Investigation of the Onset of Turbulence in Boundary Layers and the Implications for Solutions of the Navier-Stokes Equations.

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Abstract. This paper investigates the onset of turbulence in incompressible viscous fluid flow over a flat plate by looking at the pressure gradients implied by the Blasius solution for laminar fluid flow and adjusting the predicted flow, leading to a mathematically predictable flow separation in the boundary layer and the onset of turbulence (including both transition and fully turbulent regions - both with and without the presence of a flat plate). It then considers the implications for potential analytic solutions to the Navier-Stokes Equations of the fact that it is possible to predict turbulence and a singularity for many flows (at any velocity).

1. AMS Subject Classification

35Q30, 76D05, 76D10

2. Introduction

This paper investigates the onset of turbulence in incompressible viscous fluid flow over a flat plate by looking at the pressure gradients implied by the Blasius solution for laminar fluid flow and adjusting the predicted flow, leading to a mathematically predictable onset of turbulence.

Section 2 looks at the equations governing viscous incompressible flow in general and more specifically for steady state flow in a boundary layer over a flat plate. It then looks at the general characteristics of boundary layer flow over a flat plate, introduces the Blasius solution modelling this flow and notes the observed pressure drop for flow over a cylinder.

Section 3 looks at the Blasius solution in more detail, investigates the horizontal pressure gradients implied by the Blasius solution and adjusts the Blasius flow accordingly, leading to a predictable zero flow point close to the plate, resulting in flow separation and the onset of turbulence. This section also shows the applicability for the Blasius solution in free flow, without a physical plate boundary.

Section 4 looks at the conclusions to be drawn - including both for predicting turbulence in geometries where Blasius can be directly applied but also more generally in free flow regions where we can say it is possible to predict the onset of turbulence (and a singularity) as a direct consequence of flow separation for many flows (at any initial velocity) - eliminating the possibility of analytical solutions to the Navier-Stokes equations in a number of situations.

3. Preliminary - Incompressible Flow Equations and Incompressible Flow Boundary Layer Observations

3.1. Incompressible Flow Equations

The equations governing incompressible homogeneous newtonian fluid flow in all of space R^N N =2,3, are [1, p. 2]:

$\frac{Dv}{Dt} = -\Delta p + \nu \nabla v$ (Navier-Stokes for $\nu > 0$ or Euler for $\nu = 0$).

div $v = 0$ $(x, t) \in R^N \times [0, \infty)$ (Incompressibility).

$v\big|_{t=0} = v_0$, $x \in R^N$ (Initial Conditions).

where $v(x, t)$ is the fluid velocity, $p(x, t)$ is the scalar pressure, $\frac{Dv}{Dt}$ is the convective derivative (ie the derivative along the particle trajectories);

$\frac{D}{Dt} = \frac{\delta}{\delta t} + \sum_{j=1}^{N} v^j \frac{\delta}{\delta x_j}$

The gradient operator ∇ is:

$\nabla = \left(\frac{\delta}{\delta x_1}, ..., \frac{\delta}{\delta x_N}\right)^t$,

and the Laplace operator Δ is:

$\Delta = \sum_{j=1}^{N} \frac{\delta^2}{\delta x_j^2}$

$\nu = \frac{\mu}{\rho}$ is the kinematic viscosity. (μ is the viscosity, ρ the density).

For 2-dimensional steady state flow (ie no variation with time), these equations reduce to:

(using x for x_1, y for x_2, u for $\frac{\delta x_1}{\delta t}$, v for $\frac{\delta x_2}{\delta t}$)

x Navier-Stokes: $\rho\left(u\frac{\delta u}{\delta x} + v\frac{\delta u}{\delta y}\right) = -\frac{\delta p}{\delta x} + \mu\left(\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2}\right)$ (1)

y Navier-Stokes: $\rho\left(u\frac{\delta v}{\delta x} + v\frac{\delta v}{\delta y}\right) = -\frac{\delta p}{\delta y} + \mu\left(\frac{\delta^2 v}{\delta x^2} + \frac{\delta^2 u}{\delta y^2}\right)$ (2)

Incompressibility: $\frac{\delta u}{\delta x} + \frac{\delta v}{\delta y} = 0$ (3)

For most practical purposes, a scale analysis of these equations eliminates a number of terms from the above equations, resulting in a boundary layer that has no pressure variation in the y direction and a pressure variation in the x direction impressed from the external flow. In this paper, we are looking at the (small) pressure variations that do exist (and the implications) and so below we will look at a different approach.

3.2. Incompressible Flow over a Flat Plate

Firstly observing the characteristics of incompressible fluid flow over a flat plate (steady state) as shown in figure 10-81 from [2, p. 514](not reproduced here):

The key points of interest are a continually increasing boundary layer thickness (boundary layer edge being defined as the point at which u = 0.99U) with $\frac{\delta u}{\delta y}$ always the same sign, as well as the onset of turbulence as shown by the transition and turbulent regions (with corresponding increases on boundary layer thickness).

If we now refer to the paper of Blasius (NACA translation) [3, p. 3], noting that Blasius uses ϵ for the boundary layer thickness, then we see that there will be a small pressure gradient across the boundary layer (in the y direction - of the order of the square of the boundary layer thickness for a steady state flow). For most practical purposes this pressure gradient (resulting in a small pressure differential across the boundary layer) is ignored - however we will not ignore it for the purposes of this paper. More importantly, we shall investigate the pressure profile implied by the Blasius solution along the boundary layer (in the x direction).

3.3. Incompressible Flow over a cylinder

In his paper, Blasius also applied this analysis to a flow over a cylinder. In this case, there is an additional term (related to the curvature of the cylinder) which generates a larger pressure gradient.

An indication of the pressure differential in laminar flow over a cylinder is given in the widely available graphs showing the difference between theoretical and actual pressure coefficients measured along the surface of a cylinder in a moving fluid - a good example is in figure 10-64 from [2, p. 504] (not reproduced here).
The key points to notice here are the clearly increasing pressure differential between theoretical pressure coefficient and laminar flow experimental results (up to boundary layer separation. Note that the pressure is measured on the surface of the cylinder) consistent with an increasing pressure gradient in the boundary layer as well as the much smaller pressure differential seen in the turbulent boundary layer results.

4. Mechanism for the Onset of Turbulence

4.1. Blasius Solution

Blasius in his paper (see[3]) provides an approximation for laminar flow over an infinite flat plate by ignoring the small pressure variations developed along and across the boundary layer (approximating that any pressure profile is impressed on the boundary layer by the external flow). This approximation works well for many applications, but does not predict the onset of turbulence.

The reduced set of boundary layer equations that Blasius used were:

$$\rho(\frac{\delta u}{\delta t} + u\frac{\delta u}{\delta x} + v\frac{\delta u}{\delta y}) = \rho(\frac{\delta U}{\delta t} + U\frac{\delta U}{\delta x}) + \mu\frac{\delta^2 u}{\delta y^2}$$

$$\frac{\delta u}{\delta x} + \frac{\delta v}{\delta y} = 0$$

Where U is the x component of the external flow.

For the steady state situation this reduces to:

$$\rho(u\frac{\delta u}{\delta x} + v\frac{\delta u}{\delta y}) = \rho(U\frac{\delta U}{\delta x}) + \mu\frac{\delta^2 u}{\delta y^2}$$

$$\frac{\delta u}{\delta x} + \frac{\delta v}{\delta y} = 0$$

Blasius used these equations to approximate a numerical solution (based on a similarity variable approach - $\eta = y(\frac{U}{\nu x})^{\frac{1}{2}}$) for the flow velocities in a boundary layer. This solution has been shown to be usefully accurate experimentally. See figure 10-99 from [2, p. 523](not reproduced here).

We can now investigate the implied x direction pressure gradient in a boundary layer by taking the Blasius results for velocity in the x direction and (numerically) calculating the pressure profile by using the full Navier-Stokes equations (i.e. equations 1, 2 and 3 at the start of the paper).

By using the Blasius values from [4] and matlab, based on an infinite plate in water flowing at $1ms^{-1}$, over a grid of 10,000 points in the x direction (up to x=0.6 so near observed turbulence) and 1000 points in the y direction (up to y = 0.004 or the edge of the boundary layer at x=0.5) - enough points to give us a good indication of the characteristics of the pressure gradient, if not enough to give us reliably accurate values - we arrive at the following results (see figure 1 below):

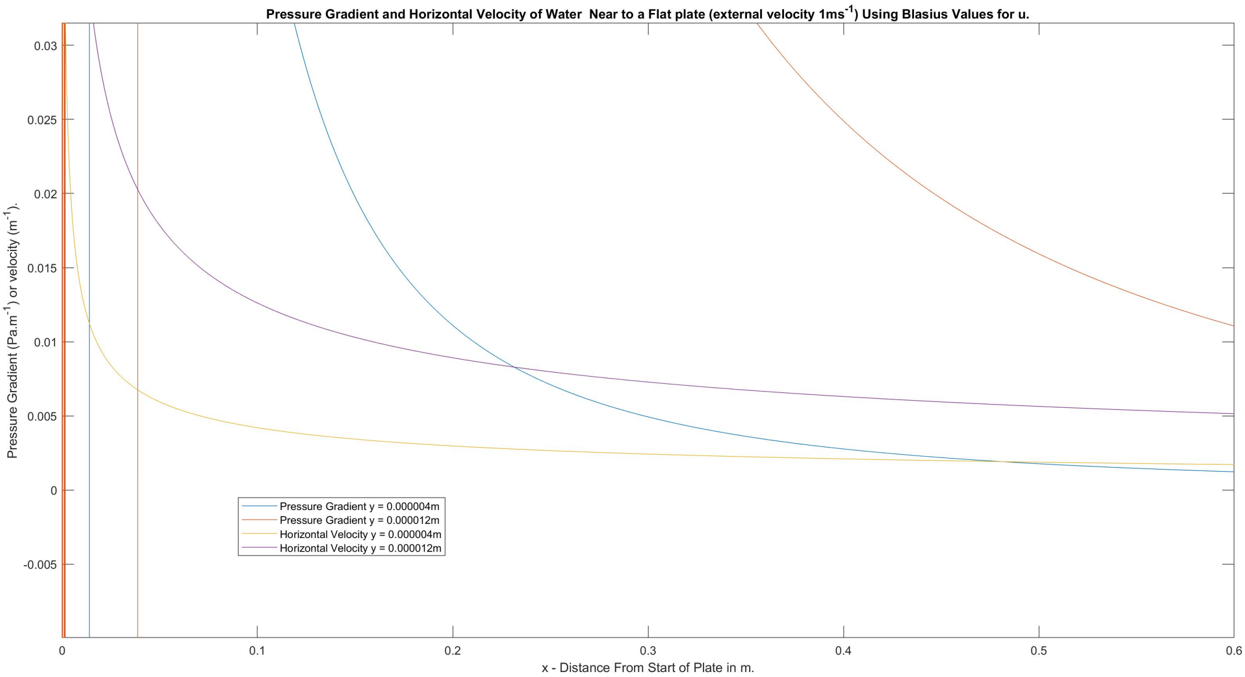

Figure 1. Calculated Pressure Gradient and Horizontal Velocity

The key points to note from this figure relevant to this paper are the continuously positive pressure gradients in the x direction (i.e continuous adverse pressure gradients - $\frac{\delta p}{\delta x} > 0$ for all values of y for large enough values of x) and the continually decreasing u value as x increases.

We can also look in more detail at the pressure profile in the region where $\eta < 2.5$ - i.e closer to the plate and away from the edge of the boundary layer (in the more 'linear' part of the Blasius graph above) - (see figure 2 below):

This detail shows more clearly the adverse pressure gradients and velocities close to the plate, away from the boundary layer edge. It is important to note that these are calculated pressure gradients and velocities from the Blasius approximation (which assumed no pressure gradients generated inside the boundary layer).

The presence of an adverse pressure gradient in boundary layer flow is a necessary but not sufficient condition for flow separation. We can visualise the pressure gradient here as a kind of virtual diffuser.

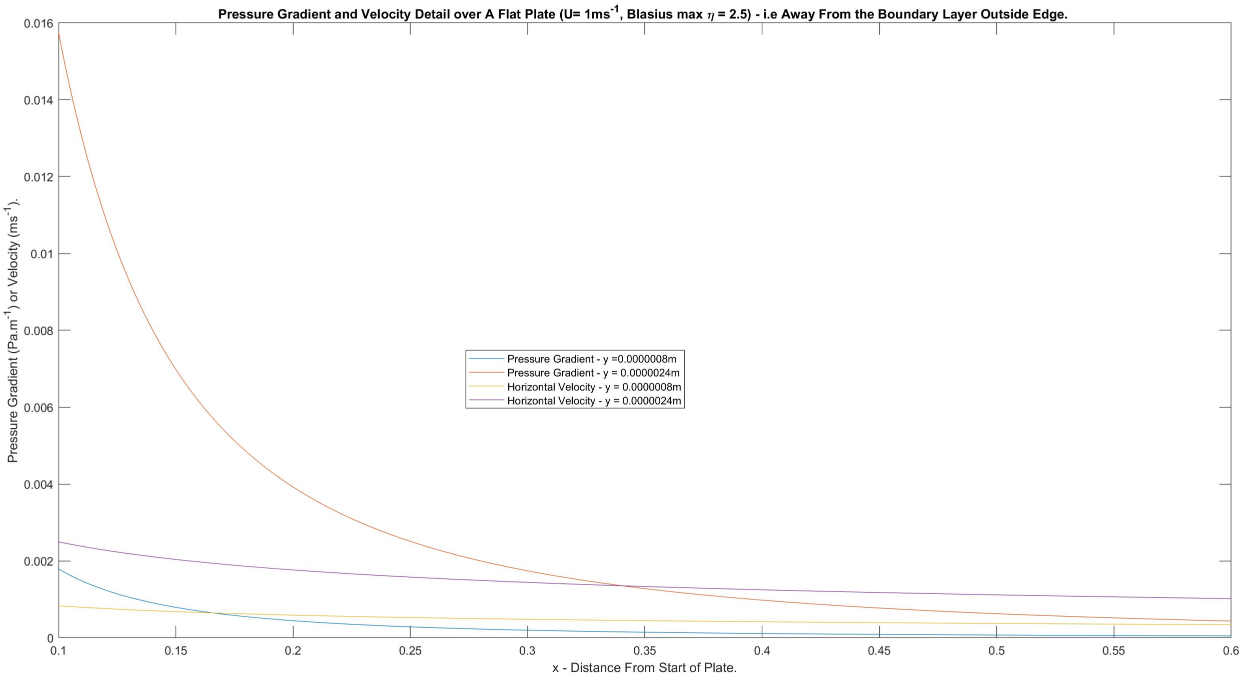

Figure 2. Calculated Pressure Gradient and Horizontal Velocity Detail $\eta < 2.5$

4.2. Adjusting the Blasius Solution with the Implied Pressure Gradients.
It is now instructive to adjust the standard Blasius Solution by adding the calculated pressure gradient and finding the implied adjusted flow velocities. Considering the full (x-direction) steady-state Navier Stokes equation:

$$\rho\big(u\tfrac{\delta u}{\delta x} + v\tfrac{\delta u}{\delta y}\big) = -\tfrac{\delta p}{\delta x} + \mu\big(\tfrac{\delta^2 u}{\delta x^2} + \tfrac{\delta^2 u}{\delta y^2}\big)$$

We can rearrange this expression to evaluate the adjusted velocity figure when we superimpose the implied pressure gradient onto the Blasius solution. It is important to note that the expression will be used to evaluate the change in variables.

Rearranging:

$$\frac{\delta u}{\delta x} = \frac{\left(-\frac{\delta p}{\delta x}+\mu\left(\frac{\delta^2 u}{\delta x^2}+\frac{\delta^2 u}{\delta y^2}\right)\right)}{(\rho * u)} - v\frac{\delta u}{\delta y} \ (4)$$

If we now look at the magnitudes of the quantities involved (note these are all close to the plate - the magnitudes are significantly different away from the plate):

Water: $\mu\ 10^{-3}\rho\ 10^3$
$\frac{\delta u}{\delta x}:\ 10^{-5}$
$\frac{\delta p}{\delta x}:\ 10^{-6}$
$\frac{\delta^2 u}{\delta x^2}:\ 10^{-5}$
$\frac{\delta^2 u}{\delta y^2}:\ 10^{-7}$ Note: Small due to the adverse the adverse pressure gradient.
$v:\ 10^{-11}$
$\frac{\delta u}{\delta y}:\ 10^{2}$

With those magnitudes in mind, a reliable first approximation for the adjusted $\frac{\delta u}{\delta x}$ value (hence adjusted u value, by numerical integration along the x direction) can be calculated numerically using the following reduced expression:

$$\frac{\delta u}{\delta x} = \frac{\left(-\frac{\delta p}{\delta x}\right)}{(\rho u)}$$

A small adjustment to this expression (including an additional factor to allow for some influence from the discarded terms) is:

$$\frac{\delta u}{\delta x} = \frac{\left(-\frac{\delta p}{\delta x}\right)}{(\rho u 1.2)}$$

The results of this calculation (using the same grid as before) are shown in figure 3 below:

The key observation from these results is that the adjusted velocity reduces to zero - first for flow adjacent to the plate with the zero flow point moving further from the plate as x increases. If we consider the concepts of marginal and massive separation (see [5, p. 403-411] for details) and use the x value as the S parameter, then we can see that the the first zero flow point is a limit of incipient separation. It marks the start of a marginal separation (separation and reattachment) region (the transitional flow region) with a change to massive separation as S becomes large enough (the fully turbulent flow region - marked with a significant increase in the boundary layer thickness and a reduced pressue drop).

In addition, we can see that the above observations are consistent with stability theory (see [5, p. 415] for details), if we note that perturbations are created by the separations and reattachments in the marginal separation region (even if there no external perturbations). Stability Theory provides a useful description of transitional and turbulent flow - not necessarily an explanation.
The above observations show us that turbulence close to a flat plate is a consequence of flow separation (marginal and massive), due to the (small) pressure gradients implied by the Blasius Solution. Below we show the Blasius Solution approach extended to a free-flow example.

The inevitability of this result can be seen in the nature of the adjustment expression above - the appearance of u in the denominator means that as u becomes small, $\frac{\delta u}{\delta x}$ increases rapidly,

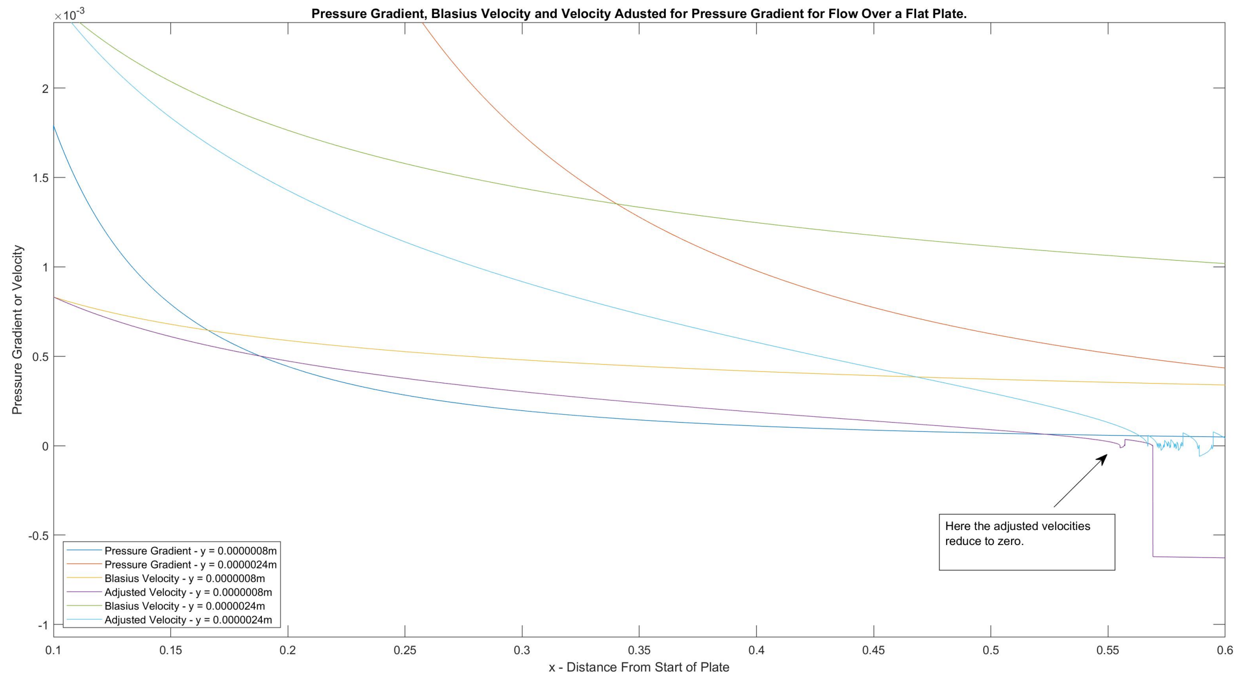

Figure 3. Pressure Gradient and Adjusted Horizontal Velocity

leading to a sudden reduction of u to zero (as can bee seen in the figure).

In short, the Blasius solution adjusted with the effects of the implied pressure gradient leads to a guaranteed (and calculable) onset of turbulence. In addition, we can see below that this result can be extended to free flowing incompressible fluid as well. The reduction of horizontal fluid velocity to zero leads to a mathematical singularity (as can be seen with u in the denominator in equation (4)), to be expected with the onset of turbulence.

4.3. Showing the Applicability of the Blasius Solution to Free Flowing Fluid (No Plates or Other Obstructions).

It is useful to show the adjusted Blasius solution above can be applied in the more general case of viscous incompressible fluid flowing without obstructions - two fluid regions flowing with different horizontal velocities (different values of u) with initially zero v velocities.

If we refer to figure 4 below showing key parameter values in the case of free-flowing incompressible fluid with the initial flow values mentioned above, with the flows assumed to start at $t = 0$ and $x = 0$:

The key points to note in the diagram are the signs of $\frac{\delta u}{\delta x}$ *and* $\frac{\delta v}{\delta y}$ at points a and b in the diagram (close to the boundary between the two layers).

The immediate conclusion is that both partial derivatives are zero at the boundary.

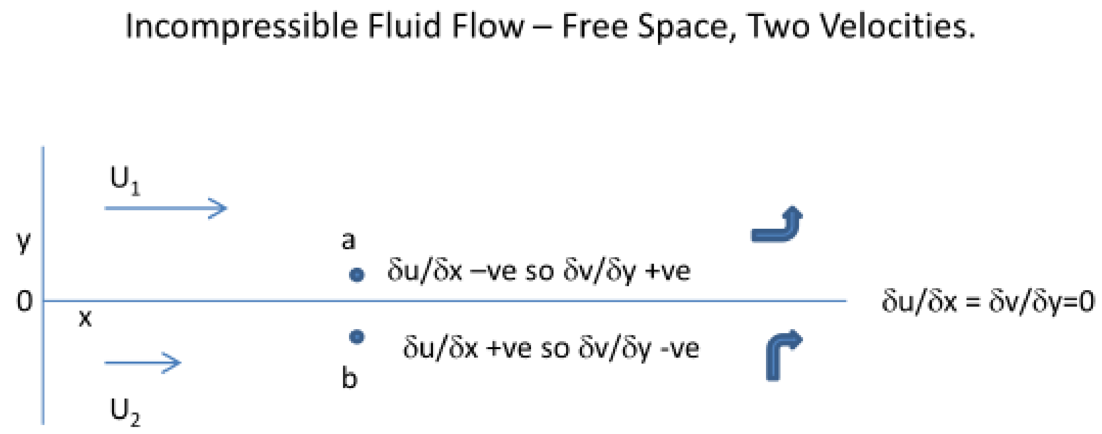

Figure 4. Blasius Free Flow Applicability.

In the case of u, this means that the value of u does not vary at the boundary, and if we accept that for large x, the value of u will approach $\frac{u_1+u_2}{2}$, then we can say that u at the boundary will be constant at $\frac{u_1+u_2}{2}$.

In the case of v, it is useful to look at the arrows in the figure for an impression of the sense of the paths of the particle flows. In the lower region (lower velocity, u increasing as we approach the boundary from below), then the value of v is positive, decreasing to zero as we approach the boundary. In the upper region (higher velocity, u decreasing as we approach the boundary from above), the value of v is also positive, increasing from zero at the boundary.

This means that we can treat the boundary as a virtual flat plate (zero thickness) with boundary layers above and below and use the Blasius and adjusted Blasius approaches detailed above for modelling the flows near the plate.

The final point to match with the earlier analyses is to set the zero u value at $\frac{u_1+u_2}{2}$, so that the the upper flow initial velocity is $\frac{u_1-u_2}{2}$ and the lower flow initial velocity is $-\frac{u_1-u_2}{2}$.

The immediate observation from the above is that there will (eventually) be turbulence for all incompressible viscid flows starting with different initial velocities.

5. Conclusions

The Blasius solution for incompressible fluid flow over a flat plate is a very useful tool for practical applications, but by eliminating pressure it eliminates the possibility of predicting the onset of turbulence . By adding back the effects of the pressure gradient, it is possible to predict mathematically the onset of turbulence as a feature of flow separation - consistent with existing approaches including marginal and massive separation and stability theory (although more work will need to be done to establish suitable accuracy).

Due to the wide applicability of the Blasius solution (for many geometries, including for flow without obstructions or plates and for flows of any velocity), this result suggests that, due to the predicted onset of turbulence and associated singularity, there may not be analytic solutions to the Navier-Stokes equations in most cases.

References

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