Generalized Formulations for the Rotational Kinetic Energies of Planets in Circular Orbits

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Abstract;
This short paper is intended to demonstrate a series of theoretical techniques that can be deployed whilst satisfying the objective of representing the rotational kinetic energies exhibited by planets in approximately circular orbits. Its premise is pursuant to Kepler's third law, and functionality determined by several ubiquitous notions of rotational KE, angular momentum, and most notably, angular velocity.

Phase 1: Begin with a simplistic derivation of Kepler's third law;

The velocity of a body in a circular orbit is represented in two modalities, the first of which is;

$$v = \frac{2\pi r}{T}\ (E1)$$

Secondly, we may equate the centripetal force experienced by a body undergoing uniformly circular motion as a consequence of its position in an external gravitational field;

$$\frac{mv^2}{r} = \frac{GMm}{r^2}$$

Simplifying, we yield:

$$\frac{v^2}{r} = \frac{GM}{r^2}$$

$$v^2 = \frac{GM}{r}$$

$$v = \sqrt{\frac{GM}{r}}\ (E2)$$

If one were to express the equivalency between E1 and E2:

$$v = \sqrt{\frac{GM}{r}} = \frac{2\pi r}{T}$$

$$\frac{GM}{r} = \frac{4\pi^2 r^2}{T^2}$$

Rearranging yields:

$$T^2 = \frac{4\pi^2 r^3}{GM}\,(Kepler's\ Third\ Law)$$

Phase 2: Consider the angular velocity of a planetary body subject to the orbital equivalency above;

$$\omega = \frac{2\pi}{T}$$

In finding the rotational kinetic energy of a body; one may use the general form:

$$KE_r = \frac{1}{2} I \omega^2$$

Wherein I represents the body's moment of inertia, and ω its angular velocity;

Substituting the first equation:

$$KE_r = \frac{1}{2} I \omega^2$$

$$\omega^2 = \frac{4\pi^2}{T^2}$$

$$KE_r = \frac{1}{2} I \frac{4\pi^2}{T^2}$$

$$KE_r = \frac{4\pi^2 I}{2T^2}$$

Since $T^2 = \frac{4\pi^2 r^3}{GM}$

$$KE_r = \frac{4\pi^2 I}{2\frac{4\pi^2 r^3}{GM}}$$

$$KE_r = \frac{GM4\pi^2 I}{8\pi^2 r^3}$$

$$\pi^2 \; cancels, leaving;$$

$$\boldsymbol{KE_r = \frac{GMI}{2r^3}}$$

The formulation above is a representation of an archetypal planet's rotational kinetic energy KE_r as a function of its mass (M), moment of inertia (I) and radius (r).

Alternatively, one may attempt to repurpose the formulation such that it tailors a given planet's angular momentum.

Phase 3: Alternative Formulations;

To commence, one must reverse back to the fundamental formulation;

$$KE_r = \frac{1}{2} I\omega^2$$

$$I\ is\ a\ tensor\ quantity\ interchangeable\ with\ the\ ratio\ \frac{L}{\omega}$$

wherein L notes a body's conserved angular momentum.

Thus, re-adjusting the expression allows one to obtain:

$$KE_r = \frac{1}{2}\frac{L}{\omega}\omega^2$$

$$KE_r = \frac{1}{2}L\omega$$

In deriving a dependent argument for ω;

$$\omega = \frac{2\pi}{T}$$

$$T^2 = \frac{4\pi^2 r^3}{GM}$$

$$T = \frac{2\pi r^{3/2}}{\sqrt{GM}}$$

Reverting;

$$\omega = \frac{2\pi}{T}$$

$$\omega = \frac{2\pi}{\frac{2\pi r^{3/2}}{\sqrt{GM}}}$$

$$\omega = \frac{2\pi\sqrt{GM}}{2\pi r^{3/2}}$$

$$2\pi\ cancels, leaving:$$

$$\omega = \frac{\sqrt{GM}}{r^{3/2}}$$

Expanding the radical procures;

$$\omega = \sqrt{\frac{GM}{r^3}}$$

Consequently; operating the alternative formulation allows one to carry out the following calculation:

$$KE_r = \frac{1}{2}L\omega$$

$$KE_r = \frac{1}{2}L\sqrt{\frac{GM}{r^3}}$$

Once more, replacing ½ outside the radical by a factor of ¼ inside it allows the mathematical continuation outlined below:

$$KE_r = L\sqrt{\frac{GM}{r^3}\frac{1}{4}}$$

$$\boldsymbol{KE_r = L\sqrt{\frac{GM}{4r^3}}}$$

Subsequently, this mechanism returns a planet's rotational kinetic energy in terms of its angular momentum (L), and the aforementioned variables listed in the first expression.

One can resolve a planet's angular momentum into its vectorial components, so as to yield a more complex dependency.