A disproof of the Riemann hypothesis by finding a solution that satisfies the conditions of Salem’s equation

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Abstract

This research deals with a conjecture equivalent to the Riemann hypothesis: "There is no non-trivial, bounded solution to a particular integral equation under a particular condition." By showing the existence of counterexamples of possible propositions, it is shown that there is a non-trivial zero of the Riemann zeta function whose real part is not equal to 1/2.

1 Introduction

1.1 Purpose of this paper

The Riemann hypothesis is a conjecture about the zero of Riemann zeta function proposed by Bernhard Riemann, a German mathematician in the 19 century. This hypothesis can be expressed as follows.

Riemann hypothesis
For the Riemann zeta function $\zeta(s)$, the real part of all non-trivial zeros is $1/2$.

Here, the Riemann zeta function is defined for a complex number $s$ as follows.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

(1)

However, since the Riemann zeta function by the above definition is a function defined only for complex numbers $s$ with real parts greater than 1, the target of the Riemann hypothesis is the Riemann zeta function $\zeta(s)$ applied analytic continuation, satisfying the following functional equality.

$$\zeta(s) = 2^s \pi^{s-1} \sin \left( \frac{\pi s}{2} \right) \Gamma(1-s) \zeta(1-s)$$

(2)

The purpose of this research is to prove a counterexample of the Riemann hypothesis, which is a claim of the same value as the Riemann hypothesis: "Is there a non-trivial and bounded solution of an integral equation?" The above proposition, which is equivalent to the Riemann hypothesis, is stated in detail as follows.

Lemma 1.1
The Riemann hypothesis and the following proposition are equivalent: Integral equation

$$\int_{0}^{\infty} \frac{e^{\delta z} \phi(z)}{e^{yz} + 1} \, dz = 0$$

(3)

does not have a non-trivial and bounded solution $\phi(z)$ for $1/2 < \delta < 1$, $y > 0$.

2010 Mathematics Subject Classification. 42A16, 45B05, 11M41, 11M26. Key Words and Phrases. Riemann Hypothesis, Fourier series, Number theory, Integral equation.
1.2 Research methods

To achieve the goal described above, we define an A-function for a prime \( p \). The representative feature of this function is that each definite integral of A-function for every \( p \) of interval width from the origin is equal to a certain number of digits of a repetend of \( 1/p \), which is the reciprocal of the prime \( p \). Furthermore, by defining A-function as a periodic function with period as a function of \( p \), we can expand A-function to a Fourier series, and then by using the above property, we can obtain the Dirichlet series \( D(s) \) for a sequence \( D(s) \) such that \( s = 1 \) has a zero. Furthermore, by applying the Melin transform to the above Dirichlet series \( D(s) \) and adding various operations to the obtained function, a function that satisfies the conditions for the integral equation of interest is found.

1.3 Results and Conclusions

By finding a solution that satisfies the conditions of the integral equation that is equivalent to the Riemann hypothesis, it is shown that there exists a non-trivial zero of the Riemann zeta function and its real part does not meet the conditions of the expectation. We obtained the following results.

Theorem 1.1
There is a non-trivial zero \( s_0 \) of the Riemann zeta function \( \zeta(s) \) whose real part is not equal to \( 1/2 \).

2 Prerequisite knowledge

Initially, we present an overview of the Riemann hypothesis and a claim by the Greek mathematician Rafael Salem, which is equivalent to the Riemann hypothesis. The ultimate goal of this paper is to show the existence of the solution of the integral equation, \( \phi(z) \), shown below. In addition, the prerequisite knowledge necessary to achieve the goal is described in the following paragraphs.

2.1 Conjecture equal to Riemann hypothesis

Lemma 2.1
The Riemann hypothesis and the following proposition are equivalent: Integral equation

\[
\int_{0}^{\infty} \frac{e^{\delta z - 1} \phi(z)}{e^{y z} + 1} \, dz = 0
\]

does not have a non-trivial and bounded solution \( \phi(z) \) for \( \frac{1}{2} < \delta < 1, y > 0 \).

2.2 Piecewise smooth function

Definition 2.1

[definition 1] If a real value function \( f(x) \) satisfies the following conditions (1), (2), then \( f(x) \) is piecewise smooth in the closed interval \([a, b]\).

(1): \( f(x) \) is differentiable over the range of all but a finite number of points in the closed interval \([a, b]\), and its derivative \( f'(x) \) is continuous.

(2): At each finite discontinuity \( t_k \) there exists a left and a right limit of the function \( f(x) \) and its derivatives \( f'(x) \), and furthermore, each limit is finite.

[definition 2] If a function \( f(x) \) is a function with period \( 2l \) and this function is piecewise smooth in the closed interval \([0, 2l]\), then \( f(x) \) is simply called a piecewise smooth function.

We denote by \( C'(X) \) the set of all piecewise smooth functions in the set \( X \).
2.3 $L^p$ space

Definition 2.2
When $p$ is $1 < p < \infty$, the real-valued measurable function $f(x)$ is $p$-th power-integrable on $S$ is defined as the following equation holds.

$$\int_S |f(x)|^p dx < \infty \quad (5)$$

Furthermore, denote by $L^p(S)$ the entire set of functions $f(x)$ satisfying the above condition for some $p$.

2.4 Fourier series

Lemma 2.2
The following equation holds if the real number function $f(x)$ is a function with a period $T$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \left( \frac{2\pi nx}{T} \right) + b_n \sin \left( \frac{2\pi nx}{T} \right) \right) \quad (6)$$

Where the sequences $\{a_n\}$ and $\{b_n\}$ are defined by

$$a_n = \frac{2}{T} \int_0^T f(x) \cos \left( \frac{2\pi nx}{T} \right) dx \quad (7)$$

$$b_n = \frac{2}{T} \int_0^T f(x) \sin \left( \frac{2\pi nx}{T} \right) dx \quad (8)$$

2.5 Midy’s theorem

Lemma 2.3
For a prime number $p$ that is not 2, 5, if the length $d_p$ of a repetend of $1/p$ is even, then let $a_2$ and $a_2$ be two values that bisect the repetend, and the following equation holds.

$$a_1 + a_2 = 10^{\frac{d_p}{2}} - 1 \quad (9)$$

2.6 Fredhorm’s theorem

Lemma 2.4
Let $K(x, y)$ be the integral kernel and consider the following first kind of Fredholm type integral equations.

$$\int_a^b K(x, y) \phi(x) dx = \lambda \phi(x) \quad (10)$$

$$\int_a^b K(x, y) \psi(x) dx = \lambda \psi(x) \quad (11)$$

In this case, one of the following two conditions holds.

1: For any $\lambda$, the two integral equations above have only the obvious solution $\phi(x) = \psi(x) = 0$.

2: For any $\lambda$, the two integral equations above have solutions $\phi_1(x), \cdots, \phi_n(x), \psi_1(x), \cdots, \psi_n(x)$ that are linearly independent.

Where $\overline{X}$ represents the complex conjugate of $X$. 
2.7 Mean-value theorem

2.7.1 Mean Value Theorem of Integral Form

Lemma 2.5
If the real number function \( f(x) \) is bounded and continuous and \( g(x) \) is a non-negative integrable function, then the product of the two functions \( f(x)g(x) \) is integrable and there exists a constant \( c \in (a, b) \) that satisfies following equality.

\[
\int_a^b f(x)g(x)dx = f(c) \cdot \int_a^b g(x)dx
\]  

(12)

The proof of this theorem will be needed later, and we leave it as an appendix.

Proof 2.1
Since \( f(x) \) is continuous in the interval \([a, b]\), it has a minimum value \( m \) and a maximum value \( M \). Hence, for such \( m, M \) and functions \( g(x) > 0 \), the following holds.

\[
m \leq f(x) \leq M
\]

(13)

\[
m \cdot g(x) \leq f(x)g(x) \leq M \cdot g(x)
\]

(14)

Integrating each side of the above equation in the interval \([a, b]\), we obtain the following.

\[
m \int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq M \int_a^b g(x)dx
\]

(15)

Since the definite integral of the function \( g(x) \) is always positive from \( g(x) > 0 \), with

\[
\lambda = \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx}
\]

(16)

the following equation holds.

\[
m \leq \lambda \leq M
\]

(17)

Here, from the intermediate value theorem, there exists a constant \( c \in (a, b) \) such that \( \lambda = f(c) \) for equation (17). Therefore, from equation (17), there exists a constant \( c \) in the interval \((a, b)\) that satisfies

\[
\int_a^b f(x)g(x)dx = f(c) \cdot \int_a^b g(x)dx
\]

(18)

\( \square \)

2.8 Bessel’s inequality

Lemma 2.6
Let \( f(x) \in L^2[-L, L] \) has a period \( 2L \). when \( f(x) \) is expanded in the Fourier series as in

\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \left( \frac{\pi nx}{L} \right) + b_n \sin \left( \frac{\pi nx}{L} \right) \right)
\]

(19)

\[
a_n = \frac{1}{L} \int_0^{2L} f(x) \cos \left( \frac{\pi nx}{L} \right) dx
\]

(20)

\[
b_n = \frac{1}{L} \int_0^{2L} f(x) \sin \left( \frac{\pi nx}{L} \right) dx
\]

(21)

the following equation holds.

\[
\frac{|a_0|}{2}^2 + \sum_{n=1}^{\infty} (|a_n|^2 + |b_n|^2) \leq \frac{1}{L} \int_{-L}^{L} |f(x)|^2 dx
\]

(22)
2.9 Comparison test

Lemma 2.7
If there are real number sequences \( \{a_n\} \) and \( \{b_n\} \) and \( 0 \leq a_n \leq b_n \) holds for all non-negative integers \( n \), then if \( \sum_{n=1}^{\infty} b_n \) converges, then \( \sum_{n=1}^{\infty} a_n \) also converges.

2.10 Mellin transform

Lemma 2.8
For a non-negative integer \( n \) and a complex sequence \( x_n \), we define the Dirichlet series \( D(s) \) as follows.

\[
D(s) = \sum_{n=1}^{\infty} \frac{x_n}{n^s}
\]  

We also define the power series \( F(z) \) for \( D(s) \) as follows.

\[
F(z) = \sum_{n=1}^{\infty} x_n z^n
\]  

In this case, in the interval of absolute convergence of \( D(s) \), the following equation holds.

\[
D(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} F(e^{-t}) t^{s-1} dt
\]

3 A-function for a prime \( p \)

In this section, we define the A-function for the prime number \( p \), which is the key to achieving the goal. In the following sections, we will develop the overall proof with the A-function as the axis, making use of the above prerequisite knowledge.

Definition 3.1
Let \( p \) be a prime number that is not 2, 5, and \( d_p \) be the repetend length of repeating decimal \( 1/p \). Let \( V \) be an infinite-dimensional linear space (vector space) and \( L^2(E) \) be the entire set of square-integrable functions in a closed interval \( E = [0, d_p \pi] \) (Lebesgue space). Let \( U \) be the open set of \( V \) and let \( C'(U) \) be the whole set of continuous and piecewise smooth functions on \( U \). Let \( f_p(x) \) be an A-function of \( p \), whose real value function \( f_p(x) \) on \( U \) satisfies the following conditions (1), (2) and (3). However, the domain of the A-function \( f_p(x) \) is defined to be the entire positive real number.

1: \( f_p(x) \geq 0 \)
2: \( f_p(x) \in C'(U) \land f_p(x) \in L^2(E) \)
3: Let \( N_p(n) \) be the number of decimal \( n \) positions of the repetend of \( 1/p \), then for any natural number \( k \), the following holds:
   - (3) - 1: The basic period of \( f_p(x) \) is \( d_p \pi \).
   - (3) - 2: The following two equations holds.

\[
\int_{0}^{2\pi} f_p(x) dx = N_p(k)
\]

\[
\int_{0}^{2\pi} f_p(x) dx = N_p \left( \frac{d_p}{2} + k \right)
\]
and (27), respectively, assuming that there exists one that satisfies the conditions of the formula (26) and (27). If the function $f_p(x)$ is defined by

$$
f_p(x) = \begin{cases} 
  u_{p,1}(x) & (2\pi(-1 + k_{p,n}) \leq x \leq \pi(-1 + 2k_{p,n})) \\
  v_{p,1}(x) & (\pi(-1 + 2k_{p,n}) < x \leq 2k_{p,n} \pi) \\
  u_{p,2}(x) & (2k_{p,n} \pi < x \leq \pi(1 + 2k_{p,n})) \\
  v_{p,2}(x) & (\pi(1 + 2k_{p,n}) < x \leq 2\pi(1 + k_{p,n})) \\
  \vdots & \\
  v_{p,s}(x) & (\pi(-1 + d_p(n+1)) < x < d_p(n+1)\pi) 
\end{cases}
$$

(28)

for any non-negative integer $n$ and a sequence $k_{p,n} = \frac{d_p n}{2} + 1$, then $f_p(x)$ satisfies the above condition (3) − 1 and is bounded in the interval $[0, d_p \pi]$, then $f_p(x) \in L^2(E)$ is trivial fact. Hence, in $f_p(x)$ above, if the function is smooth except for a finite number of points $t_k = k\pi$, the above can be said to be the definition of A-function $f_p(x)$. With that in mind, whether $u_{p,n}(x)$ and $v_{p,n}(x)$ are smoothing functions for all natural numbers $m$ such that $m$ satisfies $m \leq \frac{d_p}{2}$ for some prime $p$ is a subject for future research. However, considering the direction of this research, it is sufficient to study such primes $p$ as can define the A-function $f_p(x)$ by the above definition. However, it is necessary to extract one or more limited primes from such $p$ in order to facilitate the consideration.

4 Zero of certain Dirichlet series $D(s)$

In this section, we construct a function $D(s)$ such that it is a Dirichlet series for a certain sequence $\{c_n\}$ and has $s = 1$ at zero. Based on the following fact obtained in this section, the following sections consider the values of the sequence $D(s)$ and the absolute convergence of $D(s)$.

**Lemma 4.1**

For any non-negative integer $n$ and fixed natural number $k$, define the sequence $\{c_n\}$ as follows.

$$c_n = a_n \left( \sin \left( \frac{4kn\pi}{d_p} \right) - \sin \left( \frac{4\pi n(k-1)}{d_p} \right) - b_n \left( \cos \left( \frac{4kn\pi}{d_p} \right) - \cos \left( \frac{4\pi n(k-1)}{d_p} \right) \right) \right) \tag{29}$$

In this case, the following equation holds.

$$\sum_{n=1}^{\infty} \frac{c_n}{n} = 0 \tag{30}$$

**Proof 4.1**

If $f_p(x)$ is any A-function, the following holds.

$$a_n = \frac{2}{d_p \pi} \int_{0}^{d_p \pi} f_p(x) \cos \left( \frac{2nx}{d_p} \right) dx \tag{31}$$

$$b_n = \frac{2}{d_p \pi} \int_{0}^{d_p \pi} f_p(x) \sin \left( \frac{2nx}{d_p} \right) dx \tag{32}$$

$$f_p(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \left( \frac{2nx}{d_p} \right) + b_n \sin \left( \frac{2nx}{d_p} \right) \right) \tag{33}$$

Here, from the definition of A-function, the definite integral value in the interval $[0, d_p \pi]$ of $f_p(x)$ is equal to the sum of the digits of each digit of the repetend of the $1/p$. Hence, using Midy’s theorem, the following equation holds.

$$\frac{a_0}{2} = \frac{1}{d_p \pi} \int_{0}^{d_p \pi} f_p(x) dx = \frac{9}{2\pi} \tag{34}$$
Using the above formula, we can transform the formula for the Fourier series of A-function to obtain the following equation.

\[
f_p(x) = \frac{9}{2\pi} + \sum_{n=1}^{\infty} \left( a_n \cos \left( \frac{2nx}{d_p} \right) + b_n \sin \left( \frac{2nx}{d_p} \right) \right)
\]  

(35)

Furthermore, with \( g_p(x) = f_p(x) - 9/2 \), from Midy’s theorem and the definition of the A-function, from the fact that

\[
\int_{2\pi(-1+k)}^{\pi(-1+2k)} \left( \frac{9}{2\pi} + g_p(x) \right) \, dx = 9 + \int_{2\pi(-1+k)}^{2\pi(-1+k)} g_p(x) \, dx
\]

holds for any natural number \( k \),

\[
\int_{2\pi(-1+k)}^{2\pi(-1+k)} g_p(x) \, dx = 0
\]

(36)

follows. Furthermore, from \( f_p(x) \in C'(U) \), the Fourier series of \( f_p(x) \) converges uniformly to \( f_p(x) \). Therefore, since the extremes and the integrals are interchangeable, we can transform the equation as follows.

\[
\int_{2\pi(-1+k)}^{2\pi(-1+k)} g_p(x) \, dx = \sum_{n=1}^{\infty} \left\{ \int_{2\pi(-1+k)}^{2\pi(-1+k)} \left( a_n \cos \left( \frac{2nx}{d_p} \right) + b_n \sin \left( \frac{2nx}{d_p} \right) \right) \right\} \, dx
\]

\[
= \sum_{n=1}^{\infty} \left( a_n \int_{2\pi(-1+k)}^{2\pi(-1+k)} \cos \left( \frac{2nx}{d_p} \right) \, dx + b_n \int_{2\pi(-1+k)}^{2\pi(-1+k)} \sin \left( \frac{2nx}{d_p} \right) \, dx \right)
\]

\[
= \frac{d_p}{2} \sum_{n=1}^{\infty} c_n \frac{n}{n}
\]

where the \( \{ c_n \} \) in the above formula is given by

\[
c_n = a_n \left( \sin \left( \frac{4kn\pi}{d_p} \right) - \sin \left( \frac{4\pi(nk-1)}{d_p} \right) \right) - b_n \left( \cos \left( \frac{4kn\pi}{d_p} \right) - \cos \left( \frac{4\pi(nk-1)}{d_p} \right) \right)
\]

(37)

Thus, it was proved that the Dirichlet series \( \sum_{n=1}^{\infty} c_n/n^s \) has \( s = 1 \) at zero. From each section below, it shall be denoted as \( D(s) = \sum_{n=1}^{\infty} c_n/n^s \).

\[\square\]

5 Natural number \( n \) satisfying \( c_n \neq 0 \)

In §4 we showed that for any prime \( p \) there exists a sequence \( f_p(x) \) of A-functions \( f_p(x) \) for any prime \( p \) and that the Dirichlet series \( D(s) \) has a zero \( s = 1 \). In this section, we show that there exists a non-negative integer \( n \) satisfying \( c_n \neq 0 \). We must state this condition because if \( c_n = 0 \) for all non-negative integers \( n \), then it is a trivial fact that \( D(s) \) has a zero at \( s = 1 \).

In order to convert the phase of the trigonometric function in the above sequence \( f_p(x) \) into a simple expression, we restrict our consideration to the case of \( p = 101 \) for the A-function \( f_p(x) \).

\textbf{Proof 5.1}

In each of the following sections, including this one, the A-function for \( p = 101 \) that satisfies the condition is defined in §3 as follows.

\[
f_{101}(x) = \begin{cases} 
\frac{N(101)}{2} \sin(x) & (2\pi(-1 + k_{p,n}) \leq x \leq \pi(-1 + 2k_{p,n})) \\
-\frac{N(101)(4p + k)}{2} \sin(x) & (\pi(-1 + 2k_{p,n}) < x \leq 2k_{p,n}\pi) \\
\frac{N(101)(k)}{\pi} \sin(x) & (2k_{p,n}\pi < x \leq \pi(1 + 2k_{p,n})) \\
\frac{2N(101)(4p + k)}{\pi} \sin^2(x) & (\pi(-1 + d_p(n + 1)) < x < d_p(n + 1)\pi)
\end{cases}
\]

(38)
First, consider the case where \( n \) is an even number. \( c_n \) is expressed as follows.

\[
c_n = a_n \left( \sin \left( \frac{4kn\pi}{d_p} \right) - \sin \left( \frac{4\pi(n-1)}{d_p} \right) \right) - b_n \left( \cos \left( \frac{4kn\pi}{d_p} \right) - \cos \left( \frac{4\pi(n-1)}{d_p} \right) \right)
\]  

(39)

We are considering the case where \( p = 101 \), so \( d_p = 4 \) (the repetend length of \( 1/101 \)). Further, since \( k \) is any natural number, when \( k = 2 \), the above expression can be transformed as following.

\[
c_n = a_n \left( \sin (2n\pi) - \sin (\pi) \right) - b_n \left( \cos (2n\pi) - \cos (n\pi) \right)
\]  

(40)

However, since \( n \) is an even number, the right side of the above expression becomes 0. Hence, if \( n \) is even and \( k = 2 \), \( c_n = 0 \). Here, we need only show that there exists a non-negative integer \( n \) that does not satisfy \( c_n = 0 \), we assume that \( n \) represents an odd number and prove it by contradiction, assuming that \( c_n = 0 \).

If \( c_n = 0 \), \( n \) is odd, then in equation (40), the following equation holds.

\[
\sin (2n\pi) - \sin (n\pi) = 0
\]  

(41)

\[
\cos (2n\pi) - \cos (n\pi) \neq 0
\]  

(42)

Hence, in this case the sequence \( b_n \) must be equal to zero. Namely,

\[
b_n = \frac{1}{2\pi} \int_0^{4\pi} f_{101}(x) \sin \left( \frac{nx}{2} \right) dx = 0
\]  

(43)

If we consider equation (43) as a first kind of Fredholm type integral equation whose integral kernel is \( \sin \left( \frac{nx}{2} \right) \), we can take as an example the following sequence of functions as a solution that is not \( f_{101}(x) \).

\[
\phi_m(x) = \cos \left( \frac{mx}{2} \right)
\]  

(44)

\[
\psi_m(x) = \sin \left( \frac{mx}{2} \right)
\]  

(45)

The above example is based on the orthogonality of the trigonometric function and is a solution of the integral equation (43) for any non-negative integer \( m \). Here, from Fredholm’s theorem, \( \phi_m(x), \psi_m(x) \) and \( f_{101}(x) \) must be linearly independent of each other. However, \( f_{101}(x) \) can be expanded into a Fourier series like the above expression (33), and then \( \phi_0(x) = 1 \), so \( \frac{\phi_0(x)}{2} = \frac{\phi_0(x)}{2} \). \( f_{101}(x) \) can be thought of as a linear combination of \( \phi_m(x) \) and \( \psi_m(x) \). This is equivalent to the linear dependency of \( \phi_m(x), \psi_m(x) \), and \( f_{101}(x) \).

Hence, for \( f_{101}(x) \), since the expression (43) does not hold, if \( n \) is odd, \( b_n \neq 0 \) and \( c_n \neq 0 \) and for any natural number \( m, b_n \neq 0 \) follows as a corollary.

\( \square \)

6 Absolute convergence of \( D(s) \) at \( s = 1 \)

In the previous section, we showed that the condition "\( c_n = 0 \) for every non-negative integer \( n \)" does not hold for the general term \( c_n \) of the sequence \( \{c_n\} \). In this section, we consider the proposition "\( D(s) \) converges absolutely at \( s = 1 \)" which is a sufficient condition for the applicability of the Mellin transformation to \( D(s) \) at \( s = 1 \). After showing that this proposition about \( D(s) \) is correct, in the next section we apply the Mellin transform to \( D(s) \) and construct a function that is a solution of the integral equation (4).

Lemma 6.1

For the Dirichlet series \( D(s) \) for the sequence \( \{c_n\} \), \( D(s) \) converges absolutely at \( s = 1 \).

Proof 6.1

In the following, the constant \( k \) in \( c_n \) is assumed to be 2. If \( n \) is even, \( c_n = 0 \), then,

\[
\frac{|c_n|}{n} = 0
\]  

(46)
is trivial fact. And if \( n \) is odd, the following equation holds.

\[
|c_n| = |b_n(\cos(\pi n) - \cos(2\pi n))| = 2|b_n|
\] (47)

|c_n| = \frac{2|b_n|}{n}
\] (48)

Therefore, the following relations hold.

\[
\sum_{n=1}^{\infty} \frac{|b_n|}{n} < \infty \implies \sum_{n=1}^{\infty} \frac{|c_n|}{n} < \infty
\] (49)

Here, since \( \{b_n\} \) is one of the Fourier coefficients of \( f_{101}(x) \), from Bessel’s inequality, \( a_n, b_n \) obeys the following inequality.

\[
\frac{|a_0|^2}{2} + \sum_{n=1}^{\infty} (|a_n|^2 + |b_n|^2) \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f_{101}(x)|^2 dx
\] (50)

Furthermore, since \( f_{101}(x) \in L^2(E) \), it follows that

\[
\sum_{n=1}^{\infty} (|a_n|^2 + |b_n|^2) < \infty
\] (51)

Also, since trivial inequality \( |a_n|^2 + |b_n|^2 \geq |b_n|^2 \),

\[
\sum_{n=1}^{\infty} |b_n|^2 < \infty
\] (52)

holds from the comparison test. Here, if \(|b_n| \geq 1/n\), then \(|b_n|^2 \geq |b_n|^2/n\) is followed, and the comparison test gives the following equation

\[
\sum_{n=1}^{\infty} \frac{|b_n|}{n} < \infty
\] (53)

And, if \(|b_n| < 1/n\), multiplying both sides by \(1/n\) yields \(|b_n|/n < 1/n^2\), and taking the limit of the partial sum of both sides, the right side converges to \(\pi^2/6\) (Basel problem). Therefore, from the comparison test, the equation (53) holds in this case as well. Hence, the absolute convergence of \(\sum_{n=1}^{\infty} b_n/n\) is proved. Therefore, from equation (49), \(D(s)\) converges absolutely at \(s = 1\).

\[\square\]

### 7 Solution of integral equation (4)

In each of the previous sections, we have discussed the elements of a function (the Dirichlet series \(D(s)\)) which is the solution of the integral equation (4), and the conditions for the actual construction of a function from these elements. Finally, in this section, we derive from the Dirichlet series \(D(s)\) a solution satisfying the conditions for the integral equation (4).

**Proof 7.1**

From §5, 6, it follows that the Mellin transformation can be applied to the Dirichlet series \(D(s)\) for \(\{c_n\}\). Hence, applying the Mellin transform to the target \(D(s)\), we obtain

\[
D(s) = \sum_{n=1}^{\infty} \frac{c_n}{n^s} = \frac{1}{\Gamma(s)} \int_{0}^{\infty} F(e^{-t}) t^{s-1} dt
\] (54)

\[
F(e^{-z}) = \sum_{n=1}^{\infty} \frac{c_n}{n^z}
\] (55)
Since $D(s)$ has a zero $s = 1$, the improper integral on the right-hand side is $s = 1$, which is equal to 0. In other words,
\[
\int_0^\infty F(e^{-t}) dt = 0
\]
(56)

Here, for certain fixed $y > 0$ and any $z > 0$, we have
\[
h(z) = z^\frac{\delta}{2} (e^{yz} + 1) I(z)
\]
(57)
\[
I(z) = F(e^{-z})
\]
(58)

Therefore, we can obtain the following equation.
\[
\int_0^\infty \frac{z^{\delta-1} h(z)}{e^{yz} + 1} dz = \int_0^\infty z^{\delta-\frac{\delta}{2}} I(z) dz
\]
(59)

The above equation is equal to 0 at $1/2 < \delta = 2/3 < 1$. Hence, $h(z)$ is a non-trivial solution of the integral equation (4) that satisfies the condition for $\delta, y$.

Also, the function $f_{101}(x)$ satisfying the above definition takes a non-negative value for any $x \in [0, \infty)$ and $\sin \left( \frac{nx}{2} \right)$, $f_{101}(x)$ are continuous functions for any natural number $m$. Furthermore, for any natural number $m$, $\sin \left( \frac{nx}{2} \right)$ takes a minimum value $-1$ and a maximum value 1, so the mean value theorem can be applied. Therefore, from the Mean Value Theorem, for any non-negative integer $n$, there exists a constant $\xi_n$ that satisfies
\[
b_n = \int_0^{4\pi} f_{101}(x) \sin \left( \frac{nx}{2} \right) dx = \sin \left( \frac{n\xi_n}{2} \right) \cdot \int_0^{4\pi} f_{101}(x) dx
\]
(60)

in the open interval $(0, 4\pi)$. therefore, since
\[
\int_0^{4\pi} f_{101}(x) dx = 18
\]
(61)
holds, we can use the above $\xi_n$ and express it as
\[
b_n = 18 \sin \left( \frac{n\xi_n}{2} \right)
\]
(62)

Furthermore, when $n$ is an even number, since $c_n = 0$, the following equation holds.
\[
I(z) = \sum_{n=1}^\infty \frac{c_n}{e^{nz}} = \sum_{n=1}^\infty \frac{2b_{2n-1}}{e^{(2n-1)z}} = 36 \sum_{n=1}^\infty \sin \left( \frac{2n-1}{2} \frac{\xi_{2n-1}}{e^{(2n-1)z}} \right)
\]
(63)

Here, for any non-negative integer $n$, from the fact that
\[
-1 \leq \sin \left( \frac{n\xi_n}{2} \right) \leq 1
\]
(64)

holds, the following formula is derived.
\[
-36 \sum_{n=1}^\infty \frac{1}{e^{(2n-1)z}} \leq I(z) \leq 36 \sum_{n=1}^\infty \frac{1}{e^{(2n-1)z}}
\]
(65)

Here, by transforming the right side of the above formula, we obtain the following equation.
\[
\sum_{n=1}^\infty \frac{1}{e^{(2n-1)z}} = \sum_{n=1}^\infty \frac{e^{-z} \cdot (e^{-2x})^{n-1}}{e^{(2n-1)z}} = \sum_{n=1}^\infty \frac{e^{-z} \cdot (e^{-2x})^{n-1}}{1 - e^{-2x}} = \lim_{m \to \infty} \frac{e^z}{e^{2z} - 1}
\]
(66)

(67)

(68)
Hence, the range of the value of \( h(z) \) is expressed as
\[
\frac{36z^1(e^{y^2} + 1)e^z}{1 - e^{2z}} \leq h(z) \leq \frac{36z^1(e^{y^2} + 1)e^z}{e^{2z} - 1}
\] (69)

Furthermore, if we consider the limits at both ends of the value range of the function \( h(z) \) to proof the boundary of \( h(z) \) for \( y = 1/2 \),
\[
\lim_{z \to \infty} \frac{36z^1(e^{y^2} + 1)e^z}{1 - e^{2z}} = \lim_{z \to \infty} \frac{36z^1(e^{y^2} + 1)}{e^{2z} - 1} = 0
\] (70)
\[
\lim_{z \to \infty} \frac{36z^1(e^{y^2} + 1)e^z}{e^{2z} - 1} = \lim_{z \to \infty} \frac{36z^1(e^{y^2} + 1)}{e^z - \frac{1}{e^z}} = 0
\] (71)
holds, so from the squeeze theorem, following equation holds.
\[
\lim_{z \to \infty} h(z) = 0
\] (72)

Also, from Inequality of arithmetic and geometric means, \( |a_n|^2 + |b_n|^2 \geq 2|a_n||b_n| \) holds. Furthermore, since \( a_n \) can be transformed as in the equation (62), for a certain constant \( \mu_n \), the following equation holds.
\[
\sum_{n=1}^{\infty} (|a_n|^2 + |b_n|^2) \geq 2 \sum_{n=1}^{\infty} |a_n||b_n| = 648 \sum_{n=1}^{\infty} \cos \left( \frac{n\mu_n}{2} \right) \sin \left( \frac{n\xi_n}{2} \right)
\] (73)

And the following equation holds.
\[
648 \sum_{n=1}^{\infty} \cos \left( \frac{n\mu_n}{2} \right) \sin \left( \frac{n\xi_n}{2} \right) \leq 648 \sum_{n=1}^{\infty} \sin \left( \frac{n\xi_n}{2} \right) = 36 \sum_{n=1}^{\infty} |b_n|
\] (74)

It is also clear that the right-hand side of the above equation is less than the left-hand side of equation (73). Therefore, we have been able to show that the series \( \sum_{n=1}^{\infty} b_n \) is absolutely convergent. Hence, \( h(z) \) can also be defined by \( z = 0 \), and the following equation holds.
\[
I(0) = 2 \sum_{n=1}^{\infty} b_{2n-1} = \alpha
\] (75)
\[
h(0) = 0
\] (76)

Where \( \alpha \) is a constant. Hence, the continuous function \( h(z) \) is itself a bounded function since it is bounded at both ends of the domain. Therefore, \( h(z) \) is a solution of integral equation (4) and satisfies all the given conditions.

Thus, we obtain Theorem 1.1.

\[ \square \]

### 8 Reference


