A Mathematical Solution to Thomson’s Lamp ‘Paradox’

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Abstract: We resolve the ‘paradox’ of Thomson’s Lamp by first proving Grandi’s series is finite. We shall show why it is meaningless to ask: Is the lamp on or off at 2 minutes?

The grouping of terms in the infinite Grandi series leading to the contradiction $0 = 1$ is well known, and yet remains unresolved. Resolving this contradiction allows us to resolve the ‘paradox’ of Thomson’s Lamp showing why we cannot ask: Is the lamp on or off at 2 minutes?

Theorem: The Grandi series is finite.

Proof: (by contradiction) Assume the Grandi series is infinite, then

$$0 = (1 - 1) + (1 - 1) + (1 - 1) + \cdots$$
$$= 1 - 1 + 1 + 1 - 1 + \cdots$$
$$= 1 + (-1 + 1) + (-1 + 1) + \cdots$$
$$= 1$$

Since $0 = 1$ is a contradiction the assumption is false and Grandi’s series is finite. Hence,

$$1 - 1 + 1 - 1 + \cdots + (-1)^{n-1} = \frac{1 - (-1)^n}{2}, \quad n = 1, 2, 3, \ldots$$

Corollary: $1 - 1 + 1 - 1 + \cdots \neq 1/2$ because the L.H.S does not exist.

Resolving the Thomson’s Lamp ‘Paradox’

Suppose we switch off a lamp. After 1 minute we switch it on. 1/2 a minute later off, 1/4 on, 1/8 off, etc. Let ‘on’ be 1 and ‘off’ be 0 and put the finite Grandi series in one-to-one correspondence with the finite series

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{n-1}} = 2 - \frac{1}{2^n}, \quad n = 1, 2, 3, \ldots$$

Therefore, we can know if the lamp is on or off when it’s less than 2 minutes. But as $n = 1, 2, 3, \ldots$ continues indefinitely the term $\frac{1}{2^n}$ approaches 0 indefinitely, and since it never equals 0 it is meaningless to ask: Is the lamp on or off at 2 minutes?