Physics and Division by Zero Calculus

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Abstract: In order to show some power of the division by zero calculus we will give several simple applications to physics. Recall that Oliver Heaviside: Mathematics is an experimental science, and definitions do not come first, but later on.

Key Words: Laurent expansion, division by zero, division by zero calculus, $1/0 = 0/0 = z/0 = \tan(\pi/2) = \log 0 = 0, [z^n/n]_{n=0} = \log z, [e^{1/z}]_{z=0} = 1$, physics.

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1 Introduction

We will see the division by zero properties in various physical formulas. We found many and many division by zero phenomena in physics and others, however, we expect many publications about them by the related specialists. As the first stage, here we refer only to elementary formulas, as examples.
2 Simple introduction of the division by zero calculus

We will recall the simple background on the division by zero calculus for differentiable functions based on ([24, 25]).

For a function \( y = f(x) \) which is \( n(n > 0) \) order differentiable at \( x = a \), we will define the value of the function

\[
\frac{f(x)}{(x - a)^n}
\]

at the point \( x = a \) by the value

\[
\frac{f^{(n)}(a)}{n!}.
\]

For the important case of \( n = 1 \),

\[
\frac{f(x)}{x - a}\bigg|_{x=a} = f'(a).
\]  

In particular, the values of the functions \( y = 1/x \) and \( y = 0/x \) at the origin \( x = 0 \) are zero. We write them as \( 1/0 = 0 \) and \( 0/0 = 0 \), respectively. Of course, the definitions of \( 1/0 = 0 \) and \( 0/0 = 0 \) are not usual ones in the sense: \( 0 \cdot x = b \) and \( x = b/0 \). Our division by zero is given in this sense and is not given by the usual sense. However, we gave several definitions for \( 1/0 = 0 \) and \( 0/0 = 0 \). See, for example, [22].

In addition, when the function \( f(x) \) is not differentiable, by many meanings of zero, we should define as

\[
\frac{f(x)}{x - a}\bigg|_{x=a} = 0,
\]

for example, since 0 represents impossibility. In particular, the value of the function \( y = |x|/x \) at \( x = 0 \) is zero.

We will note its naturality of the definition.

Indeed, we consider the function \( F(x) = f(x) - f(a) \) and by the definition, we have

\[
\frac{F(x)}{x - a}\bigg|_{x=a} = F'(a) = f'(a).
\]
Meanwhile, by the definition, we have

$$\lim_{x \to a} \frac{F(x)}{x - a} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a). \quad (2.2)$$

For many applications, see the references cited in the reference.

The identity (2.1) may be regarded as an interpretation of the differential coefficient $f'(a)$ by the concept of the division by zero. Here, we do not use the concept of limitings. This means that NOT

$$\lim_{x \to a} \frac{f(x)}{x - a}$$

BUT

$$\left. \frac{f(x)}{x - a} \right|_{x=a}.$$

Note that $f'(a)$ represents the principal variation of order $x - a$ of the function $f(x)$ at $x = a$ which is defined independently of $f(a)$ in (2.2). This is a basic meaning of the division by zero calculus $\left. \frac{f(x)}{x - a} \right|_{x=a}$.

Following this idea, we can accept the formula, naturally, for also $n = 0$ for the general formula; that is,

$$\left. \frac{f(x)}{(x - a)^n} \right|_{x=a} = \frac{f^{(0)}(a)}{0!} = f(a).$$

In the expression (2.1), the value $f'(a)$ in the right hand side is represented by the point $a$, meanwhile the expression

$$\left. \frac{f(x)}{x - a} \right|_{x=a} \quad (2.3)$$

in the left hand side, is represented by the dummy variable $x - a$ that represents the property of the function around the point $x = a$ with the sense of the division

$$\frac{f(x)}{x - a}.$$

For $x \neq a$, it represents the usual division.

When we apply the relation (2.1) to the elementary formulas for differentiable functions, we can imagine some deep results. For example, in the simple formula

$$(u + v)' = u' + v',$$
we have the result
\[
\frac{u(x) + v(x)}{x - a}\bigg|_{x=a} = \frac{u(x)}{x - a}\bigg|_{x=a} + \frac{v(x)}{x - a}\bigg|_{x=a},
\]
that is not trivial in our definition. This is a result from the property of derivatives.

In the following well-known formulas, we have some **deep meanings** on the division by zero calculus.

\[
(uv)' = u'v + uv',
\]
\[
\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}
\]
and the famous laws
\[
\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt}
\]
and
\[
\frac{dy}{dx} \cdot \frac{dx}{dy} = 1.
\]
Note also the logarithm derivative, for \(u, v > 0\)

\[
(\log(uv))' = \frac{u'}{u} + \frac{v'}{v}
\]
and for \(u > 0\)

\[
(u^v)' = u^v \left( v' \log u + v \frac{u'}{u} \right).
\]

We note the basic relation for analytic functions \(f(z)\) for the analytic extension of \(f(x)\) to complex variable \(z\)

\[
\frac{f(x)}{(x - a)^n}\bigg|_{x=a} = \frac{f^{(n)}(a)}{n!} = Res_{\zeta=a} \left\{ \frac{f(\zeta)}{(\zeta - a)^{n+1}} \right\}.
\]
We therefore see the basic identities among the division by zero calculus, differential coefficients and residues in the case of analytic functions. Among these basic concepts, the differential coefficients are studied deeply and so, from the results of the differential coefficient properties, we can derive another results for the division by zero calculus and residues. See [25].
3 Examples

3.1 Bhāskara’s example – sun and shadow

We will consider the circle such that its center is the origin and its radius $R$. We consider the point S (sun) on the circle such that $\angle SOI = \theta$; $O(0,0), I(R,0)$. For fixed $d > 0$, we consider the common point $(-L, -d)$ of two line OS and $y = -d$. Then we obtain the identity

$$L = \frac{R \cos \theta}{R \sin \theta} d,$$

([5], page 77.). That is the length of the shadow of the segment of $(0,0) - (0,-d)$ onto the line $y = -d$ of the sun S.

When we consider $\theta \to +0$ we see that, of course

$$L \to \infty.$$  

Therefore, Bhāskara considered that

$$\frac{1}{0} = \infty.$$  

(3.1)

Even nowadays, our mathematics and many people consider so.

However, for $\theta = 0$, we have S=I and we can not consider any shadow on the line $y = -d$, so we should consider that $L = 0$; that is

$$\frac{1}{0} = 0.$$  

(3.2)

Nothing may be represented by zero; it will be a sense of zero.

Furthermore, for $R = 0$; that is, for S=O, we see its shadow is the point $(0, -d)$ and so $L = 0$ and

$$L = \frac{0 \cos \theta}{0 \sin \theta} d = 0;$$

that is

$$\frac{0}{0} = 0.$$  

This example shows that the division by zero calculus is not almighty.

Note that both identities (3.1) and (3.2) are right in their senses. Depending on the interpretations of $1/0$, we obtain INFINITY and ZERO, respectively.
3.1.1 Another example

We consider a triangle ABC with $AB = c$, $BC = a$, $CA = c$. Let $x_i$ be the orthogonal projections of $AB$ and $AC$ to the line $BC$. Then we have

$$x_i = \frac{1}{2} \left\{ a \pm \frac{(b + c)(b - c)}{a} \right\},$$

([5], pages 70-71.). If $b = c$, then, of course, $x_1 = x_2 = a/2$. For $a = 0$, by the division by zero, we have the reasonable value $x_1 = x_2 = 0$.

3.1.2 Remark

For the example ([5], pages 70-71.), we see that now there is no problem, because we have the relation

$$\frac{R}{R_c} = \frac{r}{R}.$$

Then, we have the right formula

$$y = r \sin \varphi.$$

3.2 In balance of a steelyard

We will consider the balance of a steelyard and then we have the equation

$$aF_a = bF_b$$

(3.3)

as the moment equality. Here, $a, b$ are the distances from a fixed point and force $F_a, F_b$ points, respectively. Then, we have

$$F_a = \frac{b}{a} F_b.$$

For $a = 0$, should be considered as $F_a = 0$ by the division by zero $b/0 = 0$?

The identity (3.3) appears in many situations, and the above result may be valid similarly.

As a typical case, we recall
Ctesibios (BC. 286-222): We consider a flow tube with some fluid. Then, when we consider some cut with a plane with its area $S$ and with its velocity $v$ of the fluid on the plane, by continuity, we see that for any cut plane, $Sv = C$; $C$ : constant. That is,

$$v = \frac{C}{S}.$$

When $S$ tends to zero, the velocity $v$ tends to infinity. However, for $S = 0$, the flow stops and so, $v = 0$. Therefore, this example shows the division by zero $C/0 = 0$ clearly. Of course, in the situation, we have $0/0 = 0$, trivially.

We can find many and many similar examples, for example, in Archimedes’ principle and Pascal’s principle.

We will state one more example:

E. Torricelli (1608 -1646): We consider some water tank and the initial high $h = h_0$ for $t = 0$ and we assume that from the bottom of the tank with a hole of area $A$, water is fall down. Then, by the law with a constant $k$

$$\frac{dh}{dt} = -\frac{k}{A}\sqrt{h},$$

we have the equation

$$h(t) = \left(\sqrt{h_0} - \frac{k}{2A}\right)^2.$$

Similarly, of course, for $A = 0$, we have

$$h(t) = h_0.$$

Even the fundamental relation among velocity $v$, time $t$ and distance $s$

$$t = \frac{s}{v},$$

we will be able to understand the division by zero

$$\frac{s}{0} = 0$$

and

$$\frac{0}{0} = 0.$$
3.3 By rotation

We will give a simple physical model showing the result $\frac{\partial}{\partial r} = 0$. We shall consider a disc with $x^2 + y^2 \leq a^2$ rowing uniformly with a positive constant angular velocity $\omega$ with its center at the origin. Then we see, at the only origin, $\omega = 0$ and at other all points, $\omega$ is a constant. Then, we see that the velocity and the radius $r$ are zero at the origin. This will mean that, in the general formula

$$v = r\omega,$$

or, in

$$\omega = \frac{v}{r}$$

at the origin,

$$\frac{0}{0} = 0.$$

We will not be able to obtain the result from

$$\lim_{r \to 0^+} \omega = \lim_{r \to 0^+} \frac{v}{r},$$

because it is the constant.

For a uniform rotation with velocity $v$ with its center $O'$ and with its radius $r$. For the angular velocity vector $\omega$ and for the moving position $P$ on the circle, we set $r = OP$. Then,

$$v = \omega \times r.$$

If $\omega \times r = 0$, then, of course, $v = 0$.

3.4 By the Newton’s law

We will recall the fundamental law by Newton:

$$F = G \frac{m_1 m_2}{r^2} \quad (3.4)$$

for two masses $m_1, m_2$ with a distance $r$ and for a constant $G$. Of course,

$$\lim_{r \to 0^+} F = \infty.$$
however, as in our fraction

$$F = 0 = G \frac{m_1 m_2}{r^2}. \quad (3.5)$$

Of course, here, we can consider the above interpretation for the mathematical formula (3.4) as the new interpretation (3.5). In the ideal case, when two masses are on one point, the force $F$ will not be positive and it will be reduced to zero.

In the Kepler (1571 - 1630) - Newton (1642 - 1727) law for central force movement of the planet,

$$\frac{d^2 \mathbf{r}}{dt^2} = - \frac{GMm}{r^3} \mathbf{r},$$

of course, we have $\mathbf{r} = 0$ for $r = 0$.

For the Coulomb’s law, see similar formulas. Indeed, in the formula

$$F = k \frac{(q)(-q)}{r^2}$$

for $r = 0$, we have $F = 0$.

In general, in the formula

$$F = k \frac{(Q_1)(Q_2)}{r^2}$$

for $r = 0$, we have $F = 0$ (S. Senuma: 2016.8.20.).

Furthermore, as well-known, the bright at a point at the distance $r$ from the origin is given by the formula

$$B = k \frac{P}{r^2},$$

where $k$ is a constant and $P$ is the amount of the light. Of course, we have, at the infinity

$$B = 0.$$

Then, meanwhile, may we consider as

$$B = 0$$

at the origin $r = 0$? Then we can obtain our formula

$$k \frac{P}{0} = 0,$$

as in our new formula.
3.5 An interpretation of $0 \times 0 = 100$ from $100/0 = 0$

The expression $100/0 = 0$ will represent some divisor by the zero in a sense that is not the usual one, and so, we will be able to consider some product sense $0 \times 0 = 100$.

We will show such an interpretation.

We shall consider same two masses $m$, however, their constant velocities $v$ for the origin are the same on the real line, in the symmetry way. We consider the moving energy product $E^2$,

$$\frac{1}{2}mv^2 \times \frac{1}{2}m(-v)^2 = E^2. $$

We shall consider at the origin and we assume that the two masses stop at the origin (possible in some case). Then, we can consider, formally

$$0 \times 0 = E^2.$$ 

The moving energies change to other energies, however, we can obtain some interpretation as in the above.

This example was discovered by M. Yamane presented in the paper [7].

3.6 Capillary pressure in a narrow capillary tube

In a narrow capillary tube saturated with fluid such as water, the capillary pressure is simply expressed as follows,

$$Pc = \frac{2\sigma}{r}$$

where $Pc$ is capillary pressure (suction pressure), $\sigma$ is surface tension, and $r$ is radius. If $r$ is zero, there is no pressure. However $Pc$ shows infinity, in the common meaning.

This simple equation is based on the Laplace-Young equation

$$P = \sigma \left( \frac{1}{R_1} + \frac{1}{R_2} \right)$$

where $R_1$ and $R_2$ are two principal radii of curvature at any point on the surface of a droplet or a bubble and in the case of spherical form $R_1 = R_2 = R$. For a spherical bubble the pressure difference across the bubble film is zero.
since the pressure is the same on both sides of the film. The Laplace-Young equation reduces to
\[ \frac{1}{R_1} + \frac{1}{R_2} = 0. \]

On other hand when diameter of a bubble is decreased and becomes \(0(R = 0)\),
the bubbles collapse and enormous energy is generated. Accumulated free
energy in the bubble is released instantaneously.

This example was discovered by M. Kuroda presented in [7].

3.7 Circles and curvature - an interpretation of the
division by zero \(r/0 = 0\)

We consider a solid body called right circular cone whose bottom is a disc
with its radius \(r_2\). We cut the body with a disc of radius \(r_1(0 < r_1 < r_2)\) that
is parallel to the bottom disc. We denote the distance by \(d\) between both
discs and \(R\) the distance between the top point of the cone and the bottom
circle on the surface of the cone. Then, \(R\) is calculated by Eko Michiwaki (8
year old daughter of H. Michiwaki) as follows:

\[ R = \frac{r_2}{r_2 - r_1} \sqrt{d^2 + (r_2 - r_1)^2}, \]

that is called EM radius, because by the rotation of the cone on the plane, the
bottom circle writes the circle of radius \(R\). We denote by \(K = K(R) = 1/R\)
the curvature of the circle with its radius \(R\). We fix the distance \(d\). Now
note that
\[ r_1 \to r_2 \implies R \to \infty. \]

This will be natural in the sense that when \(r_1 = r_2\), the circle with its radius
\(R\) becomes a line.

However, the division by zero will mean that when \(r_1 = r_2\), the above
EM radius formula makes sense and \(R = 0\). What does it mean? Here, note
that, however, then the curvature \(K = K(0) = 0\) by the division by zero
calculus; that is, the circle with its radius \(R\) becomes a line, similarly. The
curvature of a point (circle of radius zero) is zero. See [10].
3.8 Vibration

In the typical ordinary differential equation

\[ m \frac{d^2 x}{dt^2} = -kx, \]

we have a general solution

\[ x = C_1 \cos(\omega t + C_2), \quad \omega = \sqrt{\frac{k}{m}}. \]

If \( k = 0 \), that is, if \( \omega = 0 \), then the period \( T \) that is given by

\[ T = \frac{2\pi}{\omega} \]

should be understood as \( T = 0 \)?

In the typical ordinary differential equation

\[ m \frac{d^2 x}{dt^2} + kx = f \cos \omega t, \]

we have a special solution

\[ x = \frac{f}{m} \frac{1}{|\omega^2 - \omega_0^2|} \cos \omega t, \quad \omega_0 = \sqrt{\frac{k}{m}}. \]

Then, how will be the case

\[ \omega = \omega_0 \]

?

For example, for the differential equation

\[ y'' + a^2 y = b \cos \lambda x, \]

we have a special solution, with the condition \( \lambda \neq a \)

\[ y = \frac{b}{a^2 - \lambda^2} \cos \lambda x. \]

Then, when \( \lambda = a \), by the division by zero calculus, we obtain the special solution

\[ y = \frac{bx \sin(ax)}{2a} + \frac{b \cos ax}{4a^2}. \]
3.9 Spring or circuit

We will consider a spring with two spring constants \( \{k_j\} \) in a line. Then, the spring constant \( k \) of the spring is given by the formula

\[
\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2},
\]

by Hooke’s law. We know, in particular, if \( k_1 = 0 \), then

\[
\frac{1}{k} = \frac{1}{0} + \frac{1}{k_2},
\]

and by the division by zero,

\[
k = k_2,
\]

that is very reasonable. In particular, by Hooke’s law, we see that

\[
\frac{0}{0} = 0.
\]

As we saw for the case of harmonic mean, in this case \( k_1 = 0 \), the zero means that the spring does not exist.

The corresponding result for the case of Ohmu’s law is similar and valid.

3.10 Motion

A and B start at the origin on the real positive axis with, for \( t = 0 \)

\[
\frac{d^2x}{dt^2} = a, \quad \frac{dx}{dt} = u
\]

and

\[
\frac{d^2x}{dt^2} = b, \quad \frac{dx}{dt} = v,
\]

respectively. After the time \( T \) and at the distance \( X \) from the origin, if they meet, then we obtain the relations

\[
T = \frac{2(u - v)}{b - a}
\]

and

\[
X = \frac{2(u - v)(ub - va)}{(b - a)^2}.
\]
For the case $a = b$, we obtain the reasonable results $T = 0$ and $X = 0$.

We will consider the motion $(x, y)$ represented by $x = \cos \theta, y = \sin \theta$
from $(1, 0)$ to $(-1, 0)$ ($0 \leq \theta \leq \pi$) with the condition
\[ v_x = \frac{dx}{dt} = -\sin \theta \frac{d\theta}{dt} = V \quad \text{(constant)}. \]
Then, we have that
\[ v_y = \frac{dy}{dt} = -V \frac{1}{\tan \theta}, \]
and
\[ a_y = \frac{d^2y}{dt^2} = -V^2 \frac{1}{\sin^3 \theta}. \]
Then we see that
\[ v_y(1, 0) = 0, \text{ that is, } \frac{1}{\tan 0} = 0, \]
\[ v_y(-1, 0) = 0, \text{ that is, } \frac{1}{\tan \pi} = 0, \]
\[ a_y(1, 0) = 0, \text{ that is, } \frac{1}{\sin^3 0} = 0, \]
and
\[ a_y(-1, 0) = 0, \text{ that is, } \frac{1}{\sin^3 \pi} = 0. \]

### 3.11 Darcy’s law for fluid through porous media

Diffusion phenomenon and penetration phenomenon may be represented by the partial differential equations
\[ \frac{\partial u}{\partial t} = \nu \frac{\partial^2 u^m}{\partial x^2} \]
for some constants $\nu$ and $m$.

Indeed, density $u$ and pressure $p$ may be related by equation of state
\[ u = \gamma p^\alpha, \]
with some constants $\gamma$ and $\alpha$.

By the conservative law, we have, for porocity $\nu$ and velocity $v$
\[
\frac{\partial (uv)}{\partial x} = -\nu \frac{\partial u}{\partial t}..
\]

At the last, by Darcy’s law, we have for some constant \( k \)

\[
v = -k \frac{\partial p}{\partial x}.
\]

By channelling \( v, p \) from three equations we obtain

\[
\frac{\partial u}{\partial t} = \frac{k}{\nu \gamma (\alpha + 1)} \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} u^{1+1/\alpha} \right)
\]

([6], 21-22). As basic references, see ([3, 1, 2]).

Be setting \( m = 1 + 1/\alpha \), we have the desired equation.

Note that for \( \alpha = 0 \), with the division by zero \( 1/0 = 0 \), we have the right differential equation.

Meanwhile, for \( \alpha = -1 \), by the division by zero calculus, we have

\[
\frac{\partial u}{\partial t} = \frac{k}{\nu \gamma} \frac{\partial^2}{\partial x^2} (-\log u).
\]

How will be this partial differential equation?

3.12 RCL and RL circuits

We will consider an RCL circuit stated by the ordinary differential equation

\[
L \frac{di}{dt} + Ri + \frac{1}{C} \int idi = E_0 \sin \omega t,
\]

(3.6)

\[
i = \frac{E_0}{\sqrt{R^2 + ((\omega L - (1/(\omega C)))^2}} \sin (\omega t - \varphi);
\]

(3.7)

\[
\varphi = \arctan \frac{1}{R} \left( \omega L - \frac{1}{\omega C} \right).
\]

(3.8)

Here, \( E_0 \sin \omega t \) is a given AC voltage.

In this circuit, for the case \( C = 0 \) that is the condense is missing, we obtain the corresponding result precisely by the division by zero

\[
\frac{1}{C} = 0
\]

15
and
\[
\frac{1}{\omega C} = 0.
\]

We can find many and many the division by zero and division by zero calculus in physics.

References


