Investigation of the Onset of Turbulence in Boundary Layers and the Implications for Solutions of the Navier-Stokes Equations.

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Abstract. This paper investigates the onset of turbulence in incompressible viscous fluid flow over a flat plate by looking at the pressure gradients implied by the Blasius solution for laminar fluid flow and adjusting the predicted flow, leading to a mathematically predictable flow separation in the boundary layer and the onset of turbulence. It then considers the implications for potential analytic solutions to the Navier-Stokes Equations of the fact that the pressure term is needed to predict the onset of turbulence and so can not be eliminated for possible solutions.

1. AMS Subject Classification

35Q30, 76D05, 76D10

2. Introduction

This paper investigates the onset of turbulence in incompressible viscous fluid flow over a flat plate by looking at the pressure gradients implied by the Blasius solution for laminar fluid flow and adjusting the predicted flow, leading to a mathematically predictable onset of turbulence.

Section 2 looks at the equations governing viscous incompressible flow in general and more specifically for steady state flow in a boundary layer over a flat plate. It then looks at the general characteristics of boundary layer flow over a flat plate, introduces the Blasius solution modelling this flow and shows the observed pressure drop for flow over a cylinder.

Section 3 looks at the Blasius solution in more detail, investigates the horizontal pressure gradients implied by the Blasius solution and adjusts the Blasius flow accordingly, leading to a predictable zero flow point close to the plate, resulting in flow separation and the onset of turbulence.

Section 4 looks at the conclusions to be drawn - including both for predicting turbulence in geometries where Blasius can be directly applied but also more generally where we can say that techniques involving the elimination of the pressure terms from Navier Stokes in order to achieve analytical solutions are not valid.

3. Preliminary - Incompressible Flow Equations and Incompressible Flow Boundary Layer Observations

3.1. Incompressible Flow Equations

The equations governing incompressible homogeneous newtonian fluid flow in all of space R^N N =2,3, are [1, p. 2]:

$\frac{Dv}{Dt} = -\Delta p + \nu \nabla v$ (Navier-Stokes for $\nu > 0$ or Euler for $\nu = 0$).

div $v = 0$ $(x,t) \in R^N \times [0,\infty)$ (Incompressibility).

$v\big|_{t=0} = v_0$, $x \in R^N$ (Initial Conditions).

where $v(x,t)$ is the fluid velocity, $p(x,t)$ is the scalar pressure, $\frac{Dv}{Dt}$ is the convective derivative (ie the derivative along the particle trajectories);

$\frac{D}{Dt} = \frac{\delta}{\delta t} + \sum_{j=1}^{N} v^j \frac{\delta}{\delta x_j}$

The gradient operator ∇ is:

$\nabla = \left(\frac{\delta}{\delta x_1}, ..., \frac{\delta}{\delta x_N}\right)^t$,

and the Laplace operator Δ is:

$\Delta = \sum_{j=1}^{N} \frac{\delta^2}{\delta x_j^2}$

$\nu = \frac{\mu}{\rho}$ is the kinematic viscosity. (μ is the viscosity, ρ the density).

For 2-dimensional steady state flow (ie no variation with time), these equations reduce to:

(using x for x_1, y for x_2, u for $\frac{\delta x_1}{\delta t}$, v for $\frac{\delta x_2}{\delta t}$)

x Navier-Stokes: $\rho\left(u\frac{\delta u}{\delta x} + v\frac{\delta u}{\delta y}\right) = -\frac{\delta p}{\delta x} + \mu\left(\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2}\right)$ (1)

y Navier-Stokes: $\rho\left(u\frac{\delta v}{\delta x} + v\frac{\delta v}{\delta y}\right) = -\frac{\delta p}{\delta y} + \mu\left(\frac{\delta^2 v}{\delta x^2} + \frac{\delta^2 u}{\delta y^2}\right)$ (2)

Incompressibility: $\frac{\delta u}{\delta x} + \frac{\delta v}{\delta y} = 0$ (3)

For most practical purposes, a scale analysis of these equations eliminates a number of terms from the above equations, resulting in a boundary layer that has no pressure variation in the y direction and a pressure variation in the x direction impressed from the external flow. In this paper, we are looking at the (small) pressure variations that do exist (and the implications) and so below we will look at a different approach.

3.2. Incompressible Flow over a Flat Plate

Firstly observing the characteristics of incompressible fluid flow over a flat plate (steady state) as shown in figure 1 below, adapted from [2, p. 514]:

The key points of interest are a continually increasing boundary layer thickness (boundary layer edge being defined as the point at which u = 0.99U) with $\frac{\delta u}{\delta y}$ always the same sign, as well as the onset of turbulence as shown by the transition and turbulent regions (with corresponding

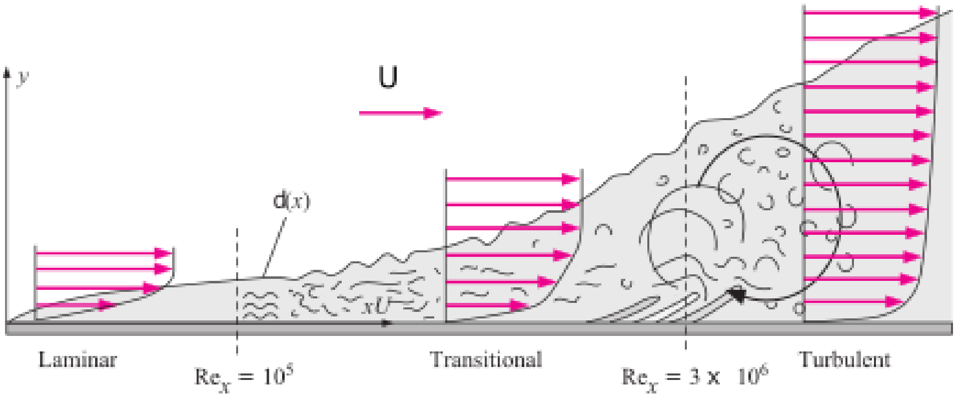


Figure 1. Incompressible Flow over a Flat Plate - not to scale.

increases on boundary layer thickness).

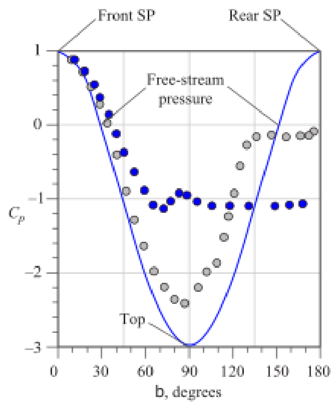
If we now refer to the paper of Blasius (NACA translation) [3, p. 3], noting that Blasius uses ϵ for the boundary layer thickness, then we see that there will be a small pressure gradient across the boundary layer (in the y direction - of the order of the square of the boundary layer thickness for a steady state flow). For most practical purposes this pressure gradient (resulting in a small pressure differential across the boundary layer) is ignored - however we will not ignore it for the purposes of this paper. More importantly, we shall investigate the pressure profile implied by the Blasius solution along the boundary layer (in the x direction).

3.3. Incompressible Flow over a cylinder
In his paper, Blasius also applied this analysis to a flow over a cylinder. In this case, there is an additional term (related to the curvature of the cylinder) which generates a larger pressure gradient.

An indication of the pressure differential in laminar flow over a cylinder is given in the widely available graphs showing the difference between theoretical and actual pressure coefficients measured along the surface of a cylinder in a moving fluid - a good example is in figure 2 below, from [2, p. 504]:
The key points to notice here are the clearly increasing pressure differential between theoretical pressure coefficient and laminar flow experimental results (up to boundary layer separation. Note that the pressure is measured on the surface of the cylinder) consistent with an increasing pressure gradient in the boundary layer as well as the much smaller pressure differential seen in the turbulent boundary layer results.

4. Mechanism for the Onset of Turbulence

4.1. Blasius Solution
Blasius in his paper (see[3]) provides an approximation for laminar flow over an infinite flat plate by ignoring the small pressure variations developed along and across the boundary layer (approximating that any pressure profile is impressed on the boundary layer by the external



Pressure coefficient as a function of angle b along the surface of a circular cylinder; the solid blue curve is the irrotational flow approximation, blue circles are from experimental data at Re − 2 * 10^5 − laminar boundary layer separation, and gray circles are from typical experimental data at Re − 7 * 10^5 − turbulent boundary layer separation.

Data from Kundu, 1990.

Figure 2. Pressure Profile for Flow over a Cylinder Theoretical and Experimental.

flow). This approximation works well for many applications, but does not predict the onset of turbulence.

The reduced set of boundary layer equations that Blasius used were:

$$\rho(\frac{\delta u}{\delta t} + u\frac{\delta u}{\delta x} + v\frac{\delta u}{\delta y}) = \rho(\frac{\delta U}{\delta t} + U\frac{\delta U}{\delta x}) + \mu\frac{\delta^2 u}{\delta y^2}$$

$$\frac{\delta u}{\delta x} + \frac{\delta v}{\delta y} = 0$$

Where U is the x component of the external flow.

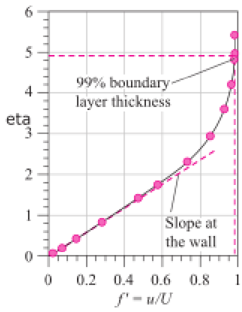
For the steady state situation this reduces to:

$$\rho(u\frac{\delta u}{\delta x} + v\frac{\delta u}{\delta y}) = \rho(U\frac{\delta U}{\delta x}) + \mu\frac{\delta^2 u}{\delta y^2}$$

$$\frac{\delta u}{\delta x} + \frac{\delta v}{\delta y} = 0$$

Blasius used these equations to approximate a numerical solution (based on a similarity variable approach - $\eta = y(\frac{U}{\nu x})^{\frac{1}{2}}$) for the flow velocities in a boundary layer. This solution has been shown to be usefully accurate experimentally. See figure 3 below, adapted from [2, p. 523]:

We can now investigate the implied x direction pressure gradient in a boundary layer by taking the Blasius results for velocity in the x direction and (numerically) calculating the pressure profile by using the full Navier-Stokes equations (i.e. equations 1, 2 and 3 at the start of the paper).



The Blasius profile in similarity variables for the boundary layer growing on a semi-infinite flat plate. Experimental data (circles) are at $Re_x = 3.64 * 10^5$.
From Panton (1996).

Figure 3. Blasius Solution for horizontal flow velocities.

By using the Blasius values from [4] and matlab, based on an infinite plate in water flowing at $1ms^{-1}$, over a grid of 10,000 points in the x direction (up to x=0.6 so near observed turbulence) and 1000 points in the y direction (up to y = 0.004 or the edge of the boundary layer at x=0.5) - enough points to give us a good indication of the characteristics of the pressure gradient, if not enough to give us reliably accurate values - we arrive at the following results (see figure 4 below):

The key points to note from this figure relevant to this paper are the continuously positive pressure gradients in the x direction (i.e continuous adverse pressure gradients - $\frac{\delta p}{\delta x} > 0$ for all values of y for large enough values of x) and the continually decreasing u value as x increases.

We can also look in more detail at the pressure profile in the region where $\eta < 2.5$ - i.e closer to the plate and away from the edge of the boundary layer (in the more 'linear' part of the Blasius graph above) - (see figure 5 below):

This detail shows more clearly the adverse pressure gradients and velocities close to the plate, away from the boundary layer edge. It is important to note that these are calculated pressure gradients and velocities from the Blasius approximation (which assumed no pressure gradients generated inside the boundary layer).

The presence of an adverse pressure gradient in boundary layer flow is a necessary but not sufficient condition for flow separation. We can visualise the pressure gradient here as a kind of virtual diffuser.

4.2. Adjusting the Blasius Solution with the Implied Pressure Gradients.
It is now instructive to adjust the standard Blasius Solution by adding the calculated pressure gradient and finding the implied adjusted flow velocities. Considering the full (x-direction) steady-state Navier Stokes equation:

$$\rho\left(u\frac{\delta u}{\delta x} + v\frac{\delta u}{\delta y}\right) = -\frac{\delta p}{\delta x} + \mu\left(\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2}\right)$$

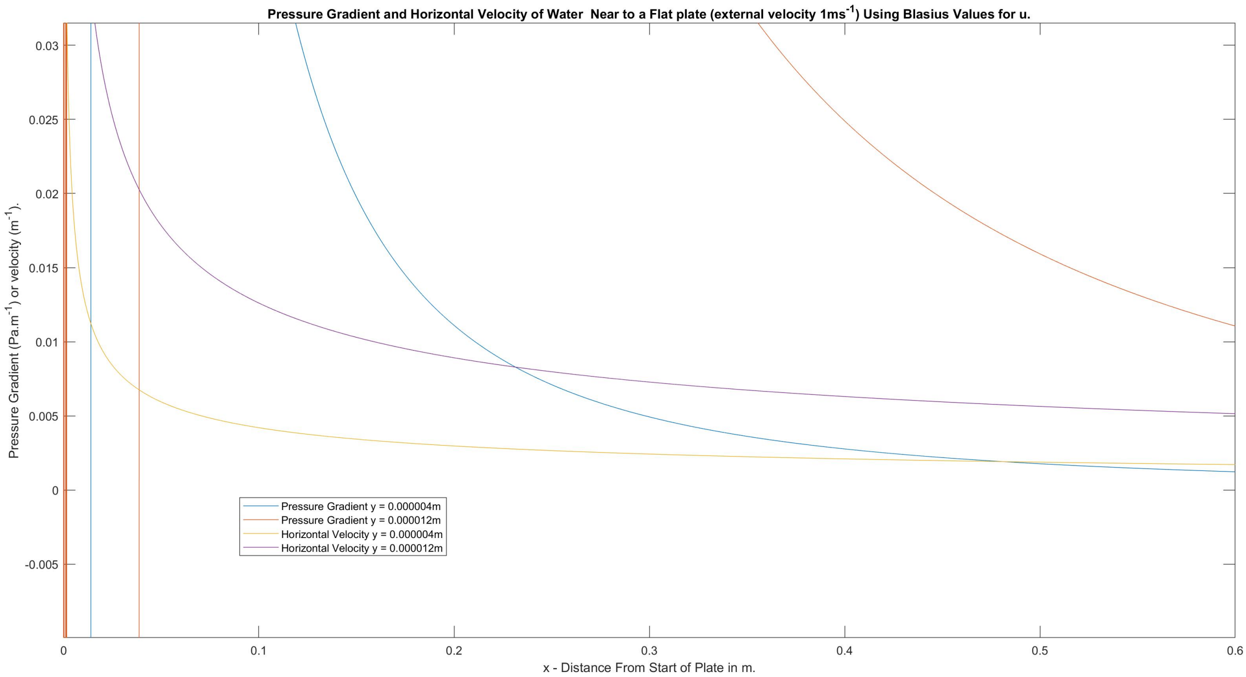


Figure 4. Calculated Pressure Gradient and Horizontal Velocity

We can rearrange this expression to evaluate the adjusted velocity figure when we superimpose the implied pressure gradient onto the Blasius solution. It is important to note that the expression will be used to evaluate the change in variables.

Rearranging:

$$\frac{\delta u}{\delta x} = \frac{\left(-\frac{\delta p}{\delta x}+\mu\left(\frac{\delta^2 u}{\delta x^2}+\frac{\delta^2 u}{\delta y^2}\right)\right)}{(\rho * u)} - v\frac{\delta u}{\delta y}$$

If we now look at the magnitudes of the quantities involved (note these are all close to the plate - the magnitudes are significantly different away from the plate):

Water: $\mu\ 10^{-3} \rho\ 10^{3}$
$\frac{\delta u}{\delta x}$: 10^{-5}
$\frac{\delta p}{\delta x}$: 10^{-6}
$\frac{\delta^2 u}{\delta x^2}$: 10^{-5}
$\frac{\delta^2 u}{\delta y^2}$: 10^{-7} Note: Small due to the adverse the adverse pressure gradient.
v : 10^{-11}
$\frac{\delta u}{\delta y}$: 10^{2}

With those magnitudes in mind, a reliable first approximation for the adjusted $\frac{\delta u}{\delta x}$ value (hence adjusted u value, by numerical integration along the x direction) can be calculated numerically using the following reduced expression:

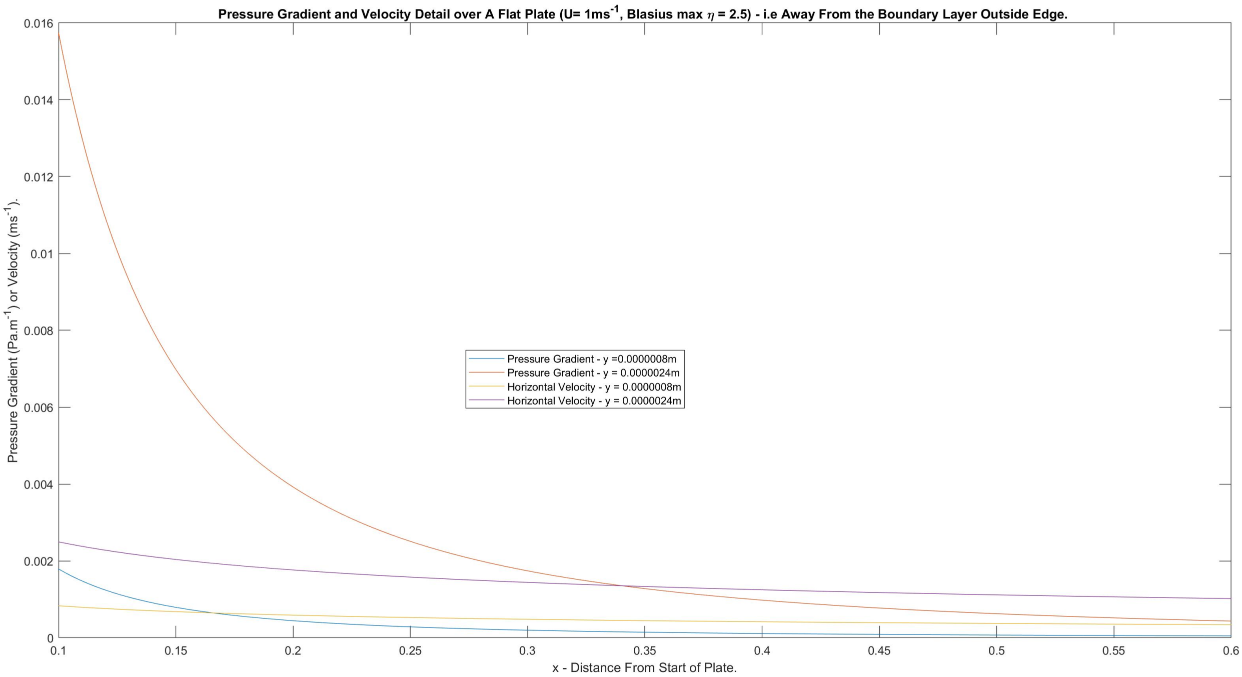


Figure 5. Calculated Pressure Gradient and Horizontal Velocity Detail $\eta < 2.5$

$$\frac{\delta u}{\delta x} = \frac{\left(-\frac{\delta p}{\delta x}\right)}{(\rho{*}u)}$$

A small adjustment to this expression (including an additional factor to allow for some influence from the discarded terms) is:

$$\frac{\delta u}{\delta x} = \frac{\left(-\frac{\delta p}{\delta x}\right)}{(\rho{*}u{*}1.2)}$$

The results of this calculation (using the same grid as before) are shown in figure 6 below:

The key observation from these results is that the adjusted velocity reduces to zero. The zero velocity causes the laminar flow to separate from the plate and creates the start of the turbulent boundary layer flow (remembering that a turbulent flow has much more difficulty separating). In effect, a virtual tripwire is created by the influence of the (small) adverse pressure gradient which leads to turbulence in the boundary layer.
The inevitability of this result can be seen in the nature of the adjustment expression above - the appearance of u in the denominator means that as u becomes small, $\frac{\delta u}{\delta x}$ increases rapidly, leading to a sudden reduction of u to zero (as can bee seen in the figure).

In short, the Blasius solution adjusted with the effects of the implied pressure gradient leads to a guaranteed (and calculable) onset of turbulence. In addition, we can see that the assumption that we can eliminate pressure from the Navier-Stokes equations is not valid if we want to predict turbulence (and find analytical solutions - which is unlikely as adjusted Blasius shows us that turbulence is inevitable at all flow speeds for a long enough plate).

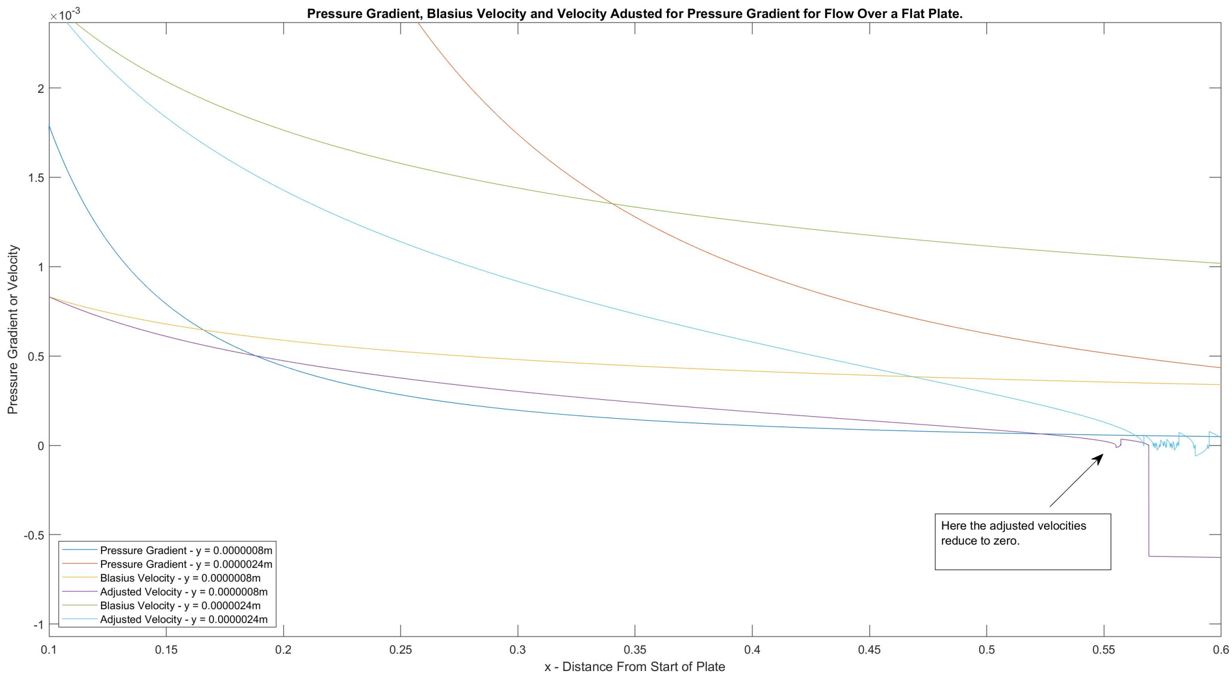


Figure 6. Pressure Gradient and Adjusted Horizontal Velocity

5. Conclusions

The Blasius solution for incompressible fluid flow over a flat plate is a very useful tool for practical applications, but by eliminating pressure it eliminates the possibility of predicting the onset of turbulence. By adding back the effects of the pressure gradient, it is possible to predict mathematically (although more work will need to be done to establish suitable accuracy).

Due to the wide applicability of the Blasius solution (for many geometries, including potentially for flow without obstructions or plates), this result suggests that the elimination of pressure from the Navier-Stokes equations for analytical solution approaches may not be valid as it eliminates the effects of pressure gradients (mathematically, the reduction from elliptic to parabolic eliminates a significant variable that is necessary for complete analytic solutions - which means that there may be no truly analytic solutions to the equations).

References

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[4] Ganapol B.D., 'Highly Accurate Solutions of the Blasius and Falkner-Skan Boundary Layer Equations via Convergence Acceleration' (arXiv:1006.3888, 2010)