_5k_2)^2 - 4a_6 (a_1k_1^2 + a_2k_2^2 + a_3k_1k_2)}}{2 (a_1k_1^2 + a_2k_2^2 + a_3k_1k_2)},
\]

(104)

\[
\lambda_3 = \frac{-a_4k_1 - a_5k_5 + \sqrt{(a_4k_1 + a_5k_5)^2 - 4a_6 (a_1k_1^2 + a_2k_5^2 + a_3k_4k_5)}}{2 (a_1k_4^2 + a_2k_5^2 + a_3k_4k_5)},
\]

(105)

\[
\lambda_4 = \frac{-a_4k_1 - a_5k_5 - \sqrt{(a_4k_1 + a_5k_5)^2 - 4a_6 (a_1k_1^2 + a_2k_5^2 + a_3k_4k_5)}}{2 (a_1k_4^2 + a_2k_5^2 + a_3k_4k_5)},
\]

(106)

where \( k_3, k_6, l_3 \) and \( l_6 \) are arbitrary constants, \( k_1, k_2, k_4, l_1, l_2, l_4 \) and \( l_5 \) are constants which satisfy

\[
l_1 = \frac{-a_3l_2 + \sqrt{a_2^2l_2^2 - 4a_1a_2l_2^2}}{2a_1}, l_4 = \frac{-a_3l_5 - \sqrt{a_4^2l_5^2 - 4a_1a_2l_5^2}}{2a_1},
\]

(107)

(108)

\[
k_1 \lambda_1 \left( 2a_1l_1 + a_3l_3 \right) + k_2 \lambda_1 \left( 2a_2l_2 + a_3l_1 \right) + a_4l_1 + a_5l_2 = 0,
\]

(109)

\[
k_1 \lambda_2 \left( 2a_1l_1 + a_3l_3 \right) + k_2 \lambda_2 \left( 2a_2l_2 + a_3l_1 \right) + a_4l_1 + a_5l_2 = 0,
\]

(110)

\[
k_4 \lambda_3 \left( 2a_1l_4 + a_3l_5 \right) + k_5 \lambda_3 \left( 2a_2l_5 + a_3l_4 \right) + a_4l_4 + a_5l_5 = 0,
\]

(111)

\[
k_4 \lambda_4 \left( 2a_1l_4 + a_3l_5 \right) + k_5 \lambda_4 \left( 2a_2l_5 + a_3l_4 \right) + a_4l_4 + a_5l_5 = 0,
\]

(112)

\[
(4a_4k_1 + a_5k_2)^2 - 4a_6 \left( a_1k_1^2 + a_2k_2^2 + a_3k_1k_2 \right) > 0,
\]

(113)

\[
(4a_4k_1 + a_5k_5)^2 - 4a_6 \left( a_1k_1^2 + a_2k_5^2 + a_3k_4k_5 \right) > 0.
\]

(114)

**Prove.** By \( Z_1 \) Transformation, set

\[
u \left( x, y \right) = f \left( v \right) = f \left( k_1x + k_2y + k_3 \right),
\]

(115)
where \(v(x, y) = k_1 x + k_2 y + k_3, k_1 - k_3\) are undetermined constants, \(f\) is an undetermined second differentiable function, so

\[
\begin{align*}
    a_1 u_{xx} + a_2 u_{yy} + a_3 u_{xy} + a_4 u_x + a_5 u_y + a_6 u \\
    = \left( a_1 k_1^2 + a_2 k_2^2 + a_3 k_1 k_2 \right) f''_v + \left( a_4 k_1 + a_5 k_2 \right) f'_v + a_6 f &= 0.
\end{align*}
\] (116)

The characteristic equation of Eq. (116) is

\[
\left( a_1 k_1^2 + a_2 k_2^2 + a_3 k_1 k_2 \right) \lambda^2 + \left( a_4 k_1 + a_5 k_2 \right) \lambda + a_6 = 0.
\] (117)

If

\[
\left( a_4 k_1 + a_5 k_2 \right)^2 - 4 a_6 \left( a_1 k_1^2 + a_2 k_2^2 + a_3 k_1 k_2 \right) > 0,
\] (118)

the particular solution of Eq. (101) is

\[
u = f = C_1 e^{\lambda_1 v} + C_2 e^{\lambda_2 v},
\] (119)

where \(C_1\) and \(C_2\) are arbitrary constants, and

\[
\lambda_1 = \frac{-a_4 k_1 - a_5 k_2 + \sqrt{\left( a_4 k_1 + a_5 k_2 \right)^2 - 4 a_6 \left( a_1 k_1^2 + a_2 k_2^2 + a_3 k_1 k_2 \right)}}{2 \left( a_1 k_1^2 + a_2 k_2^2 + a_3 k_1 k_2 \right)},
\]

\[
\lambda_2 = \frac{-a_4 k_1 - a_5 k_2 - \sqrt{\left( a_4 k_1 + a_5 k_2 \right)^2 - 4 a_6 \left( a_1 k_1^2 + a_2 k_2^2 + a_3 k_1 k_2 \right)}}{2 \left( a_1 k_1^2 + a_2 k_2^2 + a_3 k_1 k_2 \right)}.
\]

For getting the general solution of Eq. (101), using \(Z_3\) Transformation, we set

\[
u(x, y) = gh(w) = g(x, y) h(l_1 x + l_2 y + l_3),
\] (120)

where \(w(x, y) = l_1 x + l_2 y + l_3\), \(l_1 - l_3\) are undetermined constants, \(h\) and \(g\) are undetermined second differentiable functions, so

\[
\begin{align*}
    a_1 u_{xx} + a_2 u_{yy} + a_3 u_{xy} + a_4 u_x + a_5 u_y + a_6 u \\
    = a_1 \left( g_{xx} h + 2l_1 g_x h'_w + l_1^2 g'' h'_w \right) + a_2 \left( g_{yy} h + 2l_2 g_y h'_w + l_2^2 g'' h'_w \right) \\
    + a_3 \left( g_{xy} h + l_2 g_x h'_w + l_1 g_y h'_w + l_1 l_2 g'' h'_w \right) + a_4 \left( g_x h + l_1 g h'_w \right) + a_5 \left( g_y h + l_2 g h'_w \right) + a_6 g h.
\end{align*}
\]

Namely

\[
\begin{align*}
    \left( a_1 l_1^2 + a_2 l_2^2 + a_3 l_1 l_2 \right) g h''_w + (2a_1 l_1 + a_3 l_2) g_x + (2a_2 l_2 + a_3 l_1) g_y + (a_4 l_1 + a_5 l_2) g h'_w \\
    + (a_1 g_{xx} + a_2 g_{yy} + a_3 g_{xy} + a_4 g_x + a_5 g_y + a_6 g) h = 0.
\end{align*}
\] (121)

Set \(h(w)\) is an arbitrary second differentiable function, and

\[
\begin{align*}
    a_1 l_1^2 + a_3 l_2 l_1 + a_2 l_2^2 &= 0 \implies l_1 = \frac{-a_3 l_2 \pm \sqrt{a_3^2 l_2^2 - 4a_1 a_2 l_2^2}}{2a_1},
\end{align*}
\] (122)

\[
\begin{align*}
    (2a_1 l_1 + a_3 l_2) g_x + (2a_2 l_2 + a_3 l_1) g_y + (a_4 l_1 + a_5 l_2) g h'_w &= 0,
\end{align*}
\] (123)

\[
\begin{align*}
    a_1 g_{xx} + a_2 g_{yy} + a_3 g_{xy} + a_4 g_x + a_5 g_y + a_6 g &= 0.
\end{align*}
\] (124)

By Eq. (119), the particular solution of Eq. (124) is

\[
g = C_1 e^{\lambda_1 v} + C_2 e^{\lambda_2 v}.
\] (125)
Substituting (125) into (123) we get
\[
(2a_1l_1 + a_3l_2) g_x + (2a_2l_2 + a_3l_1) g_y + (a_4l_1 + a_5l_2) g
\]
\[
= (C_1k_1\lambda_1 (2a_1l_1 + a_3l_2) + C_1k_2\lambda_1 (2a_2l_2 + a_3l_1) + C_1 (a_4l_1 + a_5l_2)) e^{\lambda_1 v}
\]
\[
+ (C_2k_2\lambda_2 (2a_1l_1 + a_3l_2) + C_2k_2\lambda_2 (2a_2l_2 + a_3l_1) + C_2 (a_4l_1 + a_5l_2)) e^{\lambda_2 v} = 0.
\]
Then
\[
k_1\lambda_1 (2a_1l_1 + a_3l_2) + k_2\lambda_1 (2a_2l_2 + a_3l_1) + a_4l_1 + a_5l_2 = 0,
\]
\[
k_1\lambda_2 (2a_1l_1 + a_3l_2) + k_2\lambda_2 (2a_2l_2 + a_3l_1) + a_4l_1 + a_5l_2 = 0.
\]

Then the general solution of Eq. (101) is
\[
u = g_1 h_1 (w_1) + g_2 h_2 (w_2)
\]
\[
= (C_1e^{\lambda_1 v_1} + C_2e^{\lambda_2 v_1}) h_1 (l_1 x + l_2 y + l_3) + (C_3e^{\lambda_3 v_2} + C_4e^{\lambda_4 v_2}) h_2 (l_4 x + l_5 y + l_6),
\]
where \(v_1, v_2, \lambda_1 - \lambda_4, k_1, k_2, k_4, k_5, l_1, l_2, l_4\) and \(l_5\) satisfy Eqs. (103-114), so the theorem is proved.

In Theorem 21, if
\[
(a_4k_1 + a_5k_2)^2 - 4a_6 (a_1k_1^2 + a_2k_2^2 + a_3k_1k_2) < 0,
\]
\[
(a_4k_4 + a_5k_5)^2 - 4a_6 (a_1k_4^2 + a_2k_5^2 + a_3k_4k_5) < 0,
\]
or
\[
(a_4k_1 + a_5k_2)^2 - 4a_6 (a_1k_1^2 + a_2k_2^2 + a_3k_1k_2)
\]
\[
= (a_4k_4 + a_5k_5)^2 - 4a_6 (a_1k_4^2 + a_2k_5^2 + a_3k_4k_5) = 0.
\]

By analogous calculation, we may have Theorem 21 and 23.

**Theorem 22.** In \(\mathbb{R}^2\),
\[
a_1u_{xx} + a_2u_{yy} + a_3u_{xy} + a_4u_x + a_5u_y + a_6u = 0, \tag{101}
\]
the general solution of Eq. (101) is
\[
u = (C_1\sin q_1 v_1 + C_2\cos q_1 v_1) e^{pv_1 v_1} h_1 (l_1 x + l_2 y + l_3)
\]
\[
+ (C_3\sin q_2 v_2 + C_4\cos q_2 v_2) e^{pv_2 v_2} h_2 (l_4 x + l_5 y + l_6), \tag{126}
\]
where \(a_i\) are arbitrary known constants, \(h_1\) and \(h_2\) are arbitrary second differentiable functions, and
\[
v_1 = k_1 x + k_2 y + k_3, v_2 = k_4 x + k_5 y + k_6, \tag{103}
\]
\[
p_1 = \frac{-a_4k_1 - a_5k_2}{2 (a_1k_1^2 + a_2k_2^2 + a_3k_1k_2)}, q_1 = \frac{\sqrt{(a_4k_1 + a_5k_2)^2 - 4a_6 (a_1k_1^2 + a_2k_2^2 + a_3k_1k_2)}}{2 (a_1k_1^2 + a_2k_2^2 + a_3k_1k_2)}, \tag{127}
\]
\[
p_2 = \frac{-a_4k_4 - a_5k_5}{2 (a_1k_4^2 + a_2k_5^2 + a_3k_4k_5)}, q_2 = \frac{\sqrt{(a_4k_4 + a_5k_5)^2 - 4a_6 (a_1k_4^2 + a_2k_5^2 + a_3k_4k_5)}}{2 (a_1k_4^2 + a_2k_5^2 + a_3k_4k_5)}, \tag{128}
\]
where \(k_3, k_6, l_3\) and \(l_6\) are arbitrary constants, \(k_1, k_2, k_4, k_5, l_1, l_2, l_4\) and \(l_5\) are constants which satisfy
\[
l_1 = \frac{-a_3l_2 + \sqrt{a_4^2 l_2^2 - 4a_1a_2l_2^2}}{2a_1}, l_4 = \frac{-a_3l_5 + \sqrt{a_4^2 l_5^2 - 4a_1a_2l_5^2}}{2a_1}. \tag{108}
\]
\[(C_1 p_1 - C_2 q_1) (2a_1 k_1 l_1 + a_3 k_1 l_2 + 2a_2 k_2 l_2 + a_3 k_2 l_1) + C_1 (a_4 l_1 + a_5 l_2) = 0,\]  
\[(C_3 p_2 - C_4 q_2) (2a_1 k_4 l_1 + a_3 k_4 l_2 + 2a_2 k_5 l_2 + a_3 k_5 l_1) + C_3 (a_4 l_1 + a_5 l_2) = 0,\]  
\[(C_3 p_2 - C_4 q_2) (2a_1 k_4 l_1 + a_3 k_4 l_2 + 2a_2 k_5 l_2 + a_3 k_5 l_1) + C_3 (a_4 l_1 + a_5 l_2) = 0,\]  
\[(C_3 p_2 - C_4 q_2) (2a_1 k_4 l_1 + a_3 k_4 l_2 + 2a_2 k_5 l_2 + a_3 k_5 l_1) + C_3 (a_4 l_1 + a_5 l_2) = 0,\]  
\[(a_4 k_1 + a_5 k_2)^2 - 4a_6 (a_1 k_1^2 + a_2 k_2^2 + a_3 k_1 k_2) < 0,\]  
\[(a_4 k_4 + a_5 k_5)^2 - 4a_6 (a_1 k_4^2 + a_2 k_5^2 + a_3 k_4 k_5) < 0.\]  

**Theorem 23.** In \(\mathbb{R}^2\),
\[
a_1 u_{xx} + a_2 u_{yy} + a_3 u_{xy} + a_4 u + a_5 u + a_6 u = 0,\]
the general solution of Eq. (101) is
\[
u = C_1 v_1 e^{\lambda_1 v_1} h_1 (l_1 x + l_2 y + l_3) + C_2 v_2 e^{\lambda_2 v_2} h_2 (l_4 x + l_5 y + l_6),\]
where \(a_1\) are arbitrary known constants, \(h_1\) and \(h_2\) are arbitrary second differentiable functions, and
\[
\lambda_1 = -a_4 k_1 - a_5 k_2, \quad \lambda_2 = -a_4 k_4 - a_5 k_5,\]
where \(k_3, k_6, l_3\) and \(l_6\) are arbitrary constants, \(k_1, k_2, k_4, k_5, l_1, l_2, l_4\) and \(l_5\) are constants which satisfy
\[
l_1 = -a_3 l_2 + \frac{\sqrt{a_3^2 l_2^2 - 4a_1 a_2 l_2^2}}{2a_1}, l_4 = -a_3 l_5 - \frac{\sqrt{a_3^2 l_5^2 - 4a_1 a_2 l_5^2}}{2a_1},\]
\[(2a_1 l_1 + a_3 l_2) (k_1 + v_1 \lambda_1 k_1) + (2a_2 l_2 + a_3 l_1) (k_2 + v_1 \lambda_1 k_2) + (a_4 l_1 + a_5 l_2) v_1 = 0,\]  
\[(2a_4 l_4 + a_3 l_5) (k_4 + v_2 \lambda_2 k_4) + (2a_5 l_5 + a_3 l_4) (k_5 + v_2 \lambda_2 k_5) + (a_4 l_4 + a_5 l_5) v_2 = 0,\]  
\[(a_4 k_1 + a_5 k_2)^2 - 4a_6 (a_1 k_1^2 + a_2 k_2^2 + a_3 k_1 k_2) = 0,\]  
\[(a_4 k_4 + a_5 k_5)^2 - 4a_6 (a_1 k_4^2 + a_2 k_5^2 + a_3 k_4 k_5) = 0.\]

We propose Theorem 24 as follows.

**Theorem 24.** In \(\mathbb{R}^2\),
\[
a_1 u_{xx} + a_2 u_{yy} + a_3 u_{xy} + a_4 u + a_5 u + a_6 u = A (x, y),\]
the general solution of Eq. (141) is
\[
u = C_1 v_1 e^{\lambda_1 v_1} h_1 (l_1 x + l_2 y + l_3) + C_2 v_2 e^{\lambda_2 v_2} h_2 (l_4 x + l_5 y + l_6) + \int \int \frac{A (p, q) dp dq}{v_3 e^{\lambda_3 v_3} (2a_1 c_1 c_3 + 2a_2 c_2 c_4 + a_3 (c_1 c_4 + c_2 c_3))},\]
where \(a_1\) are arbitrary known constants, \(h_1\) and \(h_2\) are arbitrary second differentiable functions, and
\[
v_1 = k_1 x + k_2 y + k_3, v_2 = k_4 x + k_5 y + k_6, v_3 = k_7 x + k_8 y + k_9.\]
\[
\lambda_1 = \frac{-a_4k_1 - a_5k_2}{2(a_1k_1^2 + a_2k_2^2 + a_3k_1k_2)}, \quad \lambda_2 = \frac{-a_4k_4 - a_5k_5}{2(a_1k_4^2 + a_2k_5^2 + a_3k_4k_5)}, \quad \lambda_3 = \frac{-a_4k_7 - a_5k_8}{2(a_1k_7^2 + a_2k_8^2 + a_3k_7k_8)},
\]

(144)

p = c_1 x + c_2 y, \quad q = c_3 x + c_4 y,

(145)

x = \frac{pc_4 - qc_2}{c_1c_4 - c_2c_3}, \quad y = \frac{qc_1 - pc_3}{c_1c_4 - c_2c_3},

(146)

where \(C_1, C_2, k_3, k_6, k_9, l_3\) and \(l_6\) are arbitrary constants, \(k_1, k_2, k_4, k_5, k_7, k_8, l_1, l_2, l_4, l_5\) and \(c_1 - c_4\) are constants which satisfy

\[
c_1c_4 - c_2c_3 \neq 0,
\]

(147)

\[
l_1 = \frac{-a_3l_2 + \sqrt{a_3^2l_2^2 - 4a_1a_2l_2^2}}{2a_1},
\]

(148)

\[
l_4 = \frac{-a_3l_4 - \sqrt{a_3^2l_4^2 - 4a_1a_2l_4^2}}{2a_1},
\]

(149)

\[
l_1 + l_2 + l_3 + l_4 = 0
\]

(150)

\[
l_1 + l_4 + l_3 + l_5 = 0
\]

(151)

\[
l_1 + l_2 + l_3 + l_4 + l_5 = 0
\]

\[
(2a_1l_1 + a_3l_2) (k_1 + v_1 \lambda_1 k_1) + (2a_2l_2 + a_3l_1) (k_2 + v_1 \lambda_1 k_2) + (a_4l_4 + a_5l_3) v_1 = 0,
\]

(152)

\[
(2a_1l_4 + a_3l_5) (k_4 + v_2 \lambda_2 k_4) + (2a_2l_5 + a_3l_4) (k_5 + v_2 \lambda_2 k_5) + (a_4l_4 + a_5l_5) v_2 = 0,
\]

(153)

\[
(2a_1c_1 + a_3c_2) (1 + \lambda_3 v_3) k_7 + (2a_2c_2 + a_3c_1) (1 + \lambda_3 v_3) k_8 + (a_4c_1 + a_5c_2) v_3 = 0,
\]

(154)

\[
(2a_1c_3 + a_3c_4) (1 + \lambda_3 v_3) k_7 + (2a_2c_4 + a_3c_3) (1 + \lambda_3 v_3) k_8 + (a_4c_3 + a_5c_4) v_3 = 0,
\]

(155)

\[
(a_4k_1 + a_5k_2)^2 - 4a_6 (a_1k_1^2 + a_2k_2^2 + a_3k_1k_2) = 0,
\]

(156)

\[
(a_4k_4 + a_5k_5)^2 - 4a_6 (a_1k_4^2 + a_2k_5^2 + a_3k_4k_5) = 0.
\]

(157)

\[
(a_4k_7 + a_5k_8)^2 - 4a_6 (a_1k_7^2 + a_2k_8^2 + a_3k_7k_8) = 0.
\]

(158)

**Prove.** By \(Z_3\) Transformation, we set

\[
u(x, y) = g(x, y) h(p, q),
\]

(152)

where \(c_1 - c_4\) are undetermined constants, \(g\) and \(h\) are undetermined second differentiable function, and

\[
x = \frac{pc_4 - qc_2}{c_1c_4 - c_2c_3}, \quad y = \frac{qc_1 - pc_3}{c_1c_4 - c_2c_3},
\]

(153)

By Eq. (152), we get

\[
a_1 u_{xx} + a_2 u_{yy} + a_3 u_{xy} + a_4 u_x + a_5 u_y + a_6 u
\]

\[
= a_1 (h g_{xx} + 2g_x (c_1 h_p + c_3 h_q) + g (c_1^2 h_{pp} + c_2^2 h_{qq} + 2c_1 c_3 h_{pq}))
\]

\[
+ a_2 (h g_{yy} + 2g_y (c_2 h_p + c_3 h_q) + g (c_2^2 h_{pp} + c_1^2 h_{qq} + 2c_1 c_3 h_{pq}))
\]

\[
+ a_3 (h g_{xy} + g_x (c_2 h_p + c_3 h_q) + g_y (c_1 h_p + c_3 h_q) + g (c_1 c_2 h_{pp} + c_3 c_4 h_{qq} + (c_1 c_4 + c_2 c_3) h_{pq}))
\]

\[
+ a_4 (g_x h + g (c_1 h_p + c_3 h_q)) + a_5 (g_y h + g (c_2 h_p + c_4 h_q)) + a_6 g h
\]

\[
= (a_1 c_1^2 + a_2 c_2^2 + a_3 c_3 c_2) g h_{pp} + (a_1 c_3^2 + a_2 c_4^2 + a_3 c_3 c_4) g h_{qq}
\]

\[
+ (2a_1 c_1 + 2a_2 c_2 + a_3 (c_1 c_4 + c_2 c_3)) g h_{pq}
\]

\[
+ ((2a_1 c_1 + a_3 c_2) g_x + (2a_2 c_2 + a_3 c_1) g_y + (a_4 c_1 + a_5 c_2) g) h_p
\]

\[
+ ((2a_1 c_3 + a_5 c_4) g_x + (2a_2 c_4 + a_3 c_1) g_y + (a_4 c_3 + a_5 c_4) g) h_q
\]

\[
+ (a_1 g_{xx} + a_2 g_{yy} + a_3 g_{xy} + a_4 g_x + a_5 g_y + a_6 g) h = A(x, y).
\]
Set

\[ a_1g_{xx} + a_2g_{yy} + a_3g_{xy} + a_4g_x + a_5g_y + a_6g = 0, \]
\[ (2a_1c_1 + a_3c_2) g_x + (2a_2c_2 + a_3c_1) g_y + (a_4c_1 + a_5c_2) g = 0, \]
\[ (2a_1c_3 + a_3c_4) g_x + (2a_2c_4 + a_3c_3) g_y + (a_4c_3 + a_5c_4) g = 0. \]

We set \( g = g(v_3) \) and \( v_3(x, y) = k_7 x + k_8 y + k_9 \), then

\[ a_1g_{xx} + a_2g_{yy} + a_3g_{xy} + a_4g_x + a_5g_y + a_6g = (a_1 k_7^2 + a_2 k_8^2 + a_3 k_7 k_8) g_x'' + (a_4 k_7 + a_5 k_8) g_y' + a_6 g = 0. \]

If

\[ (a_4 k_7 + a_5 k_8)^2 - 4a_6 (a_1 k_7^2 + a_2 k_8^2 + a_3 k_7 k_8) = 0, \]

the particular solution of Eq. (153) is

\[ g = C_3 v_3 e^{\lambda_3 v_3}, \]

\[ \lambda_3 = \frac{-a_4 k_7 - a_5 k_8}{2 (a_1 k_7^2 + a_2 k_8^2 + a_3 k_7 k_8)}. \]

Substituting (157) into (154) we get

\[ (2a_1c_1 + a_3c_2) g_x + (2a_2c_2 + a_3c_1) g_y + (a_4c_1 + a_5c_2) g = (2a_1c_1 + a_3c_2) (1 + \lambda_3 v_3) C_3 k_7 e^{\lambda_3 v_3} + (2a_2c_2 + a_3c_1) (1 + \lambda_3 v_3) C_3 k_8 e^{\lambda_3 v_3} + (a_4c_1 + a_5c_2) C_3 v_3 e^{\lambda_3 v_3} = 0. \]

Namely

\[ (2a_1c_1 + a_3c_2) (1 + \lambda_3 v_3) k_7 + (2a_2c_2 + a_3c_1) (1 + \lambda_3 v_3) k_8 + (a_4c_1 + a_5c_2) v_3 = 0. \]

Substituting (157) into (155) we obtain

\[ (2a_1c_3 + a_3c_4) g_x + (2a_2c_4 + a_3c_3) g_y + (a_4c_3 + a_5c_4) g = (2a_1c_3 + a_3c_4) (1 + \lambda_3 v_3) C_3 k_7 e^{\lambda_3 v_3} + (2a_2c_4 + a_3c_3) (1 + \lambda_3 v_3) C_3 k_8 e^{\lambda_3 v_3} + (a_4c_3 + a_5c_4) C_3 v_3 e^{\lambda_3 v_3} = 0. \]

That is

\[ (2a_1c_3 + a_3c_4) (1 + \lambda_3 v_3) k_7 + (2a_2c_4 + a_3c_3) (1 + \lambda_3 v_3) k_8 + (a_4c_3 + a_5c_4) v_3 = 0. \]

So

\[ a_1 u_{xx} + a_2 u_{yy} + a_3 u_{xy} + a_4 u_x + a_5 u_y + a_6 u = (a_1 c_1^2 + a_2 c_2^2 + a_3 c_1 c_2) g_{pp} + (a_1 c_3^2 + a_2 c_4^2 + a_3 c_3 c_4) g_{qq} \]
\[ + (2a_1 c_1 c_3 + 2a_2 c_2 c_4 + a_3 (c_1 c_4 + c_2 c_3)) g_{pq} = A(x, y). \]

Set

\[ a_1 c_1^2 + a_2 c_2^2 + a_3 c_1 c_2 = a_1 c_3^2 + a_2 c_4^2 + a_3 c_3 c_4 = 0. \]

Namely

\[ c_1 = \frac{-a_3 c_2 \pm \sqrt{a_3^2 c_2^2 - 4a_1 a_2 c_2^2}}{2a_1}, \]
\[ c_3 = \frac{-a_3 c_4 \pm \sqrt{a_3^2 c_4^2 - 4a_1 a_2 c_4^2}}{2a_1}. \]
Then

\[ a_1 u_{xx} + a_2 u_{yy} + a_3 u_{xy} + a_4 u_x + a_5 u_y + a_6 u = (2a_1 c_1 c_3 + 2a_2 c_2 c_4 + a_3 (c_1 c_4 + c_2 c_3)) g h_{yy} = A(x, y), \]

the particular solution of \( h \) is

\[ h = \int \int \frac{A(p, q) \, dp dq}{(2a_1 c_1 c_3 + 2a_2 c_2 c_4 + a_3 (c_1 c_4 + c_2 c_3)) g}. \]  \hspace{1cm} (161)

Combining Theorem 23, we can have the general solution of Eq. (141) is

\[ u = C_1 v_1 e^{\lambda_1 v_1} h_1 (l_1 x + l_2 y + l_3) + C_2 v_2 e^{\lambda_2 v_2} h_2 (l_4 x + l_5 y + l_6) \]

\[ + v_3 e^{\lambda_3 v_3} \int \frac{A(p, q) \, dp dq}{\sqrt{e^{\lambda_3 v_3} (2a_1 c_1 c_3 + 2a_2 c_2 c_4 + a_3 (c_1 c_4 + c_2 c_3))}}. \]

So the theorem is proved. □

Eqs. (101, 141) are very important linear partial differential equations. One-dimensional homogeneous and non-homogeneous wave equations, heat equations, two-dimensional Helmholtz equations and reaction-diffusion-convection equation [23-25], etc. are all special cases of them. The general solution or exact solution of these equations can be obtained by using Theorem 21-24.

Next we propose Theorem 25.

**Theorem 25.** In \( \mathbb{R}^3 \),

\[ a_1 u_{tt} + a_2 u_{xx} + a_3 u_{yy} + a_4 u_x + a_5 u_y + a_6 u_{xy} + a_7 u_t + a_8 u_x + a_9 u_y + a_{10} u = A(t, x, y), \]  \hspace{1cm} (162)

the general solution of Eq. (162) is

\[ u = g(h_1(p, r) + h_2(q, r)) + g \int \frac{A(p, q, r) \, dp dq}{B}, \]  \hspace{1cm} (163)

where \( a_i \) are arbitrary known constants, \( h_1 \) and \( h_2 \) are arbitrary second differentiable functions, and

\[ g = C_1 e^{\lambda_1 v} + C_2 e^{\lambda_2 v}, \]

\[ \lambda_1 = \frac{-b + \sqrt{b^2 - 4aa_{10}}}{2a}, \lambda_2 = \frac{-b - \sqrt{b^2 - 4aa_{10}}}{2a}, \]

\[ a = a_1 k_1^2 + a_2 k_2^2 + a_3 k_3^2 + a_4 k_1 k_2 + a_5 k_1 k_3 + a_6 k_2 k_3, \]

\[ b = a_7 k_1 + a_8 k_2 + a_9 k_3, \]

\[ p = l_1 t + l_2 x + l_3 y, q = l_4 t + l_5 x + l_6 y, r = l_7 t + l_8 x + l_9 y, \]

\[ B = 2a_1 l_1 l_4 + 2a_2 l_2 l_5 + 2a_3 l_3 l_6 + a_4 (l_1 l_5 + l_2 l_4) + a_5 (l_1 l_6 + l_3 l_4) + a_6 (l_2 l_6 + l_3 l_5), \]  \hspace{1cm} (169)

where \( C_1, C_2 \) and \( k_4 \) are arbitrary constants, \( k_1 - k_3 \) and \( l_1 - l_9 \) are constants which satisfy

\[ b^2 - 4aa_{10} > 0, \]

\[ -l_3 l_5 l_7 + l_2 l_6 l_7 - l_3 l_4 l_8 - l_1 l_6 l_8 - l_2 l_4 l_9 - l_1 l_5 l_9 \neq 0, \]

\[ k_1 \lambda_1 (2a_1 l_1 + a_4 l_2 + a_3 l_3) + k_2 \lambda_1 (2a_2 l_2 + a_4 l_1 + a_6 l_3) + k_3 \lambda_1 (2a_3 l_3 + a_5 l_1 + a_6 l_2) \]

\[ + a_7 l_1 + a_8 l_2 + a_9 l_3 = 0, \]  \hspace{1cm} (170)
The particular solution of Eq. (162) is

\[ u(t, x, y) = f(v) = f(k_1 t + k_2 x + k_3 y + k_4), \]  

where \( v(t, x, y) = k_1 t + k_2 x + k_3 y + k_4, k_1 - k_4 \) are undetermined constants, \( f \) is an undetermined second differentiable function, so

\[
\begin{align*}
  a_1 u_{tt} + a_2 u_{xx} + a_3 u_{yy} + a_4 u_{tx} + a_5 u_{ty} + a_6 u_{xy} + a_7 u_t + a_8 u_x + a_9 u_y + a_{10} u &= 0, \\
  \left( a_1 k_1^2 + a_2 k_2^2 + a_3 k_3^2 + a_4 k_4 k_2 + a_5 k_5 k_3 + a_6 k_6 k_3 \right) \lambda^2 + \left( a_7 k_1 + a_8 k_2 + a_9 k_3 \right) \lambda + a_{10} &= 0. 
\end{align*}
\]  

The characteristic equation of Eq. (184) is

\[
\lambda = \frac{-b \pm \sqrt{b^2 - 4aa_{10}}}{2a},
\]

where

\[
\begin{align*}
  a &= a_1 k_1^2 + a_2 k_2^2 + a_3 k_3^2 + a_4 k_4 k_2 + a_5 k_5 k_3 + a_6 k_6 k_3, \\
  b &= a_7 k_1 + a_8 k_2 + a_9 k_3.
\end{align*}
\]

If \( b^2 - 4aa_{10} > 0 \), the particular solution of Eq. (162) is

\[ u = f = C_1 e^{\lambda_1 v} + C_2 e^{\lambda_2 v}, \]
where
\[
\lambda_1 = \frac{-b + \sqrt{b^2 - 4aa_10}}{2a}, \quad \lambda_2 = \frac{-b - \sqrt{b^2 - 4aa_10}}{2a}.
\]

For getting the general solution of Eq. (162), using $Z_3$ Transformation, we set
\[
u = g(t, x, y) h(p, q, r),
\]
and
\[p = l_1 t + l_2 x + l_3 y, q = l_4 t + l_5 x + l_6 y, r = l_7 t + l_8 x + l_9 y,
\]
where $l_1 - l_9$ are undetermined constants, $h$ and $g$ are undetermined second differential functions, and
\[
\frac{\partial (p, q, r)}{\partial (t, x, y)} = \begin{bmatrix} l_1 & l_2 & l_3 \\ l_4 & l_5 & l_6 \\ l_7 & l_8 & l_9 \end{bmatrix} = -l_3 l_5 l_7 + l_2 l_6 l_7 + l_3 l_4 l_8 - l_1 l_6 l_8 - l_2 l_4 l_9 + l_1 l_5 l_9 \neq 0.
\]

By Eq. (168), we get
\[
t = -\frac{rl_3 l_5 + rl_2 l_6 + ql_3 l_8 - pl_4 l_8 - ql_2 l_9 + pl_3 l_9}{l_3 l_5 l_7 - l_2 l_6 l_7 - l_3 l_4 l_8 + l_1 l_6 l_8 + l_2 l_4 l_9 - l_1 l_5 l_9},
\]
\[
x = -\frac{rl_3 l_4 - rl_1 l_6 - ql_3 l_7 + pl_4 l_7 + ql_1 l_9 - pl_4 l_9}{l_3 l_5 l_7 - l_2 l_6 l_7 - l_3 l_4 l_8 + l_1 l_6 l_8 + l_2 l_4 l_9 - l_1 l_5 l_9},
\]
\[
y = \frac{rl_2 l_4 - rl_1 l_5 - ql_2 l_7 + pl_5 l_7 + ql_1 l_9 - pl_4 l_8}{l_3 l_5 l_7 - l_2 l_6 l_7 - l_3 l_4 l_8 + l_1 l_6 l_8 + l_2 l_4 l_9 - l_1 l_5 l_9}.
\]

Then
\[
a_1 u_t + a_2 u_xx + a_3 u_yy + a_4 u_{tx} + a_5 u_{ty} + a_6 u_{xy} + a_7 u_t + a_8 u_x + a_9 u_y + a_{10} u
\]
\[= a_1 (g_{th} h + 2g_t (l_1 h_p + l_3 h_q + l_7 h_r)
\]
\[+ g (l_3^2 h_{pp} + l_3 h_{pq} + l_7 h_{pr} + l_7^2 h_{qr} + 2l_1 l_4 h_{pq} + 2l_1 l_7 h_{pr} + 2l_4 l_7 h_{qr}))
\]
\[+ a_2 (g_{xx} h + 2g_x (l_2 h_p + l_5 h_q + l_8 h_r)
\]
\[+ g (l_3^2 h_{pp} + l_3 h_{pq} + l_5 h_{qr} + l_8^2 h_{qr} + 2l_2 l_5 h_{pq} + 2l_2 l_8 h_{qr} + 2l_5 l_8 h_{qr}))
\]
\[+ a_3 (g_{yy} h + 2g_y (l_3 h_p + l_6 h_q + l_9 h_r)
\]
\[+ g (l_3^2 h_{pp} + l_3 h_{pq} + l_6 h_{qr} + l_9^2 h_{qr} + 2l_3 l_6 h_{pq} + 2l_3 l_9 h_{qr} + 2l_6 l_9 h_{qr}))
\]
\[+ a_4 (g_{tx} h + g (l_2 h_p + l_5 h_q + l_8 h_r) + g (l_1 h_p + l_4 h_q + l_7 h_r)
\]
\[+ g (l_1 l_4 h_{pp} + l_1 l_5 h_{pq} + l_7 l_8 h_{qr} + l_1 l_5 + l_2 l_4) h_{pq} + (l_1 l_6 + l_3 l_7) h_{pr} + (l_4 l_5 + l_5 l_7) h_{qr}))
\]
\[+ a_5 (g_{ty} h + g (l_3 h_p + l_6 h_q + l_9 h_r) + g (l_1 h_p + l_2 h_q + l_7 h_r)
\]
\[+ g (l_1 l_3 h_{pp} + l_1 l_6 h_{pq} + l_7 l_8 h_{qr} + l_1 l_6 + l_4 l_3) h_{pq} + (l_1 l_9 + l_3 l_7) h_{pr} + (l_4 l_9 + l_5 l_7) h_{qr}))
\]
\[+ a_6 (g_{xy} h + g (l_3 h_p + l_6 h_q + l_9 h_r) + g (l_5 l_6 h_{pq} + l_5 l_9 h_{qr} + l_6 l_9 h_{qr})
\]
\[+ g (l_2 l_5 h_{pp} + l_2 l_6 h_{pq} + l_8 l_9 h_{qr} + l_2 l_6 + l_3 l_5) h_{pq} + (l_2 l_9 + l_5 l_8) h_{pr} + (l_3 l_9 + l_6 l_8) h_{qr}))
\]
\[+ a_7 (g h + g (l_1 h_p + l_4 h_q + l_7 h_r) + a_8 (g_x h + g (l_2 h_p + l_5 h_q + l_8 h_r)
\]
\[+ a_9 (g_y h + g (l_3 h_p + l_6 h_q + l_9 h_r) ) + a_{10} g h
\]
\[= (a_1 l_3^2 + a_2 l_3^2 + a_3 l_3^2 + a_4 l_1 l_2 + a_5 l_1 l_3 + a_6 l_2 l_3) g h_{pp}
\]
\[+ (a_1 l_3^2 + a_2 l_3^2 + a_3 l_3^2 + a_4 l_1 l_4 + a_5 l_1 l_6 + a_6 l_2 l_5) g h_{pq}
\]
\[+ (a_1 l_3^2 + a_2 l_3^2 + a_3 l_3^2 + a_4 l_1 l_5 + a_5 l_1 l_3 + a_6 l_2 l_3) g h_{qr}
\]
\[+ (2a_1 l_1 l_4 + 2a_2 l_2 l_5 + 2a_3 l_3 l_6 + a_4 (l_1 l_5 + l_2 l_4) + a_5 (l_1 l_6 + l_3 l_4) + a_6 (l_2 l_6 + l_3 l_5)) g h_{pq}
\]
\[+ (2a_1 l_1 l_4 + 2a_2 l_2 l_6 + 2a_3 l_3 l_5 + a_4 (l_1 l_5 + l_2 l_7) + a_5 (l_1 l_6 + l_3 l_7) + a_6 (l_2 l_9 + l_3 l_8)) g h_{pr}
\]
\[+ (2a_1 l_2 l_5 + 2a_2 l_3 l_6 + 2a_3 l_4 l_7 + a_4 (l_1 l_7 + l_3 l_5) + a_5 (l_2 l_9 + l_3 l_8) + a_6 (l_3 l_9 + l_5 l_8)) g h_{qr}
\]
\[+ (2a_1 l_1 + a_4 l_1 + a_5 l_3) g_t + (2a_2 l_2 + a_4 l_1 + a_6 l_3) g_x + (2a_3 l_5 + a_5 l_1 + a_6 l_2) g_y
\]
\[+ (a_7 l_1 + a_7 l_2 + a_3 l_3) h_p
\]
\[+ ((2a_1 l_4 + a_5 l_5 + a_6 l_6) g_t + (2a_2 l_5 + a_4 l_1 + a_6 l_6) g_x + (2a_3 l_6 + a_5 l_1 + a_6 l_5) g_y
\]
\[+ (a_7 l_1 + a_7 l_5 + a_6 l_6) h_q
\]
\[+ ((2a_1 l_7 + a_4 l_3 + a_5 l_9) g_t + (2a_2 l_8 + a_4 l_7 + a_6 l_9) g_x + (2a_3 l_9 + a_5 l_7 + a_6 l_8) g_y
\]
\[+ (a_7 l_1 + a_7 l_9 + a_6 l_7) h_r
\]
\[+ (a_1 g_t + a_2 g_xx + a_3 g_yy + a_4 g_tx + a_5 g_ty + a_6 g_xy + a_7 g_{tx} + a_8 g_{xy} + a_9 g_y + a_{10} g) h.
\]
Set

\begin{align}
& a_1 g_{tt} + a_2 g_{xx} + a_3 g_{yy} + a_4 g_{tx} + a_5 g_{ty} + a_6 g_{xy} + a_7 g_t + a_8 g_x + a_9 g_y + a_{10} g = 0, \quad (191) \\
& (2a_1 l_1 + a_4 l_2 + a_5 l_3) g_t + (2a_2 l_2 + a_4 l_1 + a_6 l_3) g_x + (2a_3 l_3 + a_5 l_1 + a_6 l_2) g_y \\
& \quad + (a_7 l_1 + a_8 l_2 + a_9 l_3) g = 0, \quad (192) \\
& (2a_1 l_4 + a_4 l_5 + a_5 l_6) g_t + (2a_2 l_5 + a_4 l_4 + a_6 l_6) g_x + (2a_3 l_6 + a_5 l_4 + a_6 l_5) g_y \\
& \quad + (a_7 l_4 + a_8 l_5 + a_9 l_6) g = 0, \quad (193) \\
& (2a_1 t_7 + a_4 t_8 + a_5 t_9) g_t + (2a_2 t_8 + a_4 t_7 + a_6 t_9) g_x + (2a_3 t_9 + a_5 t_7 + a_6 t_8) g_y \\
& \quad + (a_7 t_7 + a_8 t_8 + a_9 t_9) g = 0, \quad (194)
\end{align}

By Eq. (186), the particular solution of Eq. (191) is

\[ g = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}, \quad (195) \]

where \( \lambda_1 \) and \( \lambda_2 \) satisfy Eq. (165). Substituting (195) into (192) we get

\[ \begin{align}
& (2a_1 l_1 + a_4 l_2 + a_5 l_3) g_t + (2a_2 l_2 + a_4 l_1 + a_6 l_3) g_x + (2a_3 l_3 + a_5 l_1 + a_6 l_2) g_y \\
& \quad + (a_7 l_1 + a_8 l_2 + a_9 l_3) g \\
& = (2a_1 l_1 + a_4 l_2 + a_5 l_3) (C_1 k_1 \lambda_1 e^{\lambda_1 t} + C_2 k_2 \lambda_2 e^{\lambda_2 t}) \\
& + (2a_2 l_2 + a_4 l_1 + a_6 l_3) (C_1 k_2 \lambda_1 e^{\lambda_1 t} + C_2 k_3 \lambda_2 e^{\lambda_2 t}) \\
& + (2a_3 l_3 + a_5 l_1 + a_6 l_2) (C_1 k_3 \lambda_1 e^{\lambda_1 t} + C_2 k_3 \lambda_2 e^{\lambda_2 t}) \\
& + (a_7 l_1 + a_8 l_2 + a_9 l_3) (C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}) \\
& = C_1 e^{\lambda_1 t} (k_1 \lambda_1 (2a_1 l_1 + a_4 l_2 + a_5 l_3) + k_2 \lambda_2 (2a_2 l_2 + a_4 l_1 + a_6 l_3) \\
& + k_3 \lambda_1 (2a_3 l_3 + a_5 l_1 + a_6 l_2) + a_7 l_1 + a_8 l_2 + a_9 l_3) \\
& + C_2 e^{\lambda_2 t} (k_1 \lambda_2 (2a_1 l_1 + a_4 l_2 + a_5 l_3) + k_2 \lambda_2 (2a_2 l_2 + a_4 l_1 + a_6 l_3) \\
& + k_3 \lambda_2 (2a_3 l_3 + a_5 l_1 + a_6 l_2) + a_7 l_1 + a_8 l_2 + a_9 l_3) = 0.
\end{align} \]

Namely

\[ \begin{align}
& k_1 \lambda_1 (2a_1 l_1 + a_4 l_2 + a_5 l_3) + k_2 \lambda_1 (2a_2 l_2 + a_4 l_1 + a_6 l_3) + k_3 \lambda_1 (2a_3 l_3 + a_5 l_1 + a_6 l_2) \\
& + a_7 l_1 + a_8 l_2 + a_9 l_3 = 0, \\
& k_1 \lambda_2 (2a_1 l_1 + a_4 l_2 + a_5 l_3) + k_2 \lambda_2 (2a_2 l_2 + a_4 l_1 + a_6 l_3) + k_3 \lambda_2 (2a_3 l_3 + a_5 l_1 + a_6 l_2) \\
& + a_7 l_1 + a_8 l_2 + a_9 l_3 = 0.
\end{align} \]

Substituting (195) into (192,193) respectively, we get

\[ \begin{align}
& k_1 \lambda_1 (2a_1 l_4 + a_4 l_5 + a_5 l_6) + k_2 \lambda_1 (2a_2 l_5 + a_4 l_4 + a_6 l_6) + k_3 \lambda_1 (2a_3 l_6 + a_5 l_4 + a_6 l_5) \\
& + a_7 l_4 + a_8 l_5 + a_9 l_6 = 0, \\
& k_1 \lambda_2 (2a_1 l_4 + a_4 l_5 + a_5 l_6) + k_2 \lambda_2 (2a_2 l_5 + a_4 l_4 + a_6 l_6) + k_3 \lambda_2 (2a_3 l_6 + a_5 l_4 + a_6 l_5) \\
& + a_7 l_4 + a_8 l_5 + a_9 l_6 = 0, \\
& k_1 \lambda_1 (2a_1 t_7 + a_4 t_8 + a_5 t_9) + k_2 \lambda_1 (2a_2 t_8 + a_4 t_7 + a_6 t_9) + k_3 \lambda_1 (2a_3 t_9 + a_5 t_7 + a_6 t_8) \\
& + a_7 t_7 + a_8 t_8 + a_9 t_9 = 0, \\
& k_1 \lambda_2 (2a_1 t_7 + a_4 t_8 + a_5 t_9) + k_2 \lambda_2 (2a_2 t_8 + a_4 t_7 + a_6 t_9) + k_3 \lambda_2 (2a_3 t_9 + a_5 t_7 + a_6 t_8) \\
& + a_7 t_7 + a_8 t_8 + a_9 t_9 = 0.
\end{align} \]
Case 1: Set
\[
\begin{align*}
\alpha_1 t^2 + \alpha_2 t^2 + \alpha_3 l^2 + \alpha_4 l t^2 + \alpha_5 1 l^3 + \alpha_6 l l^3 &= 0, \\
\alpha_1 t^2 + \alpha_2 t^2 + \alpha_3 l^2 + \alpha_4 l t^2 + \alpha_5 1 l^5 + \alpha_6 l l^5 &= 0, \\
\alpha_1 t^2 + \alpha_2 t^2 + \alpha_3 l^2 + \alpha_4 t l^3 + \alpha_5 1 l^3 + \alpha_6 l l^3 &= 0, \\
2a_1 l t^5 + 2a_2 l^5 + 2a_3 l^5 + a_4 (1 l^5 + l^2 t^2) + a_5 (1 1 l^5 + l^3 t) + a_6 (l^5 l^9 + l^5 l^9) &= 0, \\
2a_1 l t^7 + 2a_2 l^7 + 2a_3 l^7 + a_4 (1 l^7 + l^5 t^2) + a_5 (1 1 l^5 + l^3 t) + a_6 (l^5 l^9 + l^5 l^9) &= 0.
\end{align*}
\]

Then
\[
\begin{align*}
\alpha_1 u + \alpha_2 u_x + \alpha_3 u_y + \alpha_4 u_x + \alpha_5 u_y + \alpha_6 u_x + \alpha_7 u + \alpha_8 u + \alpha_9 u + \alpha_{10} u &= (2a_1 l t^4 + 2a_2 l^5 + 2a_3 l^5 + a_4 (1 l^5 + l^2 t^2) + a_5 (1 1 l^5 + l^3 t) + a_6 (l^5 l^9 + l^5 l^9)) gh_{pq} \\
&= A(p, q, r).
\end{align*}
\]

That is
\[
\begin{align*}
h_{pq} &= \frac{A(p, q, r)}{(2a_1 l t^4 + 2a_2 l^5 + 2a_3 l^5 + a_4 (1 l^5 + l^2 t^2) + a_5 (1 1 l^5 + l^3 t) + a_6 (l^5 l^9 + l^5 l^9))},
\end{align*}
\]
the general solution of Eq. (196) is
\[
\begin{align*}
h &= h_1(p, r) + h_2(q, r) + \int \frac{A(p, q, r) dp dq}{B g},
\end{align*}
\]
where
\[
B = 2a_1 l t^4 + 2a_2 l^5 + 2a_3 l^5 + a_4 (1 l^5 + l^2 t^2) + a_5 (1 1 l^5 + l^3 t) + a_6 (l^5 l^9 + l^5 l^9),
\]
So the general solution of Eq. (162) is
\[
\begin{align*}
u &= gh = g(h_1(p, r) + h_2(q, r)) + \frac{g}{B} \int \frac{A(p, q, r) dp dq}{g}.
\end{align*}
\]
Whereupon the theorem is proved. \(\square\)

Case 2: Set
\[
\begin{align*}
\alpha_1 t^2 + \alpha_2 t^2 + \alpha_3 l^2 + \alpha_4 l t^2 + \alpha_5 1 l^3 + \alpha_6 l l^3 &= 0, \\
\alpha_1 t^2 + \alpha_2 t^2 + \alpha_3 l^2 + \alpha_4 l t^2 + \alpha_5 1 l^5 + \alpha_6 l l^5 &= 0, \\
\alpha_1 t^2 + \alpha_2 t^2 + \alpha_3 l^2 + \alpha_4 l t^2 + \alpha_5 1 l^3 + \alpha_6 l l^3 &= 0, \\
2a_1 l t^5 + 2a_2 l^5 + 2a_3 l^5 + a_4 (1 l^5 + l^2 t^2) + a_5 (1 1 l^5 + l^3 t) + a_6 (l^5 l^9 + l^5 l^9) &= 0, \\
2a_1 l t^7 + 2a_2 l^7 + 2a_3 l^7 + a_4 (1 l^7 + l^5 t^2) + a_5 (1 1 l^5 + l^3 t) + a_6 (l^5 l^9 + l^5 l^9) &= 0.
\end{align*}
\]

Then
\[
\begin{align*}
\alpha_1 u + \alpha_2 u_x + \alpha_3 u_y + \alpha_4 u_x + \alpha_5 u_y + \alpha_6 u_x + \alpha_7 u + \alpha_8 u + \alpha_9 u + \alpha_{10} u &= (2a_1 l t^4 + 2a_2 l^5 + 2a_3 l^5 + a_4 (1 l^5 + l^2 t^2) + a_5 (1 1 l^5 + l^3 t) + a_6 (l^5 l^9 + l^5 l^9)) gh_{pr} \\
&= A(p, q, r).
\end{align*}
\]

Namely
\[
\begin{align*}
h_{pr} &= \frac{A(p, q, r)}{E g},
\end{align*}
\]
\[
E = 2a_1 l t^7 + 2a_2 l^5 + 2a_3 l^5 + a_4 (1 l^5 + l^2 t^2) + a_5 (1 1 l^5 + l^3 t) + a_6 (l^5 l^9 + l^5 l^9).
\]
The general solution of Eq. (198) is
\[ h = h_1(p, q) + h_2(q, r) + \int \frac{A(p, q, r) \ dp \ dr}{Eg}. \]

So the general solution of Eq. (162) is
\[ u = gh = g (h_1(p, q) + h_2(q, r)) + \frac{g}{E} \int \frac{A(p, q, r) \ dp \ dr}{g}. \]  

(200)

For other cases and \( b^2 - 4aa_{10} = 0, b^2 - 4aa_{10} < 0 \), similar calculations can be done.

According to Theorem 25, we can get Theorem 26.

**Theorem 26.** In \( \mathbb{R}^3 \),
\[ a_1u_{tt} + a_2u_{xx} + a_3u_{yy} + a_4u_{tx} + a_5u_{ty} + a_6u_{xy} + a_7u_t + a_8u_x + a_9u_y + a_{10}u = 0, \]
(201)

the general solution of Eq. (201) is
\[ u = g (h_1(p, r) + h_2(q, r)), \]
(202)

where \( a_i \) are arbitrary known constants, \( h_1 \) and \( h_2 \) are arbitrary second differentiable functions, \( g, p, q \) and \( r \) satisfy (164 – 168) and (170 – 182).

Eqs. (162, 201) are very significant linear partial differential equations. Two-dimensional wave equation, heat equation, Fokker-Planck Equation [26-28], Telegraph Equation [29-31], etc. are all special cases of them.

4. Conclusions

In this paper, we first prove a new theorem for the independent variable transformational equations, that is, the independent variable transformation not only does not change the linearity or non-linearity of the original PDEs, but also does not change their order.

We propose the concept of the banal PDE and the non-banal PDE, then use the proposed three kinds of Z Transformations to obtain the plentiful laws of general solutions of linear PDEs with variable coefficients and constant coefficients.

The characteristic equation method is a basic method to solve first order linear and quasilinear PDEs [32]. By comparing with Z Transformations, we can find that it has some limitations, such as using it cannot get the general solutions of the first order linear PDEs (25, 57, 93), cannot obtain the complete general solution of (68) and so on.

References