A SEQUENCE OF ELEMENTARY INTEGRALS RELATED TO INTEGRALS STUDIED BY GLAISHER THAT CONTAIN TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS

MARTIN NICHOLSON

Abstract. We generalize several integrals studied by Glaisher. These ideas are then applied to obtain an analog of an integral due to Ismail and Valent.

1. Introduction

The following integral
\[ \int_0^\infty \frac{\sin x \sinh(x/a)}{\cos(2x) + \cosh(2x/a)} \frac{dx}{x} = \tan^{-1} \frac{a}{2}, \]  
(1.1)
can be deduced as a particular case of entry 4.123.6 from [2]. The case \( a = 1 \) of this integral can be found in an old paper by Glaisher [1]. We are going to generalize the above integral as

**Theorem 1.** Let \( n \) be an odd integer. Then
\[ \int_0^1 \frac{\sin(n \sin^{-1} t) \sinh(n \sinh^{-1}(t/a))}{\cos(2n \sin^{-1} t) + \cosh(2n \sinh^{-1}(t/a))} \frac{dt}{t \sqrt{1 - t^2} \sqrt{1 + t^2/a^2}} = \tan^{-1} \frac{a}{2}. \]
(1.2)

When \( n \) is large, then the main contribution to the integral \( 1.2 \) comes from a small neighbourhood around \( t = 0 \) and the integral reduces to \( 1.1 \).

Another integral by Glaisher reads (equation 24 in [1])
\[ \int_0^\infty \frac{\cos x \cosh x}{\cos(2x) + \cosh(2x)} x \, dx = 0. \]
It would be generalized as

**Theorem 2.** Let \( n \) be an even integer. Then
\[ \int_0^1 \frac{\cos(n \sin^{-1} t) \cosh(n \sinh^{-1}(t/a))}{\cos(2n \sin^{-1} t) + \cosh(2n \sinh^{-1}(t/a))} \frac{t \, dt}{\sqrt{1 - t^2}} = 0. \]
(1.3)

Unfortunately there doesn’t seem to be any nice parametric extensions similar to that in Theorem 1.

A particularly interesting integral is this one due to Ismail and Valent [4]
\[ \int_{-\infty}^\infty \frac{x^{2k + 1} \, dx}{\cos x + \cosh x} = (-1)^k \pi^{2k + 2} \sum_{j=0}^{\infty} (-1)^{j} \frac{(2j + 1)^{4k + 1}}{\cos(2j + 1)} \]  
(1.4)
The next theorem gives an elementary analog of this formula:

**Theorem 3.** Let \( k \) and \( n \) be a positive integers such that \( k < \left[ \frac{n}{2} \right] \). Then
\[ \int_0^1 \frac{t^{2k}}{\cos(2n \sin^{-1} \sqrt{t}) + \cosh(2n \sinh^{-1} \sqrt{t}) \sqrt{1 - t^2}} \, dt = \pi (-1)^k \frac{n^{n/2}}{2^{2k+1} n} \sum_{j=1}^{\infty} (-1)^{j-1} \frac{\tan \left[ \frac{(2j-1)}{2n} \right]}{\cos \left[ \frac{(2j-1)}{2n} \right]} \left( \frac{\sin^2 \left[ \frac{(2j-1)}{2n} \right]}{\cos \left[ \frac{(2j-1)}{2n} \right]} \right)^{2k}. \]
(1.5)
In the last section 5, it will be explained why this form of the integral in Theorem 3 has been chosen.

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2. Proof of Theorem 1

We break the proof into a series of lemmas.

**Lemma 4.** Let \( n \) be an odd integer. Then we have the partial fractions expansion

\[
\frac{\sin(n \sin^{-1} t) \sinh(n \sin^{-1}(t/a))}{\cos(2n \sin^{-1} t) + \cosh(2n \sin^{-1}(t/a))} \frac{2n}{t^2} = \sum_{j=1}^{n} \frac{i(-1)^{j-1}}{\sin \frac{\pi(2j-1)}{2n}} \cdot \frac{-\cos \frac{\pi(2j-1)}{2n} + i \left( a + i \cos \frac{\pi(2j-1)}{2n} \right)}{t^2 \left( a^2 - 1 + 2ia \cos \frac{\pi(2j-1)}{2n} - a^2 \sin^2 \frac{\pi(2j-1)}{2n} \right)}. \tag{2.1}
\]

**Proof.** When \( n \) is an odd integer, the expressions \( 2n \sin(n \sin^{-1} t) \sinh(n \sin^{-1}(t/a))/t^2 \) and \( \cos(2n \sin^{-1} t) + \cosh(2n \sin^{-1}(t/a)) \) are polynomials in \( t^2 \) of degrees \( n-1 \) and \( n \), respectively:

\[
\frac{\sin(n \sin^{-1} t) \sinh(n \sin^{-1}(t/a))}{\cos(2n \sin^{-1} t) + \cosh(2n \sin^{-1}(t/a))} \frac{2n}{t^2} = \frac{P_{n-1}(t^2)}{Q_n(t^2)}.
\]

Let’s find the \( n \) roots of the denominator polynomial \( Q_n(x) \). \( Q_n(x) \) can be written as

\[
Q_n(x) = \cos(n \sin^{-1} \sqrt{x} + i \sinh^{-1}(\sqrt{x}/a)) \cos(n \sin^{-1} \sqrt{x} - i \sinh^{-1}(\sqrt{x}/a)),
\]

and thus its roots can be found from the equations

\[
\sin^{-1} \sqrt{x} \pm i \sinh^{-1}(\sqrt{x}/a) = \frac{\pi(2j-1)}{2n}, \quad j = 1, 2, ..., n,
\]

or equivalently from the equations

\[
\sqrt{x} \left( 1 + \frac{x}{a^2} \pm i \frac{\sqrt{x}}{a} \right) \sqrt{1-x} = \sin \frac{\pi(2j-1)}{2n}, \quad j = 1, 2, ..., n,
\]

One can get rid of the radical expressions to come to a quadratic equation wrt \( x \):

\[
x^2 \left( (1-a^2)^2 + 4a^2 \cos^2 \frac{\pi(2j-1)}{2n} \right) + 2xa^2(1-a^2) \sin^2 \frac{\pi(2j-1)}{2n} + \sin^4 \frac{\pi(2j-1)}{2n} = 0, \quad j = 1, 2, ..., n.
\]

One can easily deduce from this that the \( n \) roots of the denominator polynomial are thus

\[
x_j = \left( a^2 - 1 + 2ia \cos \frac{\pi(2j-1)}{2n} \right)^{-1} a^2 \sin \frac{\pi(2j-1)}{2n}, \quad j = 1, 2, ..., n. \tag{2.2}
\]

Now we can find the partial fractions expansion

\[
\frac{P_{n-1}(t^2)}{Q_n(t^2)} = \sum_{j=1}^{n} \frac{P_{n-1}(x_j)}{Q_n(x_j)} \frac{1}{t^2 - x_j}. \tag{2.3}
\]

A simple calculation shows that

\[
\frac{Q_n'(x_j)}{P_{n-1}(x_j)} = \sqrt{\frac{x_j}{a^2 + x_j}} \cosh(n \sin^{-1}(\sqrt{x_j}/a)) \sqrt{\frac{x_j}{1-x_j \sinh(n \sin^{-1}(\sqrt{x_j}/a))}} \cosh(n \sin^{-1}(\sqrt{x_j}/a)) - 1 - x_j \sinh(n \sin^{-1}(\sqrt{x_j}/a)) = 0 \implies \cosh(n \sin^{-1}(\sqrt{x_j}/a)) = \mu_j \sin(n \sin^{-1}(\sqrt{x_j}/a)), \quad \cos(n \sin^{-1}(\sqrt{x_j}/a)) = i \nu_j \sin(n \sin^{-1}(\sqrt{x_j}/a))
\]

where \( \mu_j = \pm, \nu_j = \pm \). To determine the signs \( \mu_j, \nu_j \), one can consider the limiting case \( a >> 1 \). We have

\[
\sqrt{x_j} = \sin \frac{\pi(2j-1)}{2n} - \frac{i}{a} \sin \frac{\pi(2j-1)}{2n} \cos \frac{\pi(2j-1)}{2n} + O(a^{-2}).
\]

This means

\[
\sin^{-1} \sqrt{x_j} = \frac{\pi(2j-1)}{2n} - \frac{i}{a} \sin \frac{\pi(2j-1)}{2n} + O(a^{-2}).
\]
From this it follows that \( \mu_j = \nu_j = (-1)^{j-1} \) and thus

\[
\frac{Q'_n(x_j)}{P_{n-1}(x_j)} = (-1)^{j-1} \left( \sqrt{\frac{x_j}{a^2 + x_j}} - i \sqrt{\frac{1 - x_j}{1 - x_j}} \right)
= i(-1)^j \frac{\sin \frac{\pi(2j-1)}{2n} \left( a^2 - 1 + 2ia \cos \frac{\pi(2j-1)}{2n} \right)}{\left( a \cos \frac{\pi(2j-1)}{2n} + i \right) \left( a + i \cos \frac{\pi(2j-1)}{2n} \right)}.
\]

Substituting this into 2.3 we get the desired result.

Lemma 5.

\[
\int_0^1 \frac{1}{t^2 \left(a^2 - 1 + 2ia \cos \frac{\pi(2j-1)}{2n} \right) - a^2 \sin^2 \frac{\pi(2j-1)}{2n} \sqrt{1 - t^2 \sqrt{1 + t^2/a^2}}} \frac{t \, dt}{\sqrt{1 - t^2 \sqrt{1 + t^2/a^2}}} = \frac{\tan^{-1} a + i \tanh^{-1} \cos \frac{\pi(2j-1)}{2n}}{i \left( a \cos \frac{\pi(2j-1)}{2n} + i \right) \left( a + i \cos \frac{\pi(2j-1)}{2n} \right)}.
\]

Proof. Composition of two substitutions \( t^2 = 1 - (1 + 1/a^2) \sin^2 \phi, \) \((0 < \phi < \tan^{-1} a)\) and \( \tan \phi = s, \) \((0 < s < a)\) reduces this integral to an integral of a rational function.

Lemma 6. For \( n \) odd, one has

\[
\sum_{j=1}^n \frac{(-1)^{j-1}}{\sin \frac{\pi(2j-1)}{2n}} = n.
\]

Proof. Put \( t = 1, a = i \) in Lemma 4.

From the three lemmas above it follows immediately that

\[
\int_0^1 \frac{\sin(n \sin^{-1} t) \sinh(n \sin^{-1}(t/a))}{\cos(2n \sin^{-1} t) + \cosh(2n \sin^{-1}(t/a))} \frac{dt}{t \sqrt{1 - t^2 \sqrt{1 + t^2/a^2}}} = \frac{\tan^{-1} a}{2} + \frac{i}{2n} \sum_{j=1}^n \frac{(-1)^{j-1}}{\sin \frac{\pi(2j-1)}{2n}} \tanh^{-1} \cos \frac{\pi(2j-1)}{2n}.
\]

To finish the proof, note that the sum in this formula is \( 0 \) because (since \( n \) is odd) \( j \)-th and \( (n + 1 - j) \)-th terms cancel each other out.

3. Proof of Theorem 2

Lemma 7. Let \( n \) be an even integer. Then

\[
\frac{\cos(n \sin^{-1} t) \cosh \left( n \sin^{-1}(t/a) \right)}{\cos(2n \sin^{-1} t) + \cosh \left( 2n \sin^{-1}(t/a) \right)} = \frac{(-1)^{n/2}}{2} \frac{a^n}{1 + a^{2n}}
+ \sum_{j=1}^n \frac{(-1)^j a^2 \sin \frac{\pi(2j-1)}{2n}}{2n \left( a^2 - 1 + 2ia \cos \frac{\pi(2j-1)}{2n} \right)} \cdot \frac{\left( a \cos \frac{\pi(2j-1)}{2n} + i \right) \left( a + i \cos \frac{\pi(2j-1)}{2n} \right)}{t^2 \left( a^2 - 1 + 2ia \cos \frac{\pi(2j-1)}{2n} \right) - a^2 \sin^2 \frac{\pi(2j-1)}{2n}}.
\]

Proof. When \( n \) is even, the functions \( \cos(n \sin^{-1} t) \) and \( \cosh \left( n \sin^{-1}(t/a) \right) \) are polynomials in \( t^2 \) of degree \( n/2 \). This means we can write

\[
\frac{\cos(n \sin^{-1} t) \cosh \left( n \sin^{-1}(t/a) \right)}{\cos(2n \sin^{-1} t) + \cosh \left( 2n \sin^{-1}(t/a) \right)} = C + \frac{R_{n-1}(t^2)}{Q_n(t^2)}
\]

where \( R_{n-1} \) is a polynomial of order \( n - 1 \) and \( Q_n \) was defined in the proof of the Lemma 4. \( Q_n(x) \) has \( n \) roots given by 2.2.

To find the constant \( C \) consider the limit \( t \to +\infty \) assuming that \( a > 0 \). In this case

\[
\sin^{-1} t = \frac{\pi}{2} - i \ln(2t) + O(t^{-1}), \quad \sinh^{-1}(t/a) = \ln(2t/a) + O(t^{-1}),
\]

and we get

\[
C = \frac{(-1)^{n/2}}{2} \frac{a^n}{1 + a^{2n}}.
\]
A calculation similar to that in Lemma 4 shows that
\[ \frac{Q_n'(x_j)}{R_n-1(x_j)} = \frac{2n}{x_j} \left( \sqrt{\frac{x_j}{a^2+x_j}} \sin(\sin^{-1}(\sqrt{x_j}/a)) - \frac{x_j}{1-x_j \cosh(\sin^{-1}(\sqrt{x_j}/a))} \right) \]
\[= (-1)^{j-1} \frac{2n}{x_j} \left( \sqrt{\frac{x_j}{a^2+x_j}} - i \sqrt{\frac{x_j}{1-x_j}} \right) \]
\[= \frac{2n(-1)^j}{a^2} \left( a^2 - 1 + 2ia \cos \frac{\pi(2j-1)}{2n} \right)^2 \left( \cos \left( \frac{\pi(2j-1)}{2n} \right) i + \frac{\pi(2j-1)}{2n} \right).
\]
This completes the proof of the lemma.  
\[\square\]

Using Lemmas 5 and 7 we find
\[
\int_0^1 \frac{\cos(n \sin^{-1} t) \cosh(n \sqrt{\sin^{-1} t/t})}{\cos(2n \sin^{-1} t) + \cosh(2n \sin^{-1} t/a)} \frac{t dt}{\sqrt{1-t^2} \sqrt{1+t^2/a^2}} \]
\[= (-1)^{n/2} \frac{an+1}{2} \tan^{-1}(1/a) + a^2 \sum_{j=1}^n (-1)^j \frac{\tan^{-1} \cos \frac{\pi(2j-1)}{2n} - i \tan^{-1} a}{2n \left( a^2 - 1 + 2ia \cos \frac{\pi(2j-1)}{2n} \right)} \sin \frac{\pi(2j-1)}{2n},
\]
and in particular when \(a = 1\)
\[
\int_0^1 \frac{\cos(n \sin^{-1} t) \cosh(n \sqrt{\sin^{-1} t/t})}{\cos(2n \sin^{-1} t) + \cosh(2n \sin^{-1} t/a)} \frac{t dt}{\sqrt{1-t^2}} = \frac{\pi}{16} (-1)^{n/2} - \sum_{j=1}^n (-1)^j \frac{\pi}{16} \frac{\tan^{-1} \cos \frac{\pi(2j-1)}{2n}}{\csc \frac{\pi(2j-1)}{2n}} \frac{\cos \frac{\pi(2j-1)}{2n}}{t \sin \frac{\pi(2j-1)}{2n}}.
\]
To calculate the sum in this expression we use Lemma 7 with \(t = 1\) and \(a \to \infty\) to get
\[
\sum_{j=1}^n (-1)^j \frac{\tan^{-1} \cos \frac{\pi(2j-1)}{2n}}{\csc \frac{\pi(2j-1)}{2n}} = (-1)^{n/2}.n.
\]
This completes the proof of the theorem.

4. PROOF OF THEOREM 3

Here we restrict the consideration to the case \(a = 1\).

Lemma 8. The following partial fraction expansion holds for positive integers \(k\) and \(n\) such that \(k < \left[ \frac{n}{2} \right]\)
\[
\frac{t^{2k}}{\cos(2n \sin^{-1} \sqrt{t}) + \cosh(2n \sin^{-1} \sqrt{t})} = \frac{(-1)^{k} \sin \frac{\pi(2j-1)}{2n}}{2^{2k} \pi} \cos \frac{\pi(2j-1)}{2n} \cos \left( \frac{\pi(2j-1)}{2n} \right) + \frac{(-1)^{j-1} \tan \frac{\pi(2j-1)}{2n}}{2^{2k} \pi} \cos \left( \frac{\pi(2j-1)}{2n} \right) + \sin \frac{\pi(2j-1)}{2n} \csc \left( \frac{\pi(2j-1)}{2n} \right) \left( \cos \left( \frac{\pi(2j-1)}{2n} \right) \right)^{2k}.
\]

Proof. From consideration of the limit \(t \to +\infty\) once can see (similarly to that in Lemma 7) that the leading coefficient of the polynomial \(Q_n(t) = \cos(2n \sin^{-1} \sqrt{t}) + \cosh(2n \sin^{-1} \sqrt{t})\) is \(2^{2n-1}(1 + (-1)^n)\) and thus that \(Q_n(t)\) is an even polynomial of degree 2 \(2^{n/2}\). Its roots are (see 2.2)
\[
x_j = -\frac{i \sin 2 \frac{\pi(2j-1)}{2n}}{2 \cos \frac{\pi(2j-1)}{2n}}, \; y_j = \frac{i \sin 2 \frac{\pi(2j-1)}{2n}}{2 \cos \frac{\pi(2j-1)}{2n}}, \; j = 1, 2, ..., \left[ \frac{n}{2} \right].
Calculations using the formulas above yield
\[ \sin^{-1}(x_j) = \xi_j - i\eta_j, \quad \sinh^{-1}(x_j) = \varphi_j - i\psi_j, \]
with \( \xi_j, \eta_j, \varphi_j, \psi_j > 0 \). Further, from elementary identities \( 1 - 2t = \cos(2\sin^{-1}\sqrt{t}) \) and \( 1 + 2t = \cosh(2\sinh^{-1}\sqrt{t}) \) one can see that
\[ \cos(2\xi_j) \cosh(2\eta_j) = \cosh(2\varphi_j) \cos(2\psi_j) = 1, \]
\[ \sin(2\xi_j) \sinh(2\eta_j) = \sinh(2\varphi_j) \sin(2\psi_j) = \frac{\sin^2 \pi(2j-1)}{2n} \cos \frac{\pi(2j-1)}{2n}. \]
These equations can be easily solved to yield
\[ \xi_j = \psi_j = \frac{\pi(2j-1)}{4n}, \quad \eta_j = \varphi_j = \frac{1}{2} \sinh^{-1} \frac{\pi(2j-1)}{2n}. \]
Thus
\[ \sin^{-1}(x_j) = \frac{\pi(2j-1)}{4n} - \frac{i}{2} \sinh^{-1} \frac{\pi(2j-1)}{2n}, \]
\[ \sinh^{-1}(x_j) = \frac{\pi(2j-1)}{4ni} + \frac{1}{2} \sinh^{-1} \frac{\pi(2j-1)}{2n}. \]
Similarly
\[ \sin^{-1}(y_j) = \frac{\pi(2j-1)}{4n} + \frac{i}{2} \sinh^{-1} \frac{\pi(2j-1)}{2n}, \]
\[ \sinh^{-1}(y_j) = -\frac{\pi(2j-1)}{4ni} + \frac{1}{2} \sinh^{-1} \frac{\pi(2j-1)}{2n}. \]
For \( k < \left[ \frac{n}{2} \right] \) we have the partial fractions expansion
\[ \frac{t^{2k}}{Q_n(t)} = \sum_{j=1}^{n/2} \left( \frac{x_j^{2k}}{Q_n(x_j)} \frac{1}{t - x_j} + \frac{y_j^{2k}}{Q_n(y_j)} \frac{1}{t - y_j} \right). \]
Calculations using the formulas above yield
\[ Q_n'(x_j) = -Q_n'(y_j) = \frac{4ni(-1)^j \cos^2 \frac{\pi(2j-1)}{2n}}{\sin \frac{\pi(2j-1)}{2n} \left( 1 + \cos^2 \frac{\pi(2j-1)}{2n} \right)}. \]
Now substitute this into the formula above.

Using Lemma 8 and the following consequence of Lemma 5
\[ \int_0^1 \frac{1}{4t^2 + \sin^4 \frac{\pi(2j-1)}{2n}} \frac{dt}{\sqrt{1 - t^2}} = \pi \cot^2 \frac{\pi(2j-1)}{2n}, \]
one can easily complete the proof of Theorem 3.

5. Discussion

Let’s introduce the notation
\[ \alpha_z = n \sinh^{-1} \sin \frac{\pi z}{2n}, \]
where we assume the principal branches of the multivalued functions of complex variable. With this definition one can rewrite the integral in Theorem 1 with \( a = 1 \) as
\[ \int_0^1 \frac{\sin(n \sin^{-1} t) \sinh(n \sinh^{-1} t)}{\cos(2n \sin^{-1} t) + \cosh(2n \sinh^{-1} t)} \frac{dt}{\sqrt{1 - t^2}} = \pi \int_0^n \frac{\sin \frac{\pi z}{2} \sin \frac{\alpha_z}{2}}{\cos \pi x + \cosh \alpha_x \sinh \frac{\alpha_x}{n}} \frac{dx}{\sin \frac{\pi y}{2} \sin \frac{\alpha_y}{2} \left( \cos \pi y + \cos(\alpha_y t) \sin \frac{\alpha_y}{n} \right)}. \]
As we will now show, the last integral has an interesting symmetry.
Let us define \( y_\star \) by the equation
\[ \alpha_{iy_\star} = \pi n, \]
and consider the integral over an interval on the imaginary axes
\[ J = \int_{iy_\star}^{iy_\star} \frac{\sin \frac{\pi z}{2} \sin \frac{\alpha_z}{2}}{\cos \pi x + \cosh \alpha_x \sinh \frac{\alpha_x}{n}} \frac{dz}{\sin \frac{\pi y}{2} \sin \frac{\alpha_y}{2} \left( \cos \pi y + \cos(\alpha_y t) \sin \frac{\alpha_y}{n} \right)} = \int_0^{y_\star} \frac{\sin \frac{\pi y}{2} \sin \frac{\alpha_y}{2}}{\cos \pi y + \cos(\alpha_y t) \sin \frac{\alpha_y}{n}} \frac{dy}{\sin \frac{\pi y}{2} \sin \frac{\alpha_y}{2} \left( \cos \pi y + \cos(\alpha_y t) \sin \frac{\alpha_y}{n} \right)}. \]
When \( y \) is real, then \( \alpha iy \) is purely imaginary, so we make change of variables \( \alpha iy = \pi is \). We get 
\[
\sin \frac{\pi s}{2n} = \sinh \frac{\pi y}{2n},
\]
which implies that 
\[
\pi y = \alpha s.
\]
Also it is easy to show that 
\[
\frac{dy}{\sin \frac{\pi x}{n}} = \frac{ds}{\sinh \frac{\alpha x}{n}}.
\]
Thus the integral under consideration becomes

\[
J = \int_0^n \frac{\sin \frac{\pi s}{2} \sinh \frac{\alpha s}{2n}}{\cos \pi s + \cosh \alpha s \sinh \frac{\alpha x}{n}} ds.
\]

To recap what we have just showed:

\[
\int_{iy}^1 \frac{\sin \frac{\pi x}{2n} \sinh \frac{\alpha x}{2n}}{\cos \pi x + \cosh \alpha x \sinh \frac{\alpha x}{n}} \frac{dz}{\cosh \pi z + \cosh \alpha z \sinh \frac{\alpha x}{n}} = \int_0^n \frac{\sin \frac{\pi s}{2} \sinh \frac{\alpha s}{2n}}{\cos \pi s + \cosh \alpha s \sinh \frac{\alpha x}{n}} ds.
\]

The integral in theorem 1 has been chosen to have the same kind of symmetry:

\[
\int_0^1 \frac{t^{2k}}{\cos(2n \sin^{-1}\sqrt{t}) + \cosh (2n \sin^{-1}\sqrt{t})} \frac{dt}{\sqrt{1 - t^2}} = \frac{\pi}{n} \int_0^n \frac{(\sin \frac{\pi x}{2n})^{4k+2}}{\cos \pi x + \cosh \alpha x \sinh \frac{\alpha x}{n}} dx,
\]

\[
\int_{iy}^1 \frac{(\sin \frac{\pi x}{2n})^{4k+2}}{\cos \pi x + \cosh \alpha x \sinh \frac{\alpha x}{n}} \frac{dz}{\cosh \pi z + \cosh \alpha x \sinh \frac{\alpha x}{n}} = \int_0^n \frac{(\sin \frac{\pi s}{2n})^{4k+2}}{\cos \pi s + \cosh \alpha s \sinh \frac{\alpha x}{n}} dx.
\]

We mention without proof an alternative representation for the sum in Theorem 3 with \( k = 0 \):

\[
\sum_{j=1}^{n/2} (-1)^{j-1} \tan \frac{\pi (2j-1)}{2n} \cosh \left( \sinh^{-1} \frac{\pi (2j-1)}{2n} \right) = \sum_{y=1}^n \coth \left( \sinh^{-1} \frac{\pi (2y-1)}{2n} \right) - \frac{n}{2}.
\]

References


