Physics and Division by Zero Calculus

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Abstract: In order to show some power of the division by zero calculus we will give several simple applications to physics. Recall that Oliver Heaviside: *Mathematics is an experimental science, and definitions do not come first, but later on.*

Key Words: Laurent expansion, division by zero, division by zero calculus, $1/0 = 0/0 = z/0 = \tan(\pi/2) = \log 0 = 0, [(z^n)/n]_{n=0} = \log z$, $[e^{(1/z)}]_{z=0} = 1$, physics.

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1 Introduction

We will see the division by zero properties in various physical formulas. We found many and many division by zero phenomena in physics and others, however, we expect many publications about them by the related specialists. As the first stage, here we refer only to elementary formulas, as examples.

2 Simple introduction of the division by zero calculus

We will recall the simple background on the division by zero calculus for differentiable functions based on ([24, 25]).

For a function $y = f(x)$ which is $n(n > 0)$ order differentiable at $x = a$, we will **define** the value of the function

$$\frac{f(x)}{(x-a)^n}$$

at the point $x = a$ by the value

$$\frac{f^{(n)}(a)}{n!}.$$

For the important case of $n = 1$,

$$\frac{f(x)}{x-a}|_{x=a} = f'(a). \tag{2.1}$$

In particular, the values of the functions $y = 1/x$ and $y = 0/x$ at the origin $x = 0$ are zero. **We write them as $1/0 = 0$ and $0/0 = 0$, respectively.** Of course, the definitions of $1/0 = 0$ and $0/0 = 0$ are not usual ones in the sense: $0 \cdot x = b$ and $x = b/0$. Our division by zero is given in this sense and is not given by the usual sense. However, we gave several definitions for $1/0 = 0$ and $0/0 = 0$. See, for example, [22].

In addition, when the function $f(x)$ is not differentiable, by many meanings of zero, we **should define** as

$$\frac{f(x)}{x-a}|_{x=a} = 0,$$

for example, since 0 **represents impossibility**. In particular, the value of the function $y = |x|/x$ at $x = 0$ is zero.

We will note its naturality of the definition.

Indeed, we consider the function $F(x) = f(x) - f(a)$ and by the definition, we have

$$\frac{F(x)}{x-a}|_{x=a} = F'(a) = f'(a).$$

Meanwhile, by the definition, we have

$$\lim_{x\to a}\frac{F(x)}{x-a}=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=f'(a). \tag{2.2}$$

For many applications, see the references cited in the reference.

The identity (2.1) may be regarded as an interpretation of the differential coefficient $f'(a)$ by the concept of the division by zero. Here, we do not use the concept of limitings. This means that NOT

$$\lim_{x\to a}\frac{f(x)}{x-a}$$

BUT

$$\frac{f(x)}{x-a}|_{x=a}.$$

Note that $f'(a)$ represents the principal variation of order $x-a$ of the function $f(x)$ at $x=a$ which is defined independently of $f(a)$ in (2.2). This is a basic meaning of the division by zero calculus $\frac{f(x)}{x-a}|_{x=a}$.

Following this idea, we can accept the formula, naturally, for also $n=0$ for the general formula; that is,

$$\frac{f(x)}{(x-a)^0}|_{x=a}=\frac{f^{(0)}(a)}{0!}=f(a).$$

In the expression (2.1), the value $f'(a)$ in the right hand side is represented by the point a, meanwhile the expression

$$\frac{f(x)}{x-a}|_{x=a} \tag{2.3}$$

in the left hand side, is represented by the dummy variable $x-a$ that represents the property of the function around the point $x=a$ with the sense of the division

$$\frac{f(x)}{x-a}.$$

For $x\neq a$, it represents the usual division.

When we apply the relation (2.1) to the elementary formulas for differentiable functions, we can imagine some deep results. For example, in the simple formula

$$(u+v)'=u'+v',$$

we have the result

$$\frac{u(x)+v(x)}{x-a}|_{x=a} = \frac{u(x)}{x-a}|_{x=a} + \frac{v(x)}{x-a}|_{x=a},$$

that is not trivial in our definition. This is a result from the property of derivatives.

In the following well-known formulas, we have some **deep meanings** on the division by zero calculus.

$$(uv)' = u'v + uv',$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

and the famous laws

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

and

$$\frac{dy}{dx} \cdot \frac{dx}{dy} = 1.$$

Note also the logarithm derivative, for $u, v > 0$

$$(\log(uv))' = \frac{u\prime}{u} + \frac{v\prime}{v}$$

and for $u > 0$

$$(u^v)' = u^v \left(v' \log u + v\frac{u'}{u}\right).$$

We note the basic relation for analytic functions $f(z)$ for the analytic extension of $f(x)$ to complex variable z

$$\frac{f(x)}{(x-a)^n}|_{x=a} = \frac{f^{(n)}(a)}{n!} = Res._{\zeta=a}\left\{\frac{f(\zeta)}{(\zeta-a)^{n+1}}\right\}.$$

We therefore see the basic identities among the division by zero calculus, differential coefficients and residues in the case of analytic functions. Among these basic concepts, the differential coefficients are studied deeply and so, from the results of the differential coefficient properties, we can derive another results for the division by zero calculus and residures. See [25].

3 Examples

3.1 Bhāskara's example – sun and shadow

We will consider the circle such that its center is the origin and its radius R. We consider the point S (sun) on the circle such that $\angle SOI = \theta$; $O(0,0), I(R,0)$. For fixed $d > 0$, we consider the common point $(-L, -d)$ of two line OS and $y = -d$. Then we obtain the identity

$$L = \frac{R\cos\theta}{R\sin\theta}d,$$

([5], page 77.). That is the length of the shadow of the segment of $(0,0) - (0,-d)$ onto the line $y = -d$ of the sun S.

When we consider $\theta \to +0$ we see that, of course

$$L \to \infty.$$

Therefore, Bhāskara considered that

$$\frac{1}{0} = \infty. \tag{3.1}$$

Even nowadays, our mathematics and many people consider so.

However, for $\theta = 0$, we have S=I and we can not consider any shadow on the line $y = -d$, so we should consider that $L = 0$; that is

$$\frac{1}{0} = 0. \tag{3.2}$$

Nothing may be represented by zero; it will be a sense of zero.

Furthermore, for $R = 0$; that is, for S=O, we see its shadow is the point $(0, -d)$ and so $L = 0$ and

$$L = \frac{0\cos\theta}{0\sin\theta}d = 0;$$

that is

$$\frac{0}{0} = 0.$$

This example shows that the division by zero calculus is not almighty.

Note that both identities (3.1) and (3.2) are right in their senses. Depending on the interpretations of 1/0, we obtain INFINITY and ZERO, respectively.

3.1.1 Another example

We consider a triangle ABC with $\overline{AB} = c$, $\overline{BC} = a$, $\overline{CA} = c$. Let x_i be the orthogonal projections of AB and AC to the line BC. Then we have

$$x_i = \frac{1}{2}\left\{a \mp \frac{(b+c)|b-c|}{a}\right\},$$

([5], pages 70-71.). If $b = c$, then, of course, $x_1 = x_2 = a/2$. For $a = 0$, by the division by zero, we have the reasonable value $x_1 = x_2 = 0$.

3.1.2 Remark

For the example ([5], pages 70-71.), we see that now there is no problem, because we have the relation

$$\frac{R}{j_c} = \frac{r}{R}.$$

Then, we have the right formula

$$y = r \sin \overline{\varphi}.$$

3.2 In balance of a steelyard

We will consider the balance of a steelyard and then we have the equation

$$aF_a = bF_b \tag{3.3}$$

as the moment equality. Here, a, b are the distances from a fixed point and force F_a, F_b points, respectively. Then, we have

$$F_a = \frac{b}{a} F_b.$$

For $a = 0$, should be considered as $F_a = 0$ by the division by zero $b/0 = 0$?

The identity (3.3) appears in many situations, and the above result may be valid similarly.

As a typical case, we recall

Ctesibios (BC. 286-222): *We consider a flow tube with some fluid. Then, when we consider some cut with a plane with its area S and with its velocity v of the fluid on the plane, by continuity, we see that for any cut plane, $Sv = C$; C : constant. That is,*

$$v = \frac{C}{S}.$$

When S tends to zero, the velocity v tends to infinity. However, for $S = 0$, the flow stops and so, $v = 0$. Therefore, this example shows the division by zero $C/0 = 0$ clearly. Of course, in the situation, we have $0/0 = 0$, trivially.

We can find many and many similar examples, for example, in Archimedes' principle and Pascal's principle.

We will state one more example:

E. Torricelli (1608 -1646): *We consider some water tank and the initial high $h = h_0$ for $t = 0$ and we assume that from the bottom of the tank with a hole of area A, water is fall down. Then, by the law with a constant k*

$$\frac{dh}{dt} = -\frac{k}{A}\sqrt{h},$$

we have the equation

$$h(t) = \left(\sqrt{h_0} - \frac{k}{2A}\right)^2.$$

Similarly, of course, for $A = 0$, we have

$$h(t) = h_0.$$

Even the fundamental relation among velocity v, time t and distance s

$$t = \frac{s}{v},$$

we will be able to understand the division by zero

$$\frac{s}{0} = 0$$

and

$$\frac{0}{0} = 0.$$

3.3 By rotation

We will give a simple physical model showing the result $\frac{0}{0} = 0$. We shall consider a disc with $x^2 + y^2 \leq a^2$ rowling uniformly with a positive constant angular velocity ω with its center at the origin. Then we see, at the only origin, $\omega = 0$ and at other all points, ω is a constant. Then, we see that the velocity and the radius r are zero at the origin. This will mean that, in the general formula

$$v = r\omega,$$

or, in

$$\omega = \frac{v}{r}$$

at the origin,

$$\frac{0}{0} = 0.$$

We will not be able to obtain the result from

$$\lim_{r \to +0} \omega = \lim_{r \to +0} \frac{v}{r},$$

because it is the constant.

For a uniform rotation with velocity $\mathbf{v}$ with its center O' and with its radius r. For the angular velocity vector ω and for the moving position P on the circle, we set $\mathbf{r} = OP$. Then,

$$\mathbf{v} = \omega \times \mathbf{r}.$$

If $\omega \times \mathbf{r} = 0$, then, of course, $\mathbf{v} = 0$.

3.4 By the Newton's law

We will recall the fundamental law by Newton:

$$F = G\frac{m_1 m_2}{r^2} \tag{3.4}$$

for two masses m_1, m_2 with a distance r and for a constant G. Of course,

$$\lim_{r \to +0} F = \infty,$$

however, as in our fraction

$$F = 0 = G\frac{m_1 m_2}{0}. \tag{3.5}$$

Of course, here, we can consider the above interpretation for the mathematical formula (3.4) as the new interpretation (3.5). In the ideal case, when two masses are on one point, the force F will not be positive and it will be reduced to zero.

In the Kepler (1571 - 1630) - Newton (1642 - 1727) law for central force movement of the planet,

$$m\frac{d^2\mathbf{r}}{dt^2} = -\frac{GmM}{r^3}\mathbf{r},$$

of course, we have $\mathbf{r} = \mathbf{0}$ for $r = 0$.

For the Coulomb's law, see similar formulas. Indeed, in the formula

$$F = k\frac{(+q)(-q)}{r^2}$$

for $r = 0$, we have $F = 0$.

In general, in the formula

$$F = k\frac{(Q_1)(Q_2)}{r^2}$$

for $r = 0$, we have $F = 0$ (S. Senuma: 2016.8.20.).

Furthermore, as well-known, the bright at a point at the distance r from the origin is given by the formula

$$B = k\frac{P}{r^2},$$

where k is a constant and P is the amount of the light. Of course, we have, at the infinity

$$B = 0.$$

Then, meanwhile, may we consider as

$$B = 0$$

at the origin $r = 0$? Then we can obtain our formula

$$k\frac{P}{0} = 0,$$

as in our new formula.

3.5 An interpretation of $0 \times 0 = 100$ from $100/0 = 0$

The expression $100/0 = 0$ will represent some divisor by the zero in a sense that is not the usual one, and so, we will be able to consider some product sense $0 \times 0 = 100$.

We will show such an interpretation.

We shall consider same two masses m, however, their constant velocities v for the origin are the same on the real line, in the symmetry way. We consider the moving energy product E^2,

$$\frac{1}{2}mv^2 \times \frac{1}{2}m(-v)^2 = E^2.$$

We shall consider at the origin and we assume that the two masses stop at the origin (possible in some case). Then, we can consider, formally

$$0 \times 0 = E^2.$$

The moving energies change to other energies, however, we can obtain some interpretation as in the above.

This example was discovered by M. Yamane presented in the paper [7].

3.6 Capillary pressure in a narrow capillary tube

In a narrow capillary tube saturated with fluid such as water, the capillary pressure is simply expressed as follows,

$$Pc = \frac{2\sigma}{r}$$

where Pc is capillary pressure (suction pressure), σ is surface tension, and r is radius. If r is zero, there is no pressure. However Pc shows infinity, in the common meaning.

This simple equation is based on the Laplace-Young equation

$$P = \sigma \left(\frac{1}{R_1} + \frac{1}{R_2} \right)$$

where R_1 and R_2 are two principal radii of curvature at any point on the surface of a droplet or a bubble and in the case of spherical form $R_1 = R_2 = R$. For a spherical bubble the pressure difference across the bubble film is zero since the pressure is the same on both sides of the film. The Laplace-Young equation reduces to

$$\frac{1}{R_1} + \frac{1}{R_2} = 0.$$

On other hand when diameter of a bubble is decreased and becomes $0 (R = 0)$, the bubbles collapse and enormous energy is generated. Accumulated free energy in the bubble is released instantaneously.

This example was discovered by M. Kuroda presented in [7].

3.7 Circles and curvature - an interpretation of the division by zero $r/0 = 0$

We consider a solid body called right circular cone whose bottom is a disc with its radius r_2. We cut the body with a disc of radius $r_1 (0 < r_1 < r_2)$ that is parallel to the bottom disc. We denote the distance by d between both discs and R the distance between the top point of the cone and the bottom circle on the surface of the cone. Then, R is calculated by Eko Michiwaki (8 year old daughter of H. Michiwaki) as follows:

$$R = \frac{r_2}{r_2 - r_1} \sqrt{d^2 + (r_2 - r_1)^2},$$

that is called *EM radius*, because by the rotation of the cone on the plane, the bottom circle writes the circle of radius R. We denote by $K = K(R) = 1/R$ the curvature of the circle with its radius R. We fix the distance d. Now note that

$$r_1 \to r_2 \Longrightarrow R \to \infty.$$

This will be natural in the sense that when $r_1 = r_2$, the circle with its radius R becomes a line.

However, the division by zero will mean that when $r_1 = r_2$, the above EM radius formula makes sense and $R = 0$. What does it mean? Here, note that, however, then the curvature $K = K(0) = 0$ by the division by zero calculus; that is, the circle with its radius R becomes a line, similarly. The curvature of a point (circle of radius zero) is zero. See [10].

3.8 Vibration

In the typical ordinary differential equation

$$m\frac{d^2x}{dt^2} = -kx,$$

we have a general solution

$$x = C_1 \cos(\omega t + C_2), \quad \omega = \sqrt{\frac{k}{m}}.$$

If $k = 0$, that is, if $\omega = 0$, then the period T that is given by

$$T = \frac{2\pi}{\omega}$$

should be understood as $T = 0$?

In the typical ordinary differential equation

$$m\frac{d^2x}{dt^2} + kx = f \cos \omega t,$$

we have a special solution

$$x = \frac{f}{m} \frac{1}{|\omega^2 - \omega_0^2|} \cos \omega t, \quad \omega_0 = \sqrt{\frac{k}{m}}.$$

Then, how will be the case

$$\omega = \omega_0$$

?

For example, for the differential equation

$$y'' + a^2 y = b \cos \lambda x,$$

we have a special solution, with the condition $\lambda \neq a$

$$y = \frac{b}{a^2 - \lambda^2} \cos \lambda x.$$

Then, when $\lambda = a$, by the division by zero calculus, we obtain the special solution

$$y = \frac{bx \sin(ax)}{2a} + \frac{b \cos ax}{4a^2}.$$

3.9 Spring or circut

We will consider a spring with two spring constants $\{k_j\}$ in a line. Then, the spring constant k of the spring is given by the formula

$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2},$$

by Hooke's law. We know, in particular, if $k_1 = 0$, then

$$\frac{1}{k} = \frac{1}{0} + \frac{1}{k_2},$$

and by the division by zero,

$$k = k_2,$$

that is very reasonable. In particular, by Hooke's law, we see that

$$\frac{0}{0} = 0.$$

As we saw for the case of harmonic mean, in this case $k_1 = 0$, the zero means that the spring does not exist.

The corresponding result for the case of Ohmu's law is similar and valid.

3.10 Motion

A and B start at the origin on the real positive axis with, for $t = 0$

$$\frac{d^2x}{dt^2} = a, \quad \frac{dx}{dt} = u$$

and

$$\frac{d^2x}{dt^2} = b, \quad \frac{dx}{dt} = v,$$

respectively. After the time T and at the distance X from the origin, if they meet, then we obtain the relations

$$T = \frac{2(u-v)}{b-a}$$

and

$$X = \frac{2(u-v)(ub-va)}{(b-a)^2}.$$

For the case $a = b$, we obtain the reasonable results $T = 0$ and $X = 0$.

We will consider the motion (x, y) represented by $x = \cos\theta, y = \sin\theta$ from $(1, 0)$ to $(-1, 0)$ $(0 \le \theta \le \pi)$ with the condition

$$v_x = \frac{dx}{dt} = -\sin\theta \frac{d\theta}{dt} = V \quad \text{(constant)}.$$

Then, we have that

$$v_y = \frac{dy}{dt} = -V\frac{1}{\tan\theta},$$

and

$$a_y = \frac{d^2y}{dt^2} = -V^2\frac{1}{\sin^3\theta}.$$

Then we see that

$$v_y(1, 0) = 0, \text{that is}, \frac{1}{\tan 0} = 0,$$

$$v_y(-1, 0) = 0, \text{that is}, \frac{1}{\tan\pi} = 0,$$

$$a_y(1, 0) = 0, \text{that is}, \frac{1}{\sin^3 0} = 0,$$

and

$$a_y(-1, 0) = 0, \text{that is}, \frac{1}{\sin^3\pi} = 0.$$

3.11 Darcy's law for fluid through porpous media

Diffusion phenomenon and penetration phenomenon may be represented by the partial differential equations

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u^m}{\partial x^2}$$

for some constants ν and m.

Indeed, density u and pressure p may be related by equation of state

$$u = \gamma p^\alpha,$$

with some constants γ and α.

By the conservative law, we have, for porocity ν and velocity v

$$\frac{\partial(uv)}{\partial x} = -\nu \frac{\partial u}{\partial t}.$$

At the last, by Darcy's law, we have for some constant k

$$v = -k \frac{\partial p}{\partial x}.$$

By chancelling v, p from three equations we obtain

$$\frac{\partial u}{\partial t} = \frac{k}{\nu\gamma(\alpha+1)} \frac{\partial}{\partial x} \left(\frac{\partial}{\partial x} u^{1+1/\alpha} \right)$$

([6], 21-22). As basic references, see ([3, 1, 2]).

Be setting $m = 1 + 1/\alpha$, we have the desired equation.

Note that for $\alpha = 0$, with the division by zero $1/0 = 0$, we have the right differential equation.

Meanwhile, for $\alpha = -1$, by the division by zero calculus, we have

$$\frac{\partial u}{\partial t} = \frac{k}{\nu\gamma} \frac{\partial^2}{\partial x^2} \left(-\log u\right).$$

How will be this partial differential equation?

3.12 RCL and RL circuts

We will consider an RCL circut stated by the ordinary differential equation

$$L\frac{di}{dt} + Ri + \frac{1}{C} \int idi = E_0 \sin \omega t, \tag{3.6}$$

$$i = \frac{E_0}{\sqrt{R^2 + ((\omega L - (1/(\omega C))^2}} \sin\left(\omega t - \varphi\right); \tag{3.7}$$

$$\varphi = \arctan \frac{1}{R} \left(\omega L - \frac{1}{\omega C} \right). \tag{3.8}$$

Here, $E_0 \sin \omega t$ is a given AC voltage.

In this circut, for the case $C = 0$ that is the condense is missing, we obtain the corresponding result precisely by the division by zero

$$\frac{1}{C} = 0$$

and

$$\frac{1}{\omega C} = 0.$$

We can find many and many the division by zero and division by zero calculus in physics.

References

[1] A. Bejan, Convection Heat Transfer, John Wiley & Sons, (1984).

[2] R. E. Cunningham and R. J. J. Williams, Diffusion in Gases and Porous Media, New York: Plenum Press (1980).

[3] H. Darcy, Les fontaines publiques de la ville de Dijon, Paris: Dalmont (1856).

[4] W. W. Däumler, H. Okumura, V. V. Puha and S. Saitoh, Horn Torus Models for the Riemann Sphere and Division by Zero, viXra:1902.0223 submitted on 2019-02-12 18:39:18.

[5] T. Hayashi, A Study of Indian Algebra, Japanese Translations with Notes of the BĪjagannita and BĪjapallava, Kouseisia Kouseikaku (2016).

[6] A. Kaneko, Introduction to partial differential equations (in Japanese), University of Tokyo Press, (1998).

[7] M. Kuroda, H. Michiwaki, S. Saitoh and M. Yamane, New meanings of the division by zero and interpretations on $100/0 = 0$ and on $0/0 = 0$, Int. J. Appl. Math. **27** (2014), no 2, pp. 191-198, DOI: 10.12732/ijam.v27i2.9.

[8] T. Matsuura and S. Saitoh, Matrices and division by zero $z/0 = 0$, Advances in Linear Algebra & Matrix Theory, **6**(2016), 51-58 Published Online June 2016 in SciRes. http://www.scirp.org/journal/alamt http://dx.doi.org/10.4236/alamt.2016.62007.

[9] T. Matsuura, H. Michiwaki and S. Saitoh, $\log 0 = \log \infty = 0$ and applications, Differential and Difference Equations with Applications, Springer Proceedings in Mathematics & Statistics, **230** (2018), 293-305.

[10] H. Michiwaki, S. Saitoh and M. Yamada, Reality of the division by zero $z/0 = 0$, IJAPM International J. of Applied Physics and Math. **6**(2015), 1–8. http://www.ijapm.org/show-63-504-1.html

[11] H. Michiwaki, H. Okumura and S. Saitoh, Division by Zero $z/0 = 0$ in Euclidean Spaces, International Journal of Mathematics and Computation, **2**8(2017); Issue 1, 1-16.

[12] H. Okumura, A Chain of Circles Touching a Circle and Its Tangent and Division by Zero, viXra:2001.0034 submitted on 2020-01-03 01:08:58.

[13] H. Okumura and S. Saitoh, The Descartes circles theorem and division by zero calculus, https://arxiv.org/abs/1711.04961 (2017.11.14).

[14] H. Okumura and S. Saitoh, Remarks for The Twin Circles of Archimedes in a Skewed Arbelos by H. Okumura and M. Watanabe, Forum Geometricorum, **18**(2018), 97-100.

[15] H. Okumura and S. Saitoh, Applications of the division by zero calculus to Wasan geometry, GLOBAL JOURNAL OF ADVANCED RESEARCH ON CLASSICAL AND MODERN GEOMETRIES" (GJARCMG), **7**(2018), 2, 44–49.

[16] H. Okumura and S. Saitoh, Wasan Geometry and Division by Zero Calculus, Sangaku Journal of Mathematics (SJM), **2** (2018), 57–73.

[17] H. Okumura, S. Saitoh, Remarks for The Twin Circles of Archimedes in a Skewed Arbelos by Okumura and Watanabe, Forum Geom., **18** (2018) 97–100.

[18] H. Okumura and S. Saitoh, Values of the Riemann Zeta Function by Means of Division by Zero Calculus viXra:1907.0437 submitted on 2019-07-23 20:48:54.

[19] H. Okumura, Pappus Chain and Division by Zero Calculus, viXra:2006.0095 submitted on 2020-06-11 17:00:55.

[20] S. Pinelas and S. Saitoh, Division by zero calculus and differential equations. Differential and Difference Equations with Applications. Springer Proceedings in Mathematics & Statistics, **230** (2018), 399-418.

[21] S. Saitoh, What Was Division by Zero?; Division by Zero Calculus and New World, viXra:1904.0408 submitted on 2019-04-22 00:32:30.

[22] S. Saitoh, Fundamental of Mathematics; Division by Zero Calculus and a New Axiom, viXra:1908.0100 submitted on 2019-08-06 20:03:01.

[23] S. Saitoh, Essential Problems on the Origins of Mathematics; Division by Zero Calculus and New World Authors, viXra:1912.0300 submitted on 2019-12-16 18:37:53.

[24] S. Saitoh, Division by Zero Calculus for Differentiable Functions - L'Hôpital's Theorem Versions, viXra:2001.0091 submitted on 2020-01-06 17:52:07.

[25] S. Saitoh, Division by Zero Calculus, Derivatives and Laurent's Expansion, viXra:2001.0586 submitted on 2020-01-27 16:28:38.

[26] S.-E. Takahasi, M. Tsukada and Y. Kobayashi, Classification of continuous fractional binary operations on the real and complex fields, Tokyo Journal of Mathematics, **38**(2015), no. 2, 369-380.