There exist infinitely many couples of primes \((p,p+2n)\), with \(2n \geq 2\) is a fixed distance between \(p\) and \(p+2n\).

**Abstract.**

For any real number \(x > 0\), let \([x]\) be the largest integer not exceeding \(x\) and \(N_{\lfloor \sqrt{x} \rfloor} = \prod_{p \leq \lfloor \sqrt{x} \rfloor, p \in \mathcal{P}} p\) is the product of all primes not exceeding \(\lfloor \sqrt{x} \rfloor\) with \(\mathcal{P}\) is the set of primes.

Let \(2n \geq 2\) denotes the distance between two primes.

For any \(n \geq 1\), there is a constant \(A(n)\) such that

\[
\Pi_{2n}(x) \geq \frac{(x-2n-\lfloor \sqrt{x} \rfloor)}{2} \prod_{p \leq \sqrt{x}, p \neq 2} \left(1 - \frac{2}{p}\right) - A(n)
\]

This result will help us to prove that, there is infinite couples of primes \((p,p+2n)\), with \(2n\) is a fixed distance between \(p\) and \(p+2n\).

We will also prove the next results:

1. there exist infinite twin primes.
2. there exist infinite cousin primes.
3. The cousin primes are equivalent to twin primes in infinity.

**Introduction.**

Let \(n > 0\) a positive integer, and \(2n\) denotes the distance between the couple of primes \((p,p+2n)\).

(Just to be obvious we don’t talk in this paper, about gabs between primes)

Let \(\Pi_{2n}(x) = \text{card}\{(p,p+2n)/ p+2n \leq x, (p,p+2n) \in \mathcal{P}^2 \}\) denotes the number of couple of primes \((p,p+2n)\) not exceeding \(x\). The aim of this paper is to prove that for any \(n \geq 1\), \(\Pi_{2n}(x) \to \infty\) when \(x \to \infty\), which means that there exists infinitely many couple of primes \((p,p+2n)\).

Furthermore we would have to special cases.

Case 1, \(2n=2\).

\(\Pi_2(x) = \text{card}\{(p,p+2)/ p+2 \leq x, (p,p+2) \in \mathcal{P}^2 \}\) will denotes the number of couples
of twin primes not exceeding x , in fact will prove that \( \Pi_2(x) \sim 2 \frac{x}{\log(x)^2} \) which means that the conjecture of twin primes is true .

Case 2 , 2n=4 .
\( \Pi_4(x) = \text{card}\{(p,p+4)/ p+4 \leq x , (p,p+4) \in \mathcal{P}^2 \} \) will denotes the number of couples of cousin primes not exceeding x , will prove also that \( \Pi_4(x) \sim 2 \frac{x}{\log(x)^2} \)

In Theorem B , we will prove an extraordinarily powerful discovery is that \(| \Pi_4(x)-\Pi_2(x)| \leq \ln(x) \) for succiently large x.

Respectively.

**Theorem A.**
let \( n \geq 1 \) and fix 2n as the distance between two primes, then

there exist infinitely many couples (p, p+2n), where p and p+2n are both prims.

**Corollary 1.**
1. \( \Pi_2(x) \sim 2 \frac{x}{\log(x)^2} \) for sufficietly large x .
2. \( \Pi_4(x) \sim 2 \frac{x}{\log(x)^2} \) for sufficietly large x .

**Theorem B.**

\(| \Pi_4(x)-\Pi_2(x)| \leq \ln(x) \) for sufficietly large x .

**Lemma 1.** For any real number \( x > 0 \), let \( \lfloor x \rfloor \) be the largest integer

not exceeding x and \( N_{\lfloor \sqrt{x} \rfloor} = \prod_{p \leq \lfloor \sqrt{x} \rfloor , p \in \mathcal{P}} p \) is the

product of all primes not exceeding \( \lfloor \sqrt{x} \rfloor \) , with \( \mathcal{P} \) is the set of primes

\( \mathcal{P} = \{2, 3, 5, 7, \ldots \} \) and let \( \gcd(a,b) \) denotes the greatest

common divisor of the elements (a,b)

then \( \lfloor \sqrt{x} \rfloor + 1 \leq n \leq x \) and \( \gcd(n, N_{\lfloor \sqrt{x} \rfloor}) = 1 \) \( \Rightarrow \) \( n \) is a prime

**Proof of Lemma 1.** let \( N_{\lfloor \sqrt{x} \rfloor} = \prod_{p \leq \lfloor \sqrt{x} \rfloor , p \in \mathcal{P}} p \)

we suppose that \( \gcd(n, N_{\lfloor \sqrt{x} \rfloor}) = 1 \).

let \( d \) be a prime divisor of \( n \) \( \Rightarrow \) \( 1 < d \leq \lfloor \sqrt{x} \rfloor \)

\( \Rightarrow \) \( d/N_{\lfloor \sqrt{x} \rfloor} \)

\( \Rightarrow \) \( \gcd(n, N_{\lfloor \sqrt{x} \rfloor}) \neq 1 \) Absurd
then n is a prime

**Lemma 2.** (see [01]) let \( \mu \) denotes the Mobius function then

\[
\sum_{d'/\gcd(n,d)} \mu(d') = \begin{cases} 
1 & \text{if } \gcd(n, d) = 1 \\
0 & \text{if not}
\end{cases}
\]

**Lemma 3.** (see [01])

let f be a multiplicative function then \( \sum_{d/n} f(d) \) is also multiplicative.

**Lemma 4.** (see [04])

\[
\prod_{p \leq x, p \neq 2} \left( 1 - \frac{2}{p} \right) \sim \frac{1}{\log(x)^2}, \text{ for all sufficiently large } x
\]

**Lemma 5.** (see [05])

Let a, b and c, any given integers and let \( ax + by = c \) be a diophantine equation, then \( ax + by = c \) has a solution iff \( \gcd(a, b)/c \).

And if \((x_0, y_0)\) is a particular solution of \( ax + by = c \)

then there exist an integer k such that \( (x_0 + \frac{kb}{\gcd(a, b)}, y_0 - \frac{ka}{\gcd(a, b)} ) \) is the set of solutions.

**Lemma 6.**

let \( N_{[\sqrt{x}]} = \prod_{p \leq \sqrt{x}, p \in \mathbb{P}} p \) and \( d_1/N_{[\sqrt{x}]} \).

\[
\text{then } \quad d_2/N_{[\sqrt{x}]}, d_1 \land d_2 = 1 \iff d_2/N_{[\sqrt{x}]}d_1,
\]

**Proof of Lemma 6.**

let \( N_{[\sqrt{x}]} = \prod_{p \leq \sqrt{x}, p \in \mathbb{P}} p \) and \( d_1/N_{[\sqrt{x}]} \).

1- we suppose that \( d_2/N_{[\sqrt{x}]}d_1 \).

we have \( d_2/N_{[\sqrt{x}]}d_1 \Rightarrow d_2d_1/N_{[\sqrt{x}]} \Rightarrow d_2/N_{[\sqrt{x}]} \)
and since $N_{\lfloor \sqrt{x} \rfloor}$ is squarefree and $d_1d_2/N_{\lfloor \sqrt{x} \rfloor}$ then $d_1 \wedge d_2 = 1$

this means that $d_2/N_{\lfloor \sqrt{x} \rfloor} \Rightarrow d_2/N_{\lfloor \sqrt{x} \rfloor} , d_1 \wedge d_2 = 1$

2- we suppose that $d_2/N_{\lfloor \sqrt{x} \rfloor} , d_1 \wedge d_2 = 1$ .

we have $d_2/N_{\lfloor \sqrt{x} \rfloor} , d_1/N_{\lfloor \sqrt{x} \rfloor} , d_1 \wedge d_2 = 1 \Rightarrow d_2d_1/N_{\lfloor \sqrt{x} \rfloor}$

$\Rightarrow d_2/N_{\lfloor \sqrt{x} \rfloor}$

then from 1 and 2 we obtain the equivalence .

$d_2/N_{\lfloor \sqrt{x} \rfloor} , d_1 \wedge d_2 = 1 \iff d_2/N_{\lfloor \sqrt{x} \rfloor}$

**Proof of theorem A.**

let $x > 9$ and fix $n \geq 1$ .

let $\Pi_{2n}(x)=\text{card}\{(p,p+2n)/ p+2n \leq x , (p,p+2n) \in P^2 \}$ denotes the number of couple of primes $(p,p+2n)$ not exceeding $x$ .

and $\Pi_{2n}'(x)=\text{card}\{(p,p+2n)/ \lfloor \sqrt{x} \rfloor < p \leq x - 2n , (p,p+2n) \in P^2 \}$

denotes the number of couple of primes $(p,p+2n)$ that are between $\lfloor \sqrt{x} \rfloor$ and $x$ .

it is evident that $\Pi_{2n}(x) \geq \Pi_{2n}'(x)$ ( $\Pi_{2n}(x)=\Pi_{2n}(\sqrt{x})+\Pi_{2n}'(x)$ ).

Then if we can prove that $\Pi_{2n}'(x) \rightarrow +\infty$ when $x \rightarrow +\infty$

this will be sufficient to prove **Theorem A** .

in fact this will be our aim for the next sections .

**Remark 1.** let $z=\lfloor \sqrt{x} \rfloor$

by Lemma 1, if $\lfloor \sqrt{x} \rfloor < p \leq x - 2n$ , $\gcd( p, N_z) = 1$ and $\gcd( p+2n, N_z) = 1$ then $(p, p+2n)$ is a couple of primes , with distance $2n$ .

we will exploit **Remark 1** to calculate $\Pi_{2n}'(x)$.

$\Pi_{2n}'(x) = \text{card}\{(p,p+2n)/ \lfloor \sqrt{x} \rfloor < p \leq x - 2n, \gcd( p, N_z) = 1, \gcd( p+2n, N_z) = 1 \}$
\[
\sum_{\gcd(p,N_z)=1, \sqrt{x} < p \leq x-2n} \sum_{\gcd(p+2n,N_z)=1, \sqrt{x} + 2n < p+2n \leq x} 1
= \sum_{\gcd(p,N_z)=1, \sqrt{x} < p \leq x-2n} 1.
\]

If we apply Lemma 2, we obtain.

\[
\Pi_{2n}'(x) = \sum_{d_1/p,N_z,d_2/(p+2n,N_z), \sqrt{x} \leq p \leq x-2n} \mu(d_1) \mu(d_2)
= \sum_{d_1/N_z,d_1/p,d_2/N_z,d_2/p+2n, \sqrt{x} \leq p \leq x-2n} \mu(d_1) \mu(d_2)
= \sum_{d_1/N_z,d_2/N_z} \mu(d_1) \mu(d_2) \sum_{d_1/p,d_2/p+2n, \sqrt{x} \leq p \leq x-2n} 1
\]

But we have the equivalence

\[
\frac{d_1}{p}, \frac{d_2}{p+2n} \iff \exists j, k \in \mathbb{N}^* \text{ such that } p = jd_1 \text{ et } p+2n = kd_2
\]

Then.

\[
\Pi_{2n}'(x) = \sum_{d_1/N_z,d_2/N_z} \mu(d_1) \mu(d_2) \sum_{p=jd_1,p+2n=kd_2, \sqrt{x} \leq p \leq x-2n} 1
= \sum_{d_1/N_z,d_2/N_z} \mu(d_1) \mu(d_2) \sum_{jd_1+2n=kd_2, \sqrt{x} \leq p=jd_1 \leq x-2n} 1
\]

**Remark 2.**

we remark that the sum \( \sum_{jd_1+2n=kd_2, \sqrt{x} \leq p=jd_1 \leq x-2n} 1 \)

depends only on the diophantine equation \( jd_1 + 2n = kd_2 \)

with \( j \) and \( k \) are the variables .

**Problem 1.**

if we want to give a explicit formula to \( \Pi_{2n}'(x) \) we would have to calculate the sum \( \sum_{jd_1+2n=kd_2, \sqrt{x} \leq p=jd_1 \leq x-2n} 1 \).

In fact we will find that if \( \gcd(d_1,d_2)/2n \) then.

\[
\sum_{jd_1+2n=kd_2, \sqrt{x} \leq p=jd_1 \leq x-2n} 1 = \frac{x-2n-\sqrt{x}}{d_1} \times \frac{\gcd(d_1,d_2)}{d_2} + 1 + O(1)
\]

**Proof of Problem 1.**

1 if the equation \( jd_1 + 2n = kd_2 \) has a solution

we set \( \delta(j,k) = \){
then $L = \sum_{jd_1 + 2n = kd_2, \lfloor \sqrt{x} \rfloor \leq p = jd_1 \leq x - 2n} 1$

If we have $\gcd(d_1, d_2) / 2n$, by Lemma 5, we will have also

$$j = j_0 + \frac{td_2}{\gcd(d_1, d_2)} \quad \text{and} \quad k = k_0 + \frac{td_1}{\gcd(d_1, d_2)}$$

and $(j_0, k_0)$ is a particular solution of $jd_1 + 2n = kd_2$

then $L = \sum_{\lfloor \sqrt{x} \rfloor \leq j = j_0 + \frac{td_2}{\gcd(d_1, d_2)} \leq x - 2n} 1,

= \sum_{\frac{\lfloor \sqrt{x} \rfloor}{d_1} \leq j_0 \leq \frac{x - 2n}{d_1}, tj \in \mathbb{N}^*} 1$

$= \sum_{\frac{\lfloor \sqrt{x} \rfloor}{d_1} \leq j_0 \leq \frac{x - 2n}{d_1}, \gcd(d_1, d_2) / 2n, tj \in \mathbb{N}^*} 1$

Result 1.

If $\gcd(d_1, d_2) / 2n$ then the sum $\sum_{jd_1 + 2n = kd_2, \lfloor \sqrt{x} \rfloor \leq p = jd_1 \leq x - 2n} 1$ is equal to

$$L = \left( \frac{x - 2n}{d_1} - j_0 \right) \times \frac{\gcd(d_1, d_2)}{d_2} - \left( \frac{\lfloor \sqrt{x} \rfloor}{d_1} - j_0 \right) \times \frac{\gcd(d_1, d_2)}{d_2} + 1$$

by Result 1 we have

$$L = \left( \frac{x - 2n}{d_1} - j_0 \right) \times \frac{\gcd(d_1, d_2)}{d_2} - \left( \frac{\lfloor \sqrt{x} \rfloor}{d_1} - j_0 \right) \times \frac{\gcd(d_1, d_2)}{d_2} + 1 + O(1)$$

then

$$L = \left( \frac{x - 2n}{d_1} \cdot \gcd(d_1, d_2) - \frac{\lfloor \sqrt{x} \rfloor}{d_1} \cdot \gcd(d_1, d_2) \right) + 1 + O(1)$$

$$= \frac{x - 2n}{d_1} \times \frac{\gcd(d_1, d_2)}{d_2} - \frac{\lfloor \sqrt{x} \rfloor}{d_1} \times \frac{\gcd(d_1, d_2)}{d_2} + 1 + O(1)$$
\[\begin{align*}
\frac{\lfloor x \rfloor}{d_1} & \cdot \gcd \left( \frac{d_1}{d_2}, \frac{d_2}{d_1} \right) + 1 + O(1) \\
\frac{x - 2n - \lfloor \sqrt{x} \rfloor}{d_1} & \cdot \gcd \left( \frac{d_1}{d_2}, \frac{d_2}{d_1} \right) + 1 + O(1) \\
\frac{x - 2n - \lfloor \sqrt{x} \rfloor}{d_1d_2} & \cdot \gcd(d_1, d_2) + 1 + O(1)
\end{align*}\]

then if \( \gcd(d_1, d_2)/2n \) we will have.

\[
\sum_{jd_1 + 2n = kd_2, \lfloor \sqrt{x} \rfloor \leq p = jd_1 \leq x - 2n} 1 = \frac{x - 2n - \lfloor \sqrt{x} \rfloor}{d_1d_2} \cdot \gcd(d_1, d_2) + 1 + O(1)
\]

Let us now return to calculate \( \Pi'_{2n}(x) \).

we have.

\[
\Pi'_{2n}(x) = \sum_{d_1/Nz, d_2/Nz} \mu(d_1) \mu(d_2) \sum_{jd_1 + 2n = kd_2, \lfloor \sqrt{x} \rfloor \leq p = jd_1 \leq x - 2n} 1
\]

By Problem 1, we will obtain.

\[
\begin{align*}
\Pi'_{2n}(x) &= \sum_{d_1/Nz, d_2/Nz} \mu(d_1) \mu(d_2) \left( x - 2n - \lfloor \sqrt{x} \rfloor \right) \gcd(d_1, d_2) + 1 + O(1) \\
&+ \sum_{d_1/Nz, d_2/Nz} \mu(d_1) \mu(d_2) \left( 1 + O(1) \right)
\end{align*}
\]

**Problem 2.** let \( \tau(n) = \sum_d 1 \) denotes the number of divisors of \( n \).

then the error term \( \sum_{d_1/Nz, d_2/Nz} \mu(d_1) \mu(d_2) \left( 1 + O(1) \right) \)

is equal to \( O(2\tau(\text{rad}(2n))) \)

**Proof of Problem 2.**

let \( K = \sum_{d_1/Nz, d_2/Nz} \mu(d_1) \mu(d_2) \left( 1 + O(1) \right) \)

\[
= \sum_{d_1/Nz} \mu(d_1) \sum_{d_2/Nz, \gcd(d_1, d_2)} 2n \mu(d_2) \left( 1 + O(1) \right)
\]

we set \( F = \{ d = d_1 \land d_2/, d_1/Nz, d_2/Nz, d/2n \} \)

then we will obtain.

\[
K = \sum_{d_1/Nz} \mu(d_1) \sum_{d_2/Nz, \gcd(d_1, d_2)} 2n \mu(d_2) \left( 1 + O(1) \right)
\]
\[= \sum_{d \in \mathcal{F}} \sum_{d_1 / N_z} \mu(d_1) \sum_{d_2 / N_z, \gcd(d_1, d_2) = d} \mu(d_2)(1 + O(1))\]

\[= \sum_{d \in \mathcal{F}} \sum_{d_1 / N_z} \mu(d_1) \sum_{d_2 / N_z} \mu(d_2)(1 + O(1))\]

By Lemma 6 we have.

\[= \sum_{d \in \mathcal{F}} \sum_{d_1 / N_z} \mu(d_1) \sum_{d_2 / d_1} \mu(d_2)\]

\[= \sum_{d \in \mathcal{F}} \sum_{d_1 / N_z} \mu(d_1) \sum_{d_2 / d_1} \mu(d_2) + \sum_{d \in \mathcal{F}} \sum_{d_1 / N_z} \mu(d_1) \sum_{d_2 / d_1} \mu(d_2) O(1)\]

now we have two sums to calculate. Let us calculate them.

let \( T = \sum_{d \in \mathcal{F}} \sum_{d_1 / N_z} \mu(d_1) \sum_{d_2 / d_1} \mu(d_2) \)

and \( R = \sum_{d \in \mathcal{F}} \sum_{d_1 / N_z} \mu(d_1) \sum_{d_2 / d_1} \mu(d_2) O(1) \)

by Lemma 2, we have \( \sum_{d_2 / d_1} \mu(d_2) = \begin{cases} 1 & \text{if } \frac{N_z}{d_1} = 1 \\ 0 & \text{if not} \end{cases} \)

then \( \sum_{d_2 / d_1} \mu(d_2) = \begin{cases} 1 & \text{if } \frac{N_z}{d} = \frac{d_1}{d} \\ 0 & \text{if not} \end{cases} \)

then we obtain \( T = \sum_{d \in \mathcal{F}} \mu\left(\frac{N_z}{d}\right) \)

\[= \sum_{d \in \mathcal{F}} \left(-1\right)^{\omega\left(\frac{N_z}{d}\right)} \]

we have also \( T \leq \sum_{d \in \mathcal{F}} \left|-\left(-1\right)^{\omega\left(\frac{N_z}{d}\right)}\right| \)

\[\leq \sum_{d \in \mathcal{F}} 1 \]

\[\leq \text{rad}(F)\]

**Remark 3.**

since \( N_z \) is squarefree then \( F = \{d = d_1 \land d_2 /, \ d_1 / N_z, d_2 / N_z, d / \text{rad}(2n)\} \)

\[= \{d / \text{rad}(2n) / d \leq \lfloor \sqrt{d} \rfloor\}\]
We have \( \tau(\text{rad}(2n)) = \text{card}\{ \rad(2n) \}
\)
\[= \text{card}\{ \frac{d}{\text{rad}(2n)} \mid d \leq \sqrt{x} \} \cup \{ \frac{d}{\text{rad}(2n)} \mid d \geq \sqrt{x} \} \]
\[= \text{card}\{ F \cup \{ \frac{d}{\text{rad}(2n)} \mid d \geq \sqrt{x} \} \} \]

Then \( F \subseteq \{ \frac{d}{\text{rad}(2n)} \} \).

Which that \( \text{rad}(F) \leq \tau(\text{rad}(2n)) \).

And if we exploit Remark 3. we will have \( T \leq \text{rad}(F) \leq \tau(\text{rad}(2n)) \)
then \( T = O(\tau(\text{rad}(2n))) \)

we remain to calculate the sum
\[ R = \sum_{d \in F} \sum_{d_1 / \text{rad}(2n) \mid d_2 / \text{rad}(2n) \mid d} \mu(d_1) \sum_{d_2 / d} \frac{N_z}{d} \mu(d_2)O(1). \]

\[ R = \sum_{d \in F} \sum_{d_1 / \text{rad}(2n) \mid d_2 / \text{rad}(2n) \mid d} \mu(d_1) \sum_{d_2 / d} \frac{N_z}{d} \mu(d_2)O(1) \]

we have \[ \sum_{d \in F} \sum_{d_1 / \text{rad}(2n) \mid d_2 / \text{rad}(2n) \mid d} \mu(d_1) \sum_{d_2 / d} \frac{N_z}{d} \mu(d_2) \leq \sum_{d \in F} \sum_{d_1 / \text{rad}(2n) \mid d_2 / \text{rad}(2n) \mid d} \mu(d_1) \sum_{d_2 / d} \frac{N_z}{d} \mu(d_2) \]
\[ \leq \sum_{d \in F} \sum_{d_1 / \text{rad}(2n) \mid d_2 / \text{rad}(2n) \mid d} \mu(d_1) \sum_{d_2 / d} \frac{N_z}{d} \mu(d_2) \]
\[ \leq \sum_{d \in F} \sum_{d_1 / \text{rad}(2n) \mid d_2 / \text{rad}(2n) \mid d} \mu(d_1) \mu(d_2) \]
\[ \leq \sum_{d \in F} \sum_{d_1 / \text{rad}(2n) \mid d_2 / \text{rad}(2n) \mid d} \mu(d_1) \mu(d_2) \]
\[ \leq \sum_{d \in F} \sum_{d_1 / \text{rad}(2n) \mid d_2 / \text{rad}(2n) \mid d} \mu(d_1) \mu(d_2) \]
\[ \leq \sum_{d \in F} \sum_{d_1 / \text{rad}(2n) \mid d_2 / \text{rad}(2n) \mid d} \mu(d_1) \mu(d_2) \]

we have \[ \sum_{d_2 / d} \frac{N_z}{d} \mu(d_2) = \sum_{d_2 / d} \frac{N_z}{d} \mu\left( \frac{d_2}{d} \right) \]
since \( \gcd(\frac{d_2}{d}, d) = 1 \), then \( \mu\left( \frac{d_2}{d} \right) = \mu\left( \frac{d_2}{d} \right) \mu(d) \)

then \[ \sum_{d \in F} \sum_{d_1 / \text{rad}(2n) \mid d_2 / \text{rad}(2n) \mid d} \mu(d_1) \mu(d_2) \leq \sum_{d \in F} \sum_{d_1 / \text{rad}(2n) \mid d_2 / \text{rad}(2n) \mid d} \mu\left( \frac{d_2}{d} \right) \mu(d) \]
\[ \leq \sum_{d \in F} \sum_{d_1/N_z} |\sum_{d_2/d} \mu \left( \frac{d_2}{d} \right)| \]

and if we apply again the Lemma 2, we obtain

\[ \sum_{d_1/d} \frac{N_z}{d} \mu \left( \frac{d_2}{d} \right) = \left| \mu \left( \frac{N_z}{d} \right) \right| = 1 \quad \text{then} \]

\[ \sum_{d \in F} \sum_{d_1/N_z} \mu(d_1) \sum_{d_2/d} \frac{N_z}{d} \mu(d_2) \leq \sum_{d \in F} \ 1 \]

we already know that \( \sum_{d \in F} 1 \leq \tau(\text{rad}(2n)) \) from the above calculations then

\[ \sum_{d \in F} \sum_{d_1/N_z} \mu(d_1) \sum_{d_2/d} \frac{N_z}{d} \mu(d_2) \leq \tau(\text{rad}(2n)) \]

then we obtain \( R = O(\tau(\text{rad}(2n))) \)

**Result 2 of Problem 2.**

we have \( T = O(\tau(\text{rad}(2n))) \) and \( R = O(\tau(\text{rad}(2n))) \)

then \( K = T + R = O(\tau(\text{rad}(2n))) + O(\tau(\text{rad}(2n))) = O(2\tau(\text{rad}(2n))) \)

now we have obtained the most interesting result in this article

\[ \sum_{d_1/N_z} \mu(d_1) \sum_{d_2/N_z, \gcd(d_1,d_2)/2n} \mu(d_2)(1 + O(1)) = O(2\tau(\text{rad}(2n))) \]

because \( O(2\tau(\text{rad}(2n))) \) will be the error term of \( \Pi'_{2n}(x) \)

in the next sections you will see that this error term is much smaller than the main term of \( \Pi_{2n}(x) \).

by Problem 2, we obtain

\[ \Pi'_{2n}(x) = \sum_{d_1/N_z, d_2/N_z, \gcd(d_1,d_2)/2n} \mu(d_1) \mu(d_2) \left( \frac{x - 2n - \lfloor \sqrt{x} \rfloor}{d_1 d_2} \gcd(d_1, d_2) \right) + O(2\tau(\text{rad}(2n))) \]

\[ = \sum_{d \in F} \sum_{d_1/N_z, d_2/N_z, \gcd(d_1,d_2)/2n} \mu(d_1) \mu(d_2) \left( \frac{x - 2n - \lfloor \sqrt{x} \rfloor}{d_1 d_2} \gcd(d_1, d_2) \right) + O(2\tau(\text{rad}(2n))) \]
\( = \sum_{d \in F} \sum_{d_1/Nz} \mu(d_1) \sum_{d_2/Nz, \gcd(d_1, d_2)} = d \mu(d_2) \left( \frac{x - 2n - \sqrt{x} - d}{d_1 d_2} + O(2 \sigma(\text{rad}(2n))) \right) \)

\( = \sum_{d \in F} \sum_{d_1/Nz} \mu(d_1) \sum_{d_2/Nz, \gcd(d_1, d_2)} = d \mu(d_2) \left( \frac{x - 2n - \sqrt{x} - d}{d_1 d_2} + O(2 \sigma(\text{rad}(2n))) \right) \)

\( = \sum_{d \in F} \sum_{d_1/Nz} \mu(d_1) \sum_{d_2/Nz, \gcd(d_1, d_2)} = d \mu(d_2) \left( \frac{x - 2n - \sqrt{x} - d}{d_1 d_2} + O(2 \sigma(\text{rad}(2n))) \right) \)

\( = (x - 2n - \sqrt{x}) \sum_{d \in F} \sum_{d_1/Nz} \mu(d_1) \mu(d_2) \sum_{d_1/Nz, \gcd(d_1, d_2)} = d \mu(d_2) \left( \frac{x - 2n - \sqrt{x} - d}{d_1 d_2} + O(2 \sigma(\text{rad}(2n))) \right) \)

\( = \sum_{d \in F} \sum_{d_1/Nz} \mu(d_1) \sum_{d_2/Nz, \gcd(d_1, d_2)} = d \mu(d_2) \left( \frac{x - 2n - \sqrt{x} - d}{d_1 d_2} + O(2 \sigma(\text{rad}(2n))) \right) \)

\( = \sum_{d \in F} \sum_{d_1/Nz} \mu(d_1) \sum_{d_2/Nz, \gcd(d_1, d_2)} = d \mu(d_2) \left( \frac{x - 2n - \sqrt{x} - d}{d_1 d_2} + O(2 \sigma(\text{rad}(2n))) \right) \)

since \( \gcd(d, d_1) = 1 \) and \( \gcd(d, d_2) = 1 \), then \( \mu(d_1) = \mu(d_1) \mu(d) \)
and \( \mu(d_2) = \mu(d_1) \mu(d) \), then we obtain.

\[ \Pi'_2n(x) = (x - 2n - \sqrt{x}) \sum_{d \in F} \sum_{d_1/Nz} \mu(d_1) \mu(d_2) \sum_{d_1/Nz, \gcd(d_1, d_2)} = d \mu(d_2) = 1 \]

we obtain.

\[ \Pi'_2n(x) = (x - 2n - \sqrt{x}) \sum_{d \in F} \sum_{d_1/Nz} \mu(d_1) \sum_{d_2/Nz, \gcd(d_1, d_2)} = d \mu(d_2) + O(2 \sigma(\text{rad}(2n))) \]

\( = (x - 2n - \sqrt{x}) \sum_{d \in F} \sum_{d_1/Nz} \mu(d_1) \sum_{d_2/Nz, \gcd(d_1, d_2)} = d \mu(d_2) + O(2 \sigma(\text{rad}(2n))) \)

\( = (x - 2n - \sqrt{x}) \sum_{d \in F} \sum_{d_1/Nz} \mu(d_1) \sum_{d_2/Nz, \gcd(d_1, d_2)} = d \mu(d_2) + O(2 \sigma(\text{rad}(2n))) \)

\( = (x - 2n - \sqrt{x}) \sum_{d \in F} \sum_{d_1/Nz} \mu(d_1) \sum_{d_2/Nz, \gcd(d_1, d_2)} = d \mu(d_2) + O(2 \sigma(\text{rad}(2n))) \)

If we apply Lemma 6 we obtain.

\[ \Pi'_2n(x) = (x - 2n - \sqrt{x}) \sum_{d \in F} \sum_{d_1/Nz} \mu(d_1) \sum_{d_2/Nz, \gcd(d_1, d_2)} = d \mu(d_2) + O(2 \sigma(\text{rad}(2n))) \]
since $\mu(\frac{d_2}{d})$ is multiplicative, then by Lemma 3, $\sum_{\frac{d_2}{d}} \frac{\mu(\frac{d_2}{d})}{N_2/d_2}$ is also multiplicative.

then $\sum_{\frac{d_2}{d}} \frac{\mu(\frac{d_2}{d})}{N_2/d_2} = \prod_{p/\frac{d_1}{d}} \frac{\mu(\frac{d_1}{d})}{p/\frac{d_1}{d}} (1-\frac{1}{p})$

we will obtain.

$\Pi_{2n}'(x) = (x-2n-\lfloor \sqrt{x} \rfloor) \sum_{d \in \mathcal{F}} \frac{1}{d} \sum_{d_1/d} \frac{\mu(\frac{d_1}{d})}{d_1/d} \prod_{p/\frac{d_1}{d}} \frac{\mu(\frac{d_1}{d})}{p/\frac{d_1}{d}} (1-\frac{1}{p}) + O(2\tau(\text{rad}(2n)))$

we apply again Lemma 3 on $\sum_{d_1/d} \frac{\mu(\frac{d_1}{d})}{d_1/d} \prod_{p/\frac{d_1}{d}} (1-\frac{1}{p})$ we obtain.

$\sum_{d_1/d} \frac{\mu(\frac{d_1}{d})}{d_1/d} \prod_{p/\frac{d_1}{d}} (1-\frac{1}{p}) = \prod_{p/\frac{d_1}{d}} (1-\frac{1}{p})$,

then

$\Pi_{2n}'(x) = (x-2n-\lfloor \sqrt{x} \rfloor) \sum_{d \in \mathcal{F}} \frac{1}{d} \prod_{p/\frac{d_1}{d}} (1-\frac{1}{p}) \prod_{p/\frac{d_1}{d}} (1-\frac{1}{p}(1-\frac{1}{p})) + O(2\tau(\text{rad}(2n)))$

$\sum_{d_1/d} \frac{\mu(\frac{d_1}{d})}{d_1/d} \prod_{p/\frac{d_1}{d}} (1-\frac{1}{p}) = \prod_{p/\frac{d_1}{d}} (1-\frac{1}{p}(1-\frac{1}{p}))$

$\Pi_{2n}'(x) = (x-2n-\lfloor \sqrt{x} \rfloor) \sum_{d \in \mathcal{F}} \frac{1}{d} \prod_{p/\frac{d_1}{d}} (1-\frac{1}{p}) \prod_{p/\frac{d_1}{d}} (1-\frac{1}{p}(1-\frac{1}{p})) + O(2\tau(\text{rad}(2n)))$

$\sum_{d_1/d} \frac{\mu(\frac{d_1}{d})}{d_1/d} \prod_{p/\frac{d_1}{d}} (1-\frac{1}{p}) = \prod_{p/\frac{d_1}{d}} (1-\frac{1}{p}(1-\frac{1}{p}))$

then

$\Pi_{2n}'(x) = (x-2n-\lfloor \sqrt{x} \rfloor) \sum_{d \in \mathcal{F}} \frac{1}{d} \prod_{p/\frac{d_1}{d}} (1-\frac{1}{p}) \prod_{p/\frac{d_1}{d}} (1-\frac{1}{p}(1-\frac{1}{p})) + O(2\tau(\text{rad}(2n)))$

We already have $\mathcal{F} = \{d=d_1 \land d_2/ , d_1/N_z, d_2/N_z, d/2n\}$
from the definition of $F$, we can deduce that.

$$F = \{1, 2, \ldots, d_r \}$$

then $\Pi'_{2n}(x) = (x - 2n - \lfloor \sqrt{x} \rfloor)(1 - \frac{2}{p}) + \sum_{d \in F, \text{d} \neq 1, \text{d} \neq 2} \frac{1}{d} \prod_{p \mid N_x} \left(1 - \frac{2}{p}\right) + O(2\tau(\text{rad}(2n)))$

since 2 is prime and $2/N_x$, then $\left( \prod_{p \mid N_x} \left(1 - \frac{2}{p}\right) = 0$.

then $\Pi'_{2n}(x) = (x - 2n - \lfloor \sqrt{x} \rfloor)(1 - \frac{2}{p}) + \sum_{d \in F, \text{d} \neq 1, \text{d} \neq 2} \frac{1}{d} \prod_{p \mid N_x} \left(1 - \frac{2}{p}\right) + O(2\tau(\text{rad}(2n)))$

**Result 3.**

$$\Pi'_{2n}(x) = (x - 2n - \lfloor \sqrt{x} \rfloor)(1 - \frac{2}{p}) + \sum_{d \in F, \text{d} \neq 1, \text{d} \neq 2} \frac{1}{d} \prod_{p \mid N_x} \left(1 - \frac{2}{p}\right) + O(2\tau(\text{rad}(2n)))$$

This result is very important we will need them to prove Corollary 1 and Theorem B.

from **Result 3** we deduce that.

$$\Pi'_{2n}(x) \geq \frac{(x - 2n - \lfloor \sqrt{x} \rfloor)}{2} \prod_{p \mid N_x} \left(1 - \frac{2}{p}\right) + O(2\tau(\text{rad}(2n)))$$

$$\geq \frac{(x - 2n - \lfloor \sqrt{x} \rfloor)}{2} \prod_{p \leq \sqrt{x}, \text{p} \neq 2} \left(1 - \frac{2}{p}\right) - 2\tau(\text{rad}(2n))$$

$$\geq \frac{(x - 2n - \lfloor \sqrt{x} \rfloor)}{2} \prod_{p \leq \sqrt{x}, \text{p} \neq 2} \left(1 - \frac{2}{p}\right) - 2\tau(\text{rad}(2n))$$

by **Lemma 4**, we have $\prod_{p \leq \sqrt{x}, \text{p} \neq 2} \left(1 - \frac{2}{p}\right) \sim \frac{1}{\log(\sqrt{x})^2}$

$$\sim \frac{4}{\log(x)^2} \text{ for sufficiently large x}.$$  

We don’t have to forget that 2n is fixed then $\tau(\text{rad}(2n))$ is also will be fix.

then $\frac{(x - 2n - \lfloor \sqrt{x} \rfloor)}{2} \prod_{p \leq \sqrt{x}, \text{p} \neq 2} \left(1 - \frac{2}{p}\right) \sim \frac{2(x - 2n - \lfloor \sqrt{x} \rfloor)}{\log(x)^2}$

$$\sim \frac{2x}{\log(x)^2} \text{ for sufficiently large x}$$

this means that $\Pi'_{2n}(x) \to +\infty$, when $x \to +\infty$ (because $\frac{2x}{\log(x)^2} \to +\infty$)
we already have $\Pi_{2n}(x) \geq \Pi'_{2n}(x)$ then $\Pi_{2n}(x) \to +\infty$ when $x \to +\infty$
this prove Theorem A.

**Proof of corollary 1.**
By Result 3 we have for any $n \geq 1$.

\[
\Pi'_{2n}(x) = (x - 2n - \lfloor \sqrt{x} \rfloor) \left( \frac{1}{2} \prod_{p / N^2} \left( 1 - \frac{2}{p} \right) + \sum_{d \neq F, d \neq 2} \frac{1}{d} \prod_{p / N^2} \left( 1 - \frac{2}{p} \right) \right) + O(2\tau(\text{rad}(2n)))
\]

**Case 1.** if $2n=2$

\[
\Pi'_2(x) = (x - 2 - \lfloor \sqrt{x} \rfloor) \left( \frac{1}{2} \prod_{p / N^2} \left( 1 - \frac{2}{p} \right) + O(2\tau(\text{rad}(2))) \right)
\]

\[
= \frac{(x - 2 - \lfloor \sqrt{x} \rfloor)}{2} \prod_{p \leq \sqrt{x}, p \neq 2} \left( 1 - \frac{2}{p} \right) + O(4)
\]

It is known that if $(p, p+2)$ is a couple of twin primes then there is no prime between them. Then, $\Pi'_2(x)$ denotes the number of twin primes not exceeding $x$.

We have $\Pi_2(x) = \Pi'_2(x) + \Pi_2(\sqrt{x})$.

then, $\Pi_2(x) = \Pi'_2(x) + \Pi'_2(\sqrt{x}) + \Pi_2(\sqrt{x})$

\[
= \Pi'_2(x) + \Pi'_2(\sqrt{x}) + \Pi'_2(\sqrt{x})
\]

let $\sqrt{x} = 4 \Rightarrow \ln(x) = 4b$

\[
\Rightarrow b = \frac{\ln(x)}{4}
\]

then we obtain, $\Pi_2(x) = \sum_{i=1}^{\frac{\ln(x)}{4}} \Pi'_2(\sqrt{x})$

\[
= \Pi'_2(x) \sum_{i=1}^{\frac{\ln(x)}{4}} \frac{\Pi'_2(\sqrt{x})}{\Pi'_2(\sqrt{x})}
\]

but from (**) we have $\Pi'_2(\sqrt{x}) = \frac{(\sqrt{x} - 2 - \lfloor \sqrt{x} \rfloor) \prod_{p \leq \sqrt{x}, p \neq 2} \left( 1 - \frac{2}{p} \right) + O(4)}{(x - 2 - \sqrt{x}) \prod_{p \leq \sqrt{x}, p \neq 2} \left( 1 - \frac{2}{p} \right) + O(4)}$

then $\Pi_2(x) = \Pi'_2(x) \sum_{i=1}^{\frac{\ln(x)}{4}} \frac{(\sqrt{x} - 2 - \lfloor \sqrt{x} \rfloor) \prod_{p \leq \sqrt{x}, p \neq 2} \left( 1 - \frac{2}{p} \right) + O(4)}{(x - 2 - \sqrt{x}) \prod_{p \leq \sqrt{x}, p \neq 2} \left( 1 - \frac{2}{p} \right) + O(4)}$

\[
= \Pi'_2(x) \left( 1 + \sum_{i=2}^{\frac{\ln(x)}{4}} \frac{(\sqrt{x} - 2 - \lfloor \sqrt{x} \rfloor) \prod_{p \leq \sqrt{x}, p \neq 2} \left( 1 - \frac{2}{p} \right) + O(4)}{(x - 2 - \sqrt{x}) \prod_{p \leq \sqrt{x}, p \neq 2} \left( 1 - \frac{2}{p} \right) + O(4)} \right)
\]
\[
\Pi'_2(x) = (\sqrt{x} - 2 - \sqrt{3x} - x) \prod_{p \leq \sqrt{x}, p \neq 2} \left( 1 - \frac{2}{p} \right) + O(1)
\]

but by Lemma 4,

\[
\frac{(\sqrt{x} - 2 + \sqrt{x}) \prod_{p \leq \sqrt{x}, p \neq 2} \left( 1 - \frac{2}{p} \right) + O(1)}{(x - 2 - \sqrt{x}) \prod_{p \leq \sqrt{x}, p \neq 2} \left( 1 - \frac{2}{p} \right) + O(1)} \sim \frac{\sqrt{x}}{\ln(x)^2}
\]

for sufficiently large \( x \) and \( \frac{\ln(x)}{4} \geq 1 \geq 2 \).

then,

\[
\frac{(\sqrt{x} - 2 - i + \sqrt{x}) \prod_{p \leq \sqrt{x}, p \neq 2} \left( 1 - \frac{2}{p} \right) + O(1)}{(x - 2 - \sqrt{x}) \prod_{p \leq \sqrt{x}, p \neq 2} \left( 1 - \frac{2}{p} \right) + O(1)} \sim \frac{\sqrt{x} \ln(x)^2}{\ln(i + \sqrt{x})^2} \\
\sim \frac{\sqrt{x} \cdot (i + 1)^2}{4} \to 0 \text{ when } x \to \infty
\]

we can deduce that

\[
\sum_{i=2}^{\ln(x)} \frac{(\sqrt{x} - 2 - i + \sqrt{x}) \prod_{p \leq \sqrt{x}, p \neq 2} \left( 1 - \frac{2}{p} \right) + O(1)}{(x - 2 - \sqrt{x}) \prod_{p \leq \sqrt{x}, p \neq 2} \left( 1 - \frac{2}{p} \right) + O(1)} \to 0 \text{ when } x \to \infty
\]

then,

\[
\Pi_2(x) \sim \Pi'_2(x)
\]

\[
\sim \frac{(x - 2 - \sqrt{x})}{2} \prod_{p \leq \sqrt{x}, p \neq 2} \left( 1 - \frac{2}{p} \right) + O(1)
\]

by Lemma 4 we have,

\[
\Pi_2(x) \sim \frac{x}{2\ln(x)^2} \text{ for sufficiently large } x.
\]

then,

\[
\Pi_2(x) \sim \frac{2x}{\ln(x)^2}
\]

Case 2. if \( 2n=4 \) \( F=\{1, 2\} \)

then,

\[
\Pi'_4(x) = (x - 4 - \sqrt{x}) \prod_{p \leq \sqrt{x}, p \neq 2} \left( 1 - \frac{2}{p} \right) + O(2\pi(\text{rad}(4)))
\]

\[
= \frac{(x - 4 - \sqrt{x})}{2} \prod_{p \leq \sqrt{x}, p \neq 2} \left( 1 - \frac{2}{p} \right) + O(1)
\]

but it is known that if \((p, p+4)\) is a couple of cousin primes then there is no prime between them, except \((3, 7)\). then \(\Pi'_4(x)\) denotes the number of
cousin primes not exceeding x.

By the seem method developped in Case 1 we can prove that $\Pi_4'(x) \sim \frac{2x}{\ln(x)^2}$.

**Proof of Theorem B.**

by the Result 3 we have .

$$\Pi_{2n}'(x) = (x - 2n - \lfloor \sqrt{x} \rfloor) \left( \frac{1}{2} \prod_{p > N_2} \left( 1 - \frac{2}{p} \right) + \sum_{d \mid F, d \neq 1, d \neq 2} \frac{1}{d} \prod_{p > N_2} \left( 1 - \frac{2}{p} \right) \right) + O(2\tau(\text{rad}(2n)))$$

**Case 1.** if $2n=2$ we have $F=\{1, 2\}$

then $\Pi_2'(x) = \frac{x - 2 - \lfloor \sqrt{x} \rfloor}{2} \prod_{p > N_2} \left( 1 - \frac{2}{p} \right) + O(2\tau(\text{rad}(2)))$

$$= \frac{x - 2 - \lfloor \sqrt{x} \rfloor}{2} \prod_{p \leq \sqrt{x}, p \neq 2} \left( 1 - \frac{2}{p} \right) + O(4)$$

**Case 2.** if $2n=4$ we have $F=\{1, 2\}$

then $\Pi_4'(x) = \frac{x - 4 - \lfloor \sqrt{x} \rfloor}{2} \prod_{p > N_2} \left( 1 - \frac{2}{p} \right) + O(2\tau(\text{rad}(4)))$

$$= \frac{x - 4 - \lfloor \sqrt{x} \rfloor}{2} \prod_{p \leq \sqrt{x}, p \neq 2} \left( 1 - \frac{2}{p} \right) + O(4)$$

From case 1 and case 2 we obtain .

$$\Pi_2'(x) - \Pi_4'(x) = \frac{x - 2 - \lfloor \sqrt{x} \rfloor}{2} \prod_{p > N_2} \left( 1 - \frac{2}{p} \right) + O(4) - \frac{x - 4 - \lfloor \sqrt{x} \rfloor}{2} \prod_{p > N_2} \left( 1 - \frac{2}{p} \right) - O(4)$$

$$= \prod_{p > N_2} \left( 1 - \frac{2}{p} \right) + O(4)$$

by Lemma 4, we have $\prod_{p \leq \sqrt{x}, p \neq 2} \left( 1 - \frac{2}{p} \right) \sim \frac{1}{\log(\sqrt{x})^2}$, for sufficiently large x.

$$\sim \frac{4}{\log(x)^2}$$, for sufficiently large x.

then $\Pi_2'(x) - \Pi_4'(x) = O(4)$, for sufficiently large x.

we have $\Pi_4(x) - \Pi_2(x) = \Pi_4'(x) - \Pi_2'(x) + \Pi_4(\sqrt{x}) - \Pi_2(\sqrt{x})$
\[=\Pi'_4(x)-\Pi'_2(x)+\Pi'_4(\sqrt{x})-\Pi'_2(\sqrt{x})+\Pi_4(\sqrt{x})-\Pi_2(\sqrt{x})\]
\[=\Pi'_4(x)-\Pi'_2(x)+\Pi'_4(\sqrt{x})-\Pi'_2(\sqrt{x})+\Pi'_4(\sqrt{x})-\Pi'_2(\sqrt{x})..\]
\[= O(4)+O(4)+O(4)...\]

let \( \sqrt{x} = 4 \Rightarrow \ln(x) = 4b \)
\[\Rightarrow b = \frac{\ln(x)}{4}\]

then \( \Pi'_4(x)-\Pi'_2(x) = \sum_{i=1}^{\frac{\ln(x)}{4}} (\Pi'_4(\sqrt{x})-\Pi'_2(\sqrt{x}))\)
\[= \sum_{i=1}^{\frac{\ln(x)}{4}} O(4)\]
\[= O(\ln(x))\]

then for sufficiently large \( x \) we have \( |\Pi'_4(x)-\Pi'_2(x)| \leq \ln(x) \)

this means that \( |\frac{\Pi_4(x)}{\Pi_2(x)} - 1| \leq \frac{\ln(x)}{\Pi_2(x)} \)

but \( \Pi_2(x) \sim \frac{2x}{\log(x)^2} \) then \( \frac{2\ln(x)}{\Pi_2(x)} \sim \frac{\ln(x)}{\frac{2x}{\log(x)^2}} \)
\[\sim \frac{\ln(x)^3}{2x} \to 0 \text{ when } x \to \infty\]

then \( \Pi'_4(x) \sim \Pi'_2(x) \) for sufficiently large \( x \).

this means that The cousin primes are equivalent to twin primes in infinity

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