A Short Proof Pertaining to the Euler/De-Moivre Complex Identity

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Abstract
This paper is a succinct demonstration of an equality derived from the differentiated expressions corresponding to the relation between Euler's and De-Moivre's formulations of complex numbers.
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This proof will demonstrate, that for any value of $\theta$; $|e^{i\theta}| = \frac{1}{2\cos \theta - e^{i\theta}}$

Phase 1: Consider the classic equivalency between De Moivre’s and Euler’s formulae

$$e^{i\theta} = \cos \theta$$
$$e^{i\theta} = \cos \theta + i\sin \theta$$
$$i\sin \theta = e^{i\theta} - \cos \theta$$

$$i = \frac{e^{i\theta} - \cos \theta}{\sin \theta}$$

Let this equation be titled $E1$.

Phase 2: Consider the same theorem; but differentiate the expression with respect to $\theta$ on both sides.

$$\frac{d[e^{i\theta}]}{d\theta} = \frac{d[\cos \theta + i\sin \theta]}{d\theta}$$

Since $\frac{d[e^{i\theta}]}{d\theta} = ie^{i\theta}$ (derivatives of exponential functions);

$$ie^{i\theta} = \frac{d[\cos \theta + i\sin \theta]}{d\theta}$$

$$ie^{i\theta} = \frac{d[\cos \theta]}{d\theta} + \frac{d[i\sin \theta]}{d\theta}$$

$$ie^{i\theta} = \frac{d[\cos \theta]}{d\theta} + i\frac{d[i\sin \theta]}{d\theta}$$

$$ie^{i\theta} = -\sin \theta + i\cos \theta$$

$$ie^{i\theta} - i\cos \theta = -\sin \theta$$

If one were to multiply both sides by a factor equivalent to -1;

$$-1(ie^{i\theta} - i\cos \theta) = -1(-\sin \theta)$$

$$i\cos \theta - ie^{i\theta} = \sin \theta$$

$$i(\cos \theta - e^{i\theta}) = \sin \theta$$

$$i = \frac{\sin \theta}{\cos \theta - e^{i\theta}}$$

Let this equation be titled $E2$. 

Phase 3: Equivalency;

Since $E1$ and $E2$ both describe $i$ in terms of trigonometric functions, they can be equated with one another;

$$i = \frac{e^{i\theta} - \cos\theta}{\sin\theta} = \frac{\sin\theta}{\cos\theta - e^{i\theta}}$$

Cross-multiplying yields;

$$(\sin\theta)(\sin\theta) = (e^{i\theta} - \cos\theta)(\cos\theta - e^{i\theta})$$

$$\sin^2 \theta = (e^{i\theta} - \cos\theta)(\cos\theta - e^{i\theta})$$

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$$\sin^2 \theta = e^{i\theta} \cos\theta - (e^{i\theta})^2 - \cos^2 \theta + e^{i\theta} \cos\theta$$

$$\sin^2 \theta = 2e^{i\theta} \cos\theta - (e^{i\theta})^2 - \cos^2 \theta$$

$$\sin^2 \theta + \cos^2 \theta = 2e^{i\theta} \cos\theta - (e^{i\theta})^2$$

Since $\sin^2 \theta + \cos^2 \theta = 1$;

$$2e^{i\theta} \cos\theta - (e^{i\theta})^2 = 1$$

$$2e^{i\theta} \cos\theta - (e^{i\theta})^2 = 1$$

$$e^{i\theta}(2\cos\theta - e^{i\theta}) = 1$$

$$e^{i\theta} = \frac{1}{2\cos\theta - e^{i\theta}}$$

On account of the original theorem, $\theta$ must be expressed in degrees in the term ‘2cos’, and radians in the term $e^{i\theta}$;

Confirming this equality: For $\theta = \pi$ radians:

$$e^{i\theta} = e^{i\pi} = -1$$

$$\frac{1}{2\cos\theta - e^{i\theta}} = \frac{1}{2\cos180 - e^{i\pi}}$$

$$= \frac{1}{-2 - (-1)}$$

$$\frac{1}{-1} = -1 = e^{i\pi}$$