AN INTEGRAL EQUATION FOR THE GRAMM SERIES AND THE PRIME COUNTING FUNCTION AND INTEGRAL TRANSFORMS

Jose Javier Garcia Moreta
Graduate student of Physics at the UPV/EHU (University of Basque country)
In Solid State Physics
Address: Address: Practicantes Adan y Grijalba 2 5 G
P.O 644 48920 Portugalete Vizcaya (Spain)
E-mail: josegarc2002@yahoo.es

ABSTRACT : In this paper we give an integral equation satisfied by the gramm series based on the use of the Borel transform

In Mathematics the Gramm series is define as the infinite series

$$G(x)=1+\sum_{n=1}^{\infty}\frac{(\log x)^{n}}{nn!\zeta(n+1)} \qquad (1)$$

Let be the following integral equation

$$-\log\left(1-\frac{1}{s}\right)=s\int_{1}^{\infty}dx\frac{g(x)}{x^{s}-1}.\frac{1}{x} \qquad (2)$$

Then with a simple change of variable $x=e^{t}$, we have the following integral equation

$$-\log\left(1-\frac{1}{s}\right)=\sum_{n=1}^{\infty}\frac{1}{n}.\frac{1}{s^{n}}=s\int_{0}^{\infty}dt\frac{g(e^{t})}{e^{st}-1} \qquad (3)$$

Using our method described in Paper [1] , which uses the Borel generalized transform to solve integral equations of this kind

$$g(s)=s\int_{0}^{\infty}dtK(st)f(t) \qquad (4)$$

With a solution given by the power series $f(t)=\sum_{n=1}^{\infty}\frac{c_{n}t^{n}}{M(n+1)}$ (5)

With $c(n)=\frac{1}{2\pi i}\int_{C}g(z)z^{n-1}$ $\quad g(s)=\sum_{n=0}^{\infty}\frac{c(n)}{s^{n}}$ $\quad M(n+1)=\int_{0}^{\infty}dtK(t)$ (6)

The Kernel, K(t) inside (4) does not need to be a smooth function, this can include step function of the form $H(ax-b)=\begin{cases}1 & \text{if } x>\frac{b}{a}\\ 0 & \text{if } x<\frac{b}{a}\end{cases}$

Then,we can find a series solution for this integral equation as follows

$$G(x)-1=g(x)=\sum_{n=1}^{\infty}\frac{(\log x)^{n}}{nn!\zeta(n+1)} \quad (7)$$

Which is precisely the Gramm series (minus a constant 1), so the Integral given in formula (3) is just the integral equation satisfied by the Gramm function

Given a function $g(s)=s\int_{0}^{\infty}dtK(st)f(t)$, then if the function $f(t)$ is analytic near 0 , then by integration by parts we can give an asymptotic expansion for the function g as

$$\frac{g(s)}{s}=\sum_{n=0}^{\infty}\frac{K_{n+1}(0)}{s^{n+1}}(-1)^{n+1}\frac{d^{n}f(0)}{dx^{n}} \quad (8)$$

The functions $K_n(t)$ are given by the recurrence equation $\frac{dK_{n+1}(t)}{dt}=K_n(t)$, and $K_0(t)=K(t)$ is the Kernel of our Integral equation.

Our method [1] can be also useful to find integral equations satisfied for some functions of series, for example for the Riesz Function

$$Riesz(x)=-\sum_{n=1}^{\infty}\frac{(-x)^{n}}{(k-1)!\zeta(2n)} \qquad 1-e^{-x}=\int_{0}^{\infty}\frac{dt}{t}\left[\sqrt{\frac{x}{t}}\right]Riesz(t) \quad (9)$$

It is not hard to prove the integral equation inside (9) just use the expansion of the function $1-e^{-\frac{1}{x}}$ in powers of $\frac{1}{x}$, and then apply the Generalized Borel transform with the Kernel $K(t)=\left[\frac{1}{\sqrt{t}}\right]$, since the Mellin transform representation of the Riemann zeta function

$$\zeta(2n)=n\int_{0}^{\infty}dt\left[\frac{1}{\sqrt{t}}\right]t^{n-1}=\sum_{k=1}^{\infty}\frac{1}{k^{2n}} \quad \text{is valid for n=1,2,3,4,5,6} \quad (10)$$

Another example of our methdo to solve integrals like (4) is related to the Stieltjes Moment problem and integral equation defined by

$$g(z)=\sum_{n=0}^{\infty}\frac{\mu(n)}{z^{n+1}}(-1)^{z}=\int_{0}^{\infty}dx\frac{\alpha(x)}{z+x} \quad (11)$$

Where $\mu(n)=\int_{0}^{\infty}dx\alpha(x)x^{n}$ are the moments of the measure $\alpha(x)$, making a change of variable $x=zu$ our integral equation becomes

$$zg(z)=\sum_{n=0}^{\infty}\frac{\mu(n)}{z^n}(-1)=z\int_0^{\infty}dx\frac{\alpha(zu)}{1+u} \qquad (12)$$

The integral equation (12) is of the form of (4) so we can get the soluton in power series as

$$f(u)=\sum_{n=1}^{\infty}\frac{c_n u^n}{M(n+1)}=\sum_{n=0}^{\infty}(-1)^n u^n=\frac{1}{1+u} \qquad (13)$$

Since $M(n+1)=\mu(n)$ and $c(n)=(-1)^n\mu(n)$

Curiosly there is a distributional solution to integral equation (11) given by

$$\alpha(x)=\sum_{n=0}^{\infty}\frac{\mu(n)}{n!}(-1)^n\delta^{(n)}(x) \qquad (14)$$

Just insert (14) inside (11) and use the properties of the delta function and of the function $(x+z)^{-1}$

$$\int_{-\infty}^{\infty}f(x)\delta^{(n)}(x)dx=(-1)^n\frac{d^n}{dx^n}f(0) \qquad \frac{\partial^n}{\partial x^n}(x+z)^{-1}\big|_{x=0}=(-1)^n\frac{n!}{z^{n+1}} \qquad (15)$$

And you get the Laurent series expansion for $g(z)$ $\quad g(z)=\sum_{n=0}^{\infty}\frac{\mu(n)}{z^n}(-1)^n$

AN INTEGRAL FOR THE PRIME COUNTING FUNCTION

For the prime counting function we have also the equivalent integral equation

$$-s\frac{\zeta'(s)}{\zeta(s)}=s\int_0^{\infty}dt\frac{d\pi(e^t)}{dt}\frac{t}{e^{st}-1} \qquad (16)$$

The proof of (16) is easy, just apply the logarithm to the Euler product $\prod_p\left(1-p^{-s}\right)^{-1}$ and then use the Abel's sum theorem with the Prime counting function in the form

$$\sum_p f(p)=-\int_0^{\infty}dxf'(x)\pi(x) \qquad (17)$$

Then make the change of variable $x=e^t$ and take $\frac{d}{ds}$. To get a solution (approximate) to (16) we need to get a rational approximation to the logarithmic derivative of the zeta function based on the Laurent series expansion of the Riemann zeta

$$\frac{\zeta'(s)}{\zeta(s)} \approx \frac{-(z-1)^{-2}+\sum_{r=1}^{M} rb_r(z-1)^{r-1}}{(z-1)^{-1}+\sum_{r=1}^{M} rb_r(z-1)^{r}} = Q(z) \qquad b_n = \frac{1}{n!}\lim_{k\to\infty}\left(\sum_{j=1}^{k}\frac{\log^n j}{j} - \frac{\log^{n+1} k}{n+1}\right) \quad (18)$$

So using our method to get the solution of integral equation into (16) we find the better solution than the Gramm series

$$\pi(x)-\pi(a) \approx \sum_{n=1}^{\infty}\frac{c_n}{nn!\zeta(n+1)}\left(\log^n(x)-\log^n(a)\right) \qquad c_n = -\frac{1}{2\pi i}\int_C zQ(z)z^{n-1} \quad (19)$$

Where $Q(z)$ is a rational approximation to the logarithmic derivative of the zeta function defined in (18) and $x > a > 1$ we need to do this in order to compute the inverse Z transform to get the coefficients .

If we have used the simpler approximation $\frac{\zeta'(s)}{\zeta(s)} \approx -\frac{1}{s-1}$, then all the constants c_n are 1 and (19) is just the Gramm series for the Prime counting function

References:

[1] Garcia J.J "Borel resummation and the solution of integral equatio· e-print vixra.org/abs/1304.0013

[2] Girling, B. "The Z Transform." In *CRC Standard Mathematical Tables, 28th ed* (Ed. W. H. Beyer). Boca Raton, FL: CRC Press, pp. 424-428, 1987.

[3] Ingham, A. E. Ch. 5 in "*The Distribution of Prime Numbers*". New York: Cambridge University Press, 1990.

[4] Weisstein, Eric W. "Gram Series." From *MathWorld*--A Wolfram Web Resource. https://mathworld.wolfram.com/GramSeries.html