On a result in the sieve method

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Abstract

In this note, we generalize a lemma of Heath-Brown in the sieve method.

Keywords: Möbius function, primes, induction.

1 Introduction

Let \( \mu(d) \) be the Möbius function, \( \Omega(n) = \sum_{p \mid n} 1 \), \( p \) is a prime variable.

In [2], Heath-Brown obtained the following result.

**Lemma A.** Let \( z_1, z_2 > 1 \), and \( Q = \prod_{p < z_1} p \).

Then

\[
\left| \sum_{\substack{d \mid (n,Q) \atop d < z_2}} \mu(d) - \sum_{\substack{d \mid (n,Q) \atop d < z_2}} \mu(d) \right| \leq \sum_{\substack{d \mid (n,Q) \atop z_2 \leq d < z_1z_2}} 1.
\]

For applications of this lemma see [1] and [2].

We give a generalization to this lemma.

Write \( \wp = \{ p : p \text{ is prime} \} \), \( z_1, z_2 > 1 \), \( \forall \mathcal{P} \subseteq \wp \), \( Q = Q(\mathcal{P}) = \prod_{p < z_1, p \in \mathcal{P}} p \).

If \( \mathcal{P} \) is empty set, that is \( \mathcal{P} = \emptyset \), then \( Q(\emptyset) = 1 \).

**Lemma B.** Let \( z_1, z_2 > 1 \), and \( \forall \mathcal{P} \subseteq \wp \), \( Q = Q(\mathcal{P}) = \prod_{p < z_1, p \in \mathcal{P}} p \).

Then

\[
\left| \sum_{\substack{d \mid (n,Q) \atop d < z_2}} \mu(d) - \sum_{\substack{d \mid (n,Q) \atop d < z_2}} \mu(d) \right| \leq \sum_{\substack{d \mid (n,Q) \atop z_2 \leq d < z_1z_2}} 1.
\]

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2 Proof of Lemma B

We use induction on \( \Omega(n) \).

The proof is the same as that in [2].

If \( n = 1 \) or \( \mathcal{P} = \emptyset \), the lemma holds.

We assume the result is true for \( n \) and for all \( \mathcal{P} \subseteq \varphi \).

We want to prove the lemma for \( pn \) and for all \( \mathcal{P} \subseteq \varphi \).

If \( p \geq z_1 \) or \( p \mid n \) or \( p \notin \mathcal{P} \), then \((np, Q) = (n, Q)\), the result follows by induction.

We suppose \( p < z_1 \), \( p \nmid n \) and \( p \in \mathcal{P} \).

\[
\sum_{d \mid (n, Q), \ d \leq z_2} \mu(d) - \sum_{d \mid (n, Q), \ d < z_2} \mu(d) = \sum_{d \mid (n, Q), \ d \geq z_2} \mu(d). \tag{2.1}
\]

We want to prove

\[
\left| \sum_{d \mid (pn, Q), \ d \geq z_2} \mu(d) \right| \leq \sum_{d \mid (pn, Q), \ z_2 \leq d < z_1 z_2} 1. \tag{2.2}
\]

We have

\[
\sum_{d \mid (pn, Q), \ d \geq z_2} \mu(d) = \sum_{d \mid (n, Q), \ d \geq z_2} \mu(d) + \sum_{pd \mid (pn, Q), \ pd \geq z_2} \mu(pd) = \sum_1 + \sum_2, \text{ say.} \tag{2.3}
\]

By the induction assumption, we have

\[
\left| \sum_1 \right| \leq \sum_{d \mid (n, Q), \ z_2 \leq d < z_1 z_2} 1. \tag{2.4}
\]

\[
\sum_2 = -\sum_{pd \mid (pn, Q), \ d \geq z_2/p} \mu(d) = -\sum_{d \mid (n, Q), \ d \geq z_2/p} \mu(d).
\]

If \( p < z_2 \), then by the induction assumption,

\[
\left| \sum_2 \right| \leq \sum_{d \mid (n, Q), \ z_2/p \leq d < z_1 z_2/p} 1.
\]

Hence

\[
\left| \sum_1 + \sum_2 \right| \leq \sum_{d \mid (n, Q), \ z_2 \leq d < z_1 z_2} 1 + \sum_{d \mid (n, Q), \ z_2/p \leq d < z_1 z_2/p} 1 = \sum_{d \mid (pn, Q), \ z_2 \leq d < z_1 z_2} 1. \tag{2.5}
\]
If $p \geq z_2$ and $(n, Q) > 1$, in this case $z_2 < z_1$, since $p \nmid n$ and $p \in \mathcal{P}$, then

$$|\sum_2| = | - \sum_{d|(n, Q)} \mu(d) | = 0 \leq \sum_{pd|(n, Q), z_2 \leq pd < z_1 z_2} 1.$$ 

Thus, unless $p \geq z_2$ and $(n, Q) = 1$, we obtain

$$|\sum_1 + \sum_2| \leq \sum_{d|(pn, Q), z_2 \leq d < z_1 z_2} 1.$$ 

If $p \geq z_2$ and $(n, Q) = 1$, since $p < z_1$ and $p \in \mathcal{P}$, in this case $z_2 < z_1$, we have

$$\sum_{d|(pn, Q), d \geq z_2} \mu(d) = \sum_{d|(p, Q), d \geq z_2} \mu(d) = -1,$$

hence

$$| \sum_{d|(pn, Q), d \geq z_2} \mu(d) | \leq \sum_{d|(pn, Q), z_2 \leq d < z_1 z_2} 1.$$ 

The lemma follows.

**References**
