Integration technique using Laplace transforms: A Generilized form of the Dirichlet integral

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Abstract: the problem of integration technique over integrands of the form f (t)/ t^n, can be solved by differentiation (n-times) by using Leibniz's rule to get rid of t^n, that leads to integrate back (n-times) to end the game, wich it's harder than the original problem. This work focuses on the derivation of the formula (***Pagano's theorem***) wich is a perfect tool to avoid that hard task. It allows to change the difficult n iterated integrals into a more outstanding easier problem wich consists of n-1 derivatives. The ***Pagano's theorem*** is a generalization of the Dirichlet integral.

Key words: Integration, Dirchlet integral ,Laplace Transform, iterated intergals, multiple integration, integration by parts, Pagano's theorem

1 Hypothesis

H1) Let f: $[0;\infty) \rightarrow$ R be a continuous piecewise function admitting n-2 derivatives at the origin.

H2) $\int_0^\infty \frac{f(t)}{t^n} \mathrm{d}t$ is convergent $\forall n \in N$

H3) $\lim\limits_{t \to 0^+} \frac{f(t)}{t^n} = \gamma$, finite.

H4) let $f(t)$ and its first n-2 derivatives to be bounded functions.

2 Thesis

$$\int_0^\infty \frac{f\ (t)}{t^n} \mathrm{d}t = \lim_{s \to 0} \int_s^\infty \frac{1}{\Pi_{(n-1)}} \mathcal{L}(D^{n-1} f\ (t))\, \mathrm{d}\sigma$$

3 Demonstration

factor	D	I
(+1)	f(t)	t^{-n}
(-1)	$D^1 f(t)$	$\frac{(-1)t^{-n+1}}{(n-1)}$
(+1)	$D^2 f(t)$	$\frac{t^{-n+2}}{(n-1)(n-2)}$
(-1)	$D^3 f(t)$	$\frac{(-1)t^{-n+3}}{(n-1)(n-2)(n-3)}$
……	….	…….
$(-1)^k$	$D^k f(t)$	$\frac{(-1)^k t^{-n+k}}{\prod_{j=1}^{k}(n-j)}$
……	…..	…….
$(-1)^{n-1}$	$D^{n-1} f(t)$	$\frac{(-1)^{n-1}}{(n-1)!\,t}$

$$\forall n \epsilon N \geq 2, \int_0^\infty \frac{f(t)}{t^n} dt = -\frac{f(t)t^{-n+1}}{(n-1)} - \frac{(D^1 f(t))t^{-n+2}}{(n-1)(n-2)} - \frac{(D^2 f(t))t^{-n+3}}{(n-1)(n-2)(n-3)} - \cdots \frac{(D^{n-2} f(t))}{(n-1)!\,t} \mid_0^\infty + \int_0^\infty \frac{D^{n-1} f(t)}{(n-1)!\,t} dt$$

$$\int_0^\infty \frac{f(t)}{t^n} dt = -\lim_{t\to\infty} \sum_{k=1}^{n-1} \frac{(D^{k-1} f(t))t^{-n+k}}{\prod_{j=1}^{k}(n-j)} + \lim_{t\to 0^+} \sum_{k=1}^{n-1} \frac{(D^{k-1} f(t))t^{-n+k}}{\prod_{j=1}^{k}(n-j)} + \int_0^\infty \frac{1}{(n-1)!} \frac{D^{n-1} f(t)}{t} dt$$

remark: the first sum is canceled because f(t) as well as its first n-2 derivatives are bounded functions (H4)

$$\mid D^{k-1} f(t) \mid \leq M_{k-1}, \forall k \epsilon [1; n-1]$$

$$0\leq \lim_{t\to\infty} \left| \sum_{k=1}^{n-1} \frac{\left(D^{k-1}f(t)\right)t^{-n+k}}{\prod_{j=1}^{k}(n-j)} \right| = \lim_{t\to\infty} \left| \sum_{k=1}^{n-1} \frac{\left(D^{k-1}f(t)\right)}{t^{n-k}\prod_{j=1}^{k}(n-j)} \right| \leq \lim_{t\to\infty} \sum_{k=1}^{n-1} \frac{|\, D^{k-1}f(t)\,|}{t^{n-k}\prod_{j=1}^{k}(n-j)} \leq$$

$$\lim_{t\to\infty} \sum_{k=1}^{n-1} \frac{M_{k-1}}{t^{n-k}\prod_{j=1}^{k}(n-j)} = 0$$

$$\lim_{t\to\infty} \sum_{k=1}^{n-1} \frac{(D^{k-1}f(t))t^{-n+k}}{\prod_{j=1}^{k}(n-j)} = 0$$

remark: the second sum is canceled because $\lim_{t\to 0^+} \frac{f(t)}{t^n} = \gamma$ existst (H3) and by using the L'Hopital's rule n-2 times (H1)

$$\lim_{t\to 0^+} \frac{f(t)}{t^n} = \lim_{t\to 0^+} \frac{D^1 f(t)}{n.t^{n-1}} = \lim_{t\to 0^+} \frac{D^2 f(t)}{n(n-1).t^{n-2}} = \ldots = \lim_{t\to 0^+} \frac{D^k f(t)}{t^{n-k}\prod_{j=0}^{k-1}(n-j)} = \lim_{t\to 0^+} \frac{D^{n-2} f(t)}{t^2\prod_{j=0}^{n-3}(n-j)} = \gamma$$

$$\lim_{t\to 0^+} \frac{D^k f(t)}{t^{n-k}} = \gamma.\prod_{j=0}^{k-1}(n-j) = C_k$$

$$\lim_{t\to 0^+} \sum_{k=1}^{n-1} \frac{(D^{k-1}f(t))t^{-n+k}}{\prod_{j=1}^{k}(n-j)} = \lim_{t\to 0^+} \sum_{k=1}^{n-1} \frac{D^{k-1}f(t)}{t^{n-k}\prod_{j=1}^{k}(n-j)} =$$

$$\lim_{t\to 0^+} \sum_{k=1}^{n-1} \frac{\text{t. } (D^{k-1}f(t))}{t^{n-(k-1)}\prod_{j=1}^{k}(n-j)} = \lim_{t\to 0^+} \sum_{k=1}^{n-1} \frac{\text{t. } C_{k-1}}{\prod_{j=1}^{k}(n-j)} = 0$$

$$\lim_{t\to 0+} \sum_{k=1}^{n-1} \frac{(D^{k-1}f(t))t^{-n+k}}{\prod_{j=1}^{k}(n-j)} = 0$$

$$\therefore \int_0^\infty \frac{f(t)}{t^n}\mathrm{d}t = \int_0^\infty \frac{1}{(n-1)!}\frac{D^{n-1}f(t)}{t}\mathrm{d}t$$

remark : valid for limit case n=1

therefore the left integral converges (H2), so does the right one.

$$\int_0^\infty \frac{f(t)}{t^n}\mathrm{d}t = \int_0^\infty \frac{1}{(n-1)!}\frac{D^{n-1}f(t)}{t}\mathrm{d}t \ ,\forall\, n \in N$$

$$\lim_{\sigma\to 0}\int_0^\infty \frac{f(t)}{t^n}e^{-\sigma t}\,\mathrm{d}t = \lim_{\sigma\to 0}\int_0^\infty \frac{1}{(n-1)!}\frac{D^{n-1}f(t)}{t}e^{-\sigma t}\,\mathrm{d}t$$

definition: let $g(t) = \frac{1}{(n-1)!}\frac{D^{n-1}f(t)}{t} \ \therefore \ G(\sigma) = \mathcal{L}\left[\frac{1}{(n-1)!}\frac{D^{n-1}f(t)}{t}\right] = \int_0^\infty \frac{1}{(n-1)!}\frac{D^{n-1}f(t)}{t}e^{-\sigma t}\,\mathrm{d}t$

remark : $\lim_{\sigma\to\infty} G(\sigma)=0$

$$\frac{D[G(\sigma)]}{\mathrm{d}\sigma} = \int_0^\infty \partial_\sigma \frac{1}{(n-1)!}\frac{D^{n-1}f(t)}{t}e^{-\sigma t}\,\mathrm{d}t = (-1)\int_0^\infty \frac{1}{(n-1)!}D^{n-1}f(t)e^{-\sigma t}\,\mathrm{d}t = \frac{-1}{(n-1)!}\mathcal{L}[D^{n-1}f(t)]$$

$$\mathrm{d}G = \frac{-1}{(n-1)!}\mathcal{L}[D^{n-1}f(t)]\mathrm{d}\sigma$$

$$\lim_{\sigma\to 0}\int_{G(\sigma)}^{0}\mathrm{d}G = \lim_{s\to 0}\int_s^\infty \frac{-1}{(n-1)!}\mathcal{L}[D^{n-1}f(t)]\,\mathrm{d}\sigma$$

$$0-\lim_{\sigma\to 0} G(\sigma)=\lim_{s\to 0}\int_s^{\infty}\frac{-1}{(n-1)!}\mathcal{L}[D^{n-1}f(t)]\,\mathrm{d}\,\sigma$$

$$\lim_{\sigma\to 0} G(\sigma)=\lim_{s\to 0}\int_s^{\infty}\frac{1}{(n-1)!}\mathcal{L}[D^{n-1}f(t)]\,\mathrm{d}\sigma$$

$$\lim_{\sigma\to 0}\int_0^{\infty}\frac{1}{(n-1)!}\frac{D^{n-1}f(t)}{t}e^{-\sigma t}\,\mathrm{d}t=\lim_{s\to 0}\int_s^{\infty}\frac{1}{(n-1)!}\mathcal{L}[D^{n-1}f(t)]\,\mathrm{d}\sigma$$

$$\lim_{\sigma\to 0}\int_0^{\infty}\frac{f(t)}{t^n}e^{-\sigma t}\,\mathrm{d}t=\lim_{s\to 0}\int_s^{\infty}\frac{1}{(n-1)!}\mathcal{L}[D^{n-1}f(t)]\,\mathrm{d}\,\sigma$$

$$\int_0^{\infty}\frac{f(t)}{t^n}\,\mathrm{d}t=\lim_{s\to 0}\int_S^{\infty}\frac{1}{(n-1)!}\mathcal{L}[D^{n-1}f(t)]\,\mathrm{d}\,\sigma$$

$$\int_0^{\infty}\frac{f(t)}{t^n}\,\mathrm{d}t=\lim_{s\to 0}\int_S^{\infty}\frac{1}{\Pi_{(n-1)}}\mathcal{L}[D^{n-1}f(t)]\,\mathrm{d}\sigma$$

$$\int_0^{\infty}\frac{\mathrm{f(t)}}{\mathrm{t^n}}\,\mathrm{dt}=\lim_{\mathrm{s}\to 0}\int_{\mathrm{S}}^{\infty}\frac{1}{\Pi_{(\mathrm{n}-1)}}\mathcal{L}[\mathrm{D^{n-1}f(t)}]\,\mathrm{d}\sigma=\lim_{\sigma\longrightarrow 0}\int_{\sigma}^{\infty}\int_{\sigma_1}^{\infty}\ldots\int_{\sigma_{n-1}}^{\infty}F(\sigma_n)\,\mathrm{d}\sigma_n\ldots\mathrm{d}\sigma_2\,\mathrm{d}\sigma_1 \quad (2017)$$

remark: As an alternative way to approach the same result we can make use of the Laplace transforms properties:

known

$$\mathcal{L}(D^{n-1}f(t)) = \sigma^{n-1}\mathcal{L}(f(t)) - \sigma^{n-2}f(0) - \sigma^{n-3}Df(0) - \ldots\ldots D^{n-2}f(0)$$

If we apply H3, f(t) and "at least" its first n-2 derivatives cancel at the origin, because γ is finite.

$$f(0) = Df(0) = D^2f(0) = \ldots = D^{n-2}f(0) = 0$$

$$\therefore$$

$$\mathcal{L}(D^{n-1}f(t)) = \sigma^{n-1}\mathcal{L}(f(t))$$

$$\lim_{s\to 0}\int_s^{\infty}\mathcal{L}\big(D^{n-1}f(t)\big)\,\mathrm{d}\sigma = \lim_{s\to 0}\int_s^{\infty}\sigma^{n-1}\mathcal{L}(f(t))\,\mathrm{d}\sigma =$$

$$\lim_{s\to 0}\int_s^{\infty}\int_0^{\infty}\sigma^{n-1}f(t)e^{-\sigma t}\,\mathrm{d}t\,\mathrm{d}\sigma = \int_0^{\infty} f(t)\lim_{s\to 0}\int_s^{\infty}\sigma^{n-1}e^{-t\sigma}\,\mathrm{d}\sigma\,\mathrm{d}t =$$

$$\int_0^{\infty} f(t)\frac{(n-1)!}{t^n}\,\mathrm{d}t = \lim_{s\to 0}\int_s^{\infty}\mathcal{L}\big(D^{n-1}f(t)\big)\,\mathrm{d}\sigma =$$

$$\int_0^{\infty} f(t)\,\frac{\Pi_{(n-1)}}{t^n}\,\mathrm{d}t = \lim_{s\to 0}\int_s^{\infty}\mathcal{L}\big(D^{n-1}f(t)\big)\,\mathrm{d}\sigma =$$

$$\int_0^{\infty}\frac{f(t)}{t^n}\,\mathrm{d}t = \lim_{s\to 0}\int_s^{\infty}\frac{1}{\Pi_{(n-1)}}\mathcal{L}\big(D^{n-1}f(t)\big)\,\mathrm{d}\sigma$$

4 Examples

Example 4.1: $\int_0^\infty \frac{(\mathrm{Sin}[t])^6}{t^6}\mathrm{d}t = \frac{11\pi}{40}$ (WolframAlpha)

$$\int_0^\infty \frac{(\mathrm{Sin}[t])^6}{t^6}\mathrm{d}t = \lim_{s\to 0}\int_s^\infty \frac{\mathcal{L}(D^5((\mathrm{Sin}[t])^6))}{5!}\mathrm{d}\sigma = \lim_{s\to 0}\int_s^\infty \frac{\mathcal{L}(D^5((\frac{e^{\mathrm{i}t}-e^{-\mathrm{i}t}}{2i})^6))}{5!}\mathrm{d}\sigma =$$

$$\lim_{s\to 0}\int_s^\infty \frac{\mathcal{L}(D^5(\frac{e^{\mathrm{i}6t}-6e^{i4t}+15e^{\mathrm{i}2t}-20+15e^{-\mathrm{i}2t}-6e^{-\mathrm{i}4t}+e^{-\mathrm{i}6t}}{-64})}{5!}\mathrm{d}\sigma =$$

$$\lim_{s\to 0}\int_s^\infty \mathcal{L}(\frac{-81ie^{\mathrm{i}6t}}{80}+\frac{4ie^{\mathrm{i}4t}}{5}-\frac{ie^{\mathrm{i}2t}}{16}+\frac{ie^{-\mathrm{i}2t}}{16}-\frac{4ie^{-\mathrm{i}4t}}{5}+\frac{81ie^{-\mathrm{i}6t}}{80})\,\mathrm{d}\sigma =$$

$$\lim_{s\to 0}\int_s^\infty \mathcal{L}(\frac{81e^{\mathrm{i}6t}}{80i}-\frac{4e^{\mathrm{i}4t}}{5i}+\frac{e^{\mathrm{i}2t}}{16i}-\frac{e^{-\mathrm{i}2t}}{16i}+\frac{4e^{-\mathrm{i}4t}}{5i}-\frac{81e^{-\mathrm{i}6t}}{80i})\,\mathrm{d}\sigma =$$

$$\lim_{s\to 0}\int_s^\infty \mathcal{L}(\frac{81}{40}\mathrm{Sin}[6t]-\frac{8}{5}\mathrm{Sin}[4t]+\frac{1}{8}\mathrm{Sin}[2t])\,\mathrm{d}\sigma =$$

$$\lim_{s\to 0}\int_s^\infty \frac{81}{40}\frac{6}{(\sigma^2+36)}-\frac{8}{5}\frac{4}{(\sigma^2+16)}+\frac{1}{8}\frac{2}{(\sigma^2+4)}\mathrm{d}\sigma = \lim_{s\to 0}\int_s^\infty \frac{81}{40}\frac{1/6}{((\frac{\sigma}{6})^2+1)}-\frac{8}{5}\frac{1/4}{((\frac{\sigma}{4})^2+1)}+\frac{1}{8}\frac{1/2}{((\frac{\sigma}{2})^2+1)}\mathrm{d}\sigma$$

$$\lim_{b\to\infty}(\frac{81}{40}\mathrm{Arctan}\left(\frac{b}{6}\right)-\frac{8}{5}\mathrm{Arctan}\left(\frac{b}{4}\right)+\frac{1}{8}\mathrm{Arctan}\left(\frac{b}{2}\right)) - \lim_{s\to 0}(\frac{81}{40}\mathrm{Arctan}\left(\frac{s}{6}\right)-\frac{8}{5}\mathrm{Arctan}\left(\frac{s}{4}\right)+\frac{1}{8}\mathrm{Arctan}\left(\frac{s}{2}\right))=$$

$$=\frac{\pi}{2}\left(\frac{81}{40}-\frac{8}{5}+\frac{1}{8}\right)=\frac{11\pi}{40}$$

Example 4.2: $\int_0^{\infty} \frac{(e^{-2t}-e^{-3t})^4}{t^4} dt$ =0.100336374512 (WolframAlpha)

$$\lim_{s\to 0}\int_s^{\infty} \frac{\mathcal{L}(D^3((e^{-2t}-e^{-3t})^4)}{3!}\,d\sigma = \lim_{s\to 0}\int_s^{\infty} \frac{\mathcal{L}(D^3(e^{-8t}-4e^{-9t}+6e^{-10t}-4e^{-11t}+e^{-12t})}{3!}\,d\sigma =$$

$$\lim_{s\to 0}\int_s^{\infty} \mathcal{L}(\frac{-256}{3}e^{-8t}+486e^{-9t}-1000e^{-10t}+\frac{2662}{3}e^{-11t}-288e^{-12t})\,d\sigma =$$

$$\lim_{s\to 0}\int_s^{\infty} \frac{-256}{3(\sigma+8)}+\frac{486}{(\sigma+9)}-\frac{1000}{(\sigma+10)}+\frac{2662}{3(\sigma+11)}-\frac{288}{\sigma+12}\,d\sigma =$$

$$\lim_{b\to\infty}\mathrm{Ln}\left(\frac{(b+9)^{486}(b+11)^{\frac{2662}{3}}}{(b+10)^{1000}(b+8)^{\frac{256}{3}}(b+12)^{288}}\right)-\lim_{s\to 0}\mathrm{Ln}(\frac{(s+9)^{486}(s+11)^{\frac{2662}{3}}}{(s+10)^{1000}(s+8)^{\frac{256}{3}}(s+12)^{288}})=$$

$$\lim_{s\to 0}\mathrm{Ln}(\frac{(s+10)^{1000}(s+8)^{\frac{256}{3}}(s+12)^{288}}{(s+9)^{486}(s+11)^{\frac{2662}{3}}})=\mathrm{Ln}(\frac{(10)^{1000}(8)^{\frac{256}{3}}(12)^{288}}{(9)^{486}(11)^{\frac{2662}{3}}})=0.100336374512$$

$$\int_0^{\infty}\frac{(e^{-2t}-e^{-3t})^4}{t^4}\,dt = \lim_{s\to 0}\int_s^{\infty}\frac{\mathcal{L}(D^3((e^{-2t}-e^{-3t})^4)}{3!}\,d\sigma = 0.100336374512$$

Example 4.3: $\int_0^\infty \frac{(1-\text{Cos}[t])^2}{t^4}\text{dt} = \frac{\pi}{6}$ (WolframAlpha)

$$\int_0^\infty \frac{(1-\text{Cos}[t])^2}{t^4}\text{dt} = \lim_{s\to 0}\int_s^\infty \frac{\mathcal{L}(D^3((1-\text{Cos}[t])^2))}{3!}\text{d}\sigma = \lim_{s\to 0}\int_s^\infty \frac{\mathcal{L}(D^3(1-2\text{Cos}[t]+(\text{Cos}[t])^2))}{6}\text{d}\sigma =$$

$$\lim_{s\to 0}\int_s^\infty \frac{\mathcal{L}(D^3(1-2\text{Cos}[t]+(\text{Cos}[t])^2))}{6}\text{d}\sigma = \lim_{s\to 0}\int_s^\infty \frac{\mathcal{L}(D^3(\frac{3}{2}-2\text{Cos}[t]+\frac{1}{2}\text{Cos}[2t]))}{6}\text{d}\sigma =$$

$$\lim_{s\to 0}\int_s^\infty \frac{\mathcal{L}(-2\text{Sin}[t]+4\text{Sin}[2t])}{6}\text{d}\sigma = \lim_{s\to 0}\int_s^\infty \left(\frac{-1}{3(\sigma^2+1)} + \frac{4}{3(\sigma^2+4)}\right)\text{d}\sigma = \lim_{s\to 0}\int_s^\infty \left(\frac{-1}{3(\sigma^2+1)} + \frac{1}{3((\frac{\sigma}{2})^2+1)}\right)\text{d}\sigma =$$

$$\lim_{s\to 0}\int_s^\infty \left(\frac{-1}{3(\sigma^2+1)} + \frac{2}{3}\frac{\frac{1}{2}}{((\frac{\sigma}{2})^2+1)}\right)\text{d}\sigma = \lim_{b\to\infty} \frac{-1}{3}\text{ArcTan}[b] + \frac{2}{3}\text{ArcTan}\left[\frac{b}{2}\right] = \left(-\frac{1}{3}+\frac{2}{3}\right)\frac{\pi}{2} = \frac{\pi}{6}$$

Example 4.4: $\int_0^\infty \frac{\text{Sin}[t]}{t}\text{dt} = \frac{\pi}{2}$ (The Dirichlet integral)

$$\int_0^\infty \frac{\text{Sin}[t]}{t}\text{dt} = \lim_{s\to 0}\int_s^\infty \mathcal{L}(\text{Sin}[t])\,\text{d}\sigma = \lim_{s\to 0}\int_s^\infty \frac{1}{(\sigma^2+1)}\text{d}\sigma = \lim_{b\to\infty}\text{ArcTan}[b] = \frac{\pi}{2}$$

5 Conclusion: this theorem allows to integrate special functions by using Laplace transforms. For low values of n, in the factor t^n we could solve the problem integrating a few times the laplace transform of f(t), but when n goes high it turns very tough, in many cases impossible to solve this way. Notice the strong conection with Cauchy's residue theorem which involves (n-1) derivatives.