On Jordan-Clifford Algebras, Three Fermion Generations with Higgs Fields and a $SU(3) \times SU(2)_L \times SU(2)_R \times U(1)$ model

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Abstract

It is shown how the algebra $\mathbf{J}_3[\mathbf{C} \otimes \mathbf{O}] \otimes Cl(4, \mathbf{C})$ based on the tensor product of the complex Exceptional Jordan $\mathbf{J}_3[\mathbf{C} \otimes \mathbf{O}]$, and the complex Clifford algebra $Cl(4, \mathbf{C})$, can describe all of the spinorial degrees of freedom of three generations of fermions in four-spacetime dimensions, and, in addition, to include the degrees of freedom of three sets of pairs of complex scalar Higgs-doublets $\{\mathbf{H}^i_L, \mathbf{H}^i_R\}; i = 1, 2, 3$, and their conjugates. A close inspection of the fermion structure of each generation reveals that it fits naturally with the **16** complex-dimensional representation of the internal left/right symmetric gauge group $G_{LR} = SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)$. It is reviewed how the latter group emerges from the intersection of $SO(10)$ and $SU(3) \times SU(3) \times SU(3)$ in E_6. In the concluding remarks we briefly discuss the role that the extra Higgs fields may have as dark matter candidates; the construction of Chern-Simons-like matrix cubic actions; hexaquarks and Clifford bundles over the complex-octonionic projective plane $(\mathbf{C} \otimes \mathbf{O})\mathbf{P}^2$ whose isometry group is E_6.

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1 Introduction

Dixon [1] many years ago proposed an algebraic design of the fundamental particles in Physics showing the key role that the composition algebra (the Dixon algebra) $\mathbf{T} = \mathbf{R} \otimes \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$ had in the architecture of the Standard Model. More recently, it has been shown by Furey how the $\mathbf{C} \otimes \mathbf{O}$ algebra acting on itself allows to find the Standard Model particle representations [2]. A geometric approach to the physics of the standard model and gravity based on Noncommutative Geometry can be found in [14]. Other original work on the role of Octonions, Exceptional Jordan algebras in Physics can be found in [4], [5], [9], [12], [18], [23], [**?**], [19].

A geometric basis for the Standard Model gauge group based on the real $Cl(7,R)$ algebra was found earlier on by [15]. Since dim $Cl(7,R) = 128$ coincides with the real dimension of the complex $Cl(6,C)$ algebra, it was shown later on by [2], [16] that the eight minimal left ideals of the algebra of 8×8 complex matrices $C(8) \cong Cl(6,C)$ contains the 64 elementary fermion states of one generation of fixed spin, including their antiparticles.

In a recent paper [13] the basis states of the minimal left ideals of the complex Clifford algebra $Cl(8,C)$ were shown to contain three generations of Standard Model fermion states, with full Lorentzian, right and left chiral, weak isospin, spin, and electro-color degrees of freedom. The left adjoint action algebra of $Cl(8,C) \cong C(16)$ on its minimal left ideals contains the Dirac algebra, weak isopin and spin transformations. The right adjoint action algebra on the other hand encodes the electro-color $U(3)$ symmetries.

These results extend earlier work in the literature that shows that the eight minimal left ideals of $Cl(6,C) \cong C(8)$ contain the quark and lepton states of one generation of fixed spin. The key behind the construction of [13] was the *triality* automorphism of the $Cl(8)$ algebra which allows the extension from a single generation of fermions to exactly three generations. This triality automorphism of $Spin(8)$ permutes the two spinor and fundamental vector representations, all three of which are eight-dimensional.

The authors [13] displayed in detail how the sixteen minimal left ideals (spinors) of the 2^8 complex-dimensional Clifford algebra $Cl(8,C)$ can be represented by sixteen *column* vectors in the isomorphic matrix algebra $C(16)$ of 16×16 matrices over the complex numbers C. They showed how the action of left and right matrix multiplication differs. Left multiplication of a minimal left ideal column-matrix (a square matrix with one non-zero column) interchanges rows (along the column), and hence produces transformations within the minimal left ideals themselves. In contrast, right multiplication of a minimal left ideal by an arbitrary matrix in $C(16)$ interchanges columns, and hence transforms between different minimal left ideals.

The $Cl(8,C)$ spinor index A in $\Psi_A, 1, 2, \cdots, 16$ can be decomposed into a $Cl(4,C) \otimes Cl(2,C) \otimes Cl(2,C)$ form as follows $\Psi_{\alpha,a,b}$ with $\alpha = 1,2,3,4$; $a = 1,2$; $b = 1,2$. The spinorial index $\alpha = 1,2,3,4$ is associated to the complex Dirac algebra $Cl(4,C)$. The index $a = 1,2$ belonging to the first $Cl(2,C)$ algebra corresponds to the weak isospin $SU(2)$; and the index $b = 1, 2$ belonging to the second $Cl(2, C)$ algebra labels a family-doublet. There are 3 pairings of family-doublets among the 3 generations given by the electron/muon $\Psi^{(e,\mu)}_{\alpha,a}$, electron/tau $\Psi^{(e,\tau)}_{\alpha,a}$, and muon/tau $\Psi^{(\mu,\tau)}_{\alpha,a}$ family doublets, respectively. Under the *triality* automorphism of the $Cl(8, C)$ algebra, these 3 family doublets are rotated into each other in a cyclical way. For further details we refer to [13].

Having outlined some key results involving Clifford algebras let us turn now to the role of Jordan algebras. The authors [17] have shown that the Exceptional Jordan algebra $J_3[\mathbf{O}]$ of Hermitian 3×3 octonionic matrices can describe the *internal* space of the fundamental fermions of the Standard Model with 3 generations. An *additional* conjugate Jordan algebra $\bar{J}_3[\mathbf{O}]$ must be introduced in order to describe their *antiparticles*. The pair of Jordan algebras, $J_3[\mathbf{O}]$ and its conjugate $\bar{J}_3[\mathbf{O}]$, globally behave like the $\mathbf{3}, \bar{\mathbf{3}}$ dimensional representations of the complex $su(3)$ algebra.

The Jordan algebra $J_2[\mathbf{O}]$ of Hermitian 2×2 octonionic matrices is relevant for the description of the internal space of the fundamental fermions of one generation. Once again, *triality* was instrumental to incorporate the internal space of the 3 generations which avoids the introduction of new fundamental fermions and where there is no problem with respect to the electroweak symmetry [17].

The 3 subalgebras $\mathcal{J}_i, i = 1, 2, 3$ of $J_3[\mathbf{O}]$ isomorphic to $J_2[\mathbf{O}]$ were associated to the 3 complete generations of fundamental fermions. $\mathcal{J}_1$ consists of the matrices of $J_3[\mathbf{O}]$ having vanishing elements in the first row and the first column. $\mathcal{J}_2$ consists of the matrices having vanishing elements in the second row and the second column while $\mathcal{J}_3$ consists of the matrices having vanishing elements in the third row and the third column.

$\mathcal{J}_1$ is associated to the first generation containing the leptons e and ν_e. $\mathcal{J}_2$ is associated to the second generation containing the leptons μ and ν_μ, and $\mathcal{J}_3$ is associated to the third generation containing the leptons τ and ν_τ.

The automorphism groups of $J_3[\mathbf{O}]$ and $\mathcal{J}_2[\mathbf{O}]$ are F_4 and $Spin(9)$ respectively. The intersection in F_4 of $Spin(9)$ with $SU(3) \times SU(3)/Z_3$ is precisely the Standard Model group $G_{SM} = SU(3) \times SU(2) \times U(1)/Z_6$ [17]. The first $SU(3)$ factor is common for all $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3$ and is the color group $SU(3)_c$. The second $SU(3)$ factor projects for each of the 3 generations to its electroweak symmetry $U(2)$. It is why its natural interpretation is that of the extended electroweak symmetry of the Standard Model with 3 generations and was denoted by $SU(3)_{ew}$ in [17].

In this work we will combine Clifford algebras with Exceptional Jordan algebras and arrive at some interesting results. The main one is that it provides an algebraic-geometric framework where to accommodate three fermion generations, Higgs fields, and the group $SU(3) \times SU(2)_L \times SU(2)_R \times U(1)$. In doing so, we extend our earlier work [3] based on Jordan-Clifford algebras and Exceptional Periodicity [25], [26], [27]. With this brief introduction we turn next to our main construction.

2 The $J_3[\mathbf{C}\otimes\mathbf{O}]\otimes\mathbf{Cl}(4,\mathbf{C})$ Algebra, Three Fermion Generations and Higgs Fields

We show next that the model described in this letter can account for all the degrees of freedom of three generations of fermions in addition to the Standard Model gauge symmetries. Our novel approach in this work is based on the algebra $J_3[\mathbf{C}\otimes\mathbf{O}]\otimes\mathbf{Cl(4,C)}$, where we include explicitly the Dirac spinorial degrees of freedom of the fermions, and their antiparticles in $4D$. Let us denote the 3×3 **block** matrix by $\mathbf{J}_{3,ij}$

$$\mathbf{J}_{3,ij} \;\equiv\; \begin{pmatrix} \lambda_1^A\;\Gamma_{ij}^A & \Phi_1^A\;\Gamma_{ij}^A & \bar\Phi_2^A\;\Gamma_{ij}^A \\ \bar\Phi_1^A\;\Gamma_{ij}^A & \lambda_2^A\;\Gamma_{ij}^A & \Phi_3^A\;\Gamma_{ij}^A \\ \Phi_2^A\;\Gamma_{ij}^A & \bar\Phi_3^A\;\Gamma_{ij}^A & \lambda_3^A\;\Gamma_{ij}^A \end{pmatrix}. \quad A=1,2,\cdots,16;\quad i,j=1,2,3,4 \tag{1}$$

where each block is comprised of 4×4 matrices belonging to the $Cl(4,\mathbf{C})$ algebra. The components $\Phi_1^A,\Phi_2^A,\Phi_3^A$ are complex-octonionic-valued, and $\lambda_1^A,\lambda_2^A,\lambda_3^A$ are complex-valued. A summation over the A index from 1 to 16 is implied in (1).

The bar operation in the entries of the matrix in eq-(1) denotes *octonionic* conjugation $e_a\to -e_a, a=1,2,\cdots,7$ associated with the 7 imaginary units of the octonions. We should emphasize that there is **no** complex conjugation appearing in the entries of (1). For example, there is no complex conjugation in the 16 Gamma matrices Γ_{ij}^A associated with the 16-dim Clifford algebra $Cl(4,\mathbf{C})$, nor in the diagonal coefficients $\lambda_1^A,\lambda_2^A,\lambda_3^A$.

We shall show that the three generations of fermions can be assigned to the complex-octonionic entries of $\mathbf{J}_{3,ij}$. In order to incorporate their *antiparticles* one requires to include the conjugate $\bar{\mathbf{J}}_{3,ij}$ matrix which is obtained by taking the ordinary complex conjugate of the entries of $\mathbf{J}_{3,ij}$ in eq-(1). In order to establish this correspondence one needs to start with the following spinors (one for each generation) with complex-valued entries such that

$$(\xi\;\xi^T)_{ij} = \begin{pmatrix}\xi_1\\ \xi_2\\ \xi_3\\ \xi_4\end{pmatrix}\begin{pmatrix}\xi_1 & \xi_2 & \xi_3 & \xi_4\end{pmatrix} \;\leftrightarrow\; \sum_{A=1}^{16}\lambda^A\;\Gamma_{ij}^A \tag{2}$$

and the spinors (one for each generation) with complex-octonionic entries such that

$$(\Psi\;\Psi^T)_{ij} \;=\; \begin{pmatrix}\Psi_1\\ \Psi_2\\ \Psi_3\\ \Psi_4\end{pmatrix}\begin{pmatrix}\Psi_1 & \Psi_2 & \Psi_3 & \Psi_4\end{pmatrix} \;\leftrightarrow\; \sum_{A=1}^{16}\Phi^A\;\Gamma_{ij}^A \tag{3}$$

The 4×4 matrix $\mathbf{M}$ obtained in the left hand side of eq-(2) (after performing the product of a column and a row) is of rank-one (its determinant is zero). By following the elementary row operations of multiplying the first row of $\mathbf{M}$ by $-\frac{\xi_2}{\xi_1}$ and adding it to the second row; multiplying the first row by $-\frac{\xi_3}{\xi_1}$ and adding it to the third row; and multiplying the first row by $-\frac{\xi_4}{\xi_1}$ and adding it to the fourth row, one arrives at a *new* matrix $\mathbf{M}'$ whose second, third and fourth rows are zero, and leaving a non-vanishing first row (a rank-one matrix) whose entries are, respectively,

$$\xi_1\ \xi_1, \quad \xi_1\ \xi_2, \quad \xi_1\ \xi_3, \quad \xi_1\ \xi_4 \tag{4}$$

This procedure can also be applied to the four columns instead leading to three zero columns and a non-vanishing first column; i.e. a rank-one matrix.

By equating the rank-one 4×4 matrix $\mathbf{M}'$ to the 4×4 matrix $\sum_{A=1}^{16} \lambda^A\ \Gamma^A_{ij}$, there will be three rows of zero entries leading to a net number of $3\times 4 = 12$ *null* linear conditions imposed on the 16 complex variables $\lambda^A = \{\lambda^1, \lambda^2, \cdots, \lambda^{16}\}$. Consequently, out of the 16 complex scalar variables λ^A, only $16 - 12 = 4$ of them are truly linearly independent. It is precisely these 4 linearly independent complex scalar variables which will correspond to the 4 complex Higgs scalars inside the two complex scalar Higgs *doublets* $\mathbf{H}_L, \mathbf{H}_R$ and belonging to an appealing version of the left/right minimal symmetric extension of the standard model [6] proposed by [7].

As strongly emphasized by [11] this model proposed by [7] is not only experimentally viable, but can simultaneously: (i) explain the vanishing of the Higgs coupling at 10^{10} GeV; (ii) provide an elegant solution to the strong-CP problem; (iii) give precise gauge-coupling unification; and (iv) account for dark matter and the cosmological matter/anti-matter asymmetry.

In the case of the matrix obtained in the left hand side of eq-(3), and whose entries are products of complex octonions, the above procedure to obtain a rank-one matrix by multiplying the first row on the left by $(\Psi_i\Psi_1^{-1})$ with $i = 2,3,4$ will no longer work. Firstly, there is no natural way to define a determinant for (square) quaternionic, octonionic matrices so that the values of the determinant are quaternions, octonions. Secondly, due to the noncommutativity of octonions one has

$$\Psi_i\ \Psi_j\ \neq\ \Psi_j\ \Psi_i, \quad \Psi_i\ \Psi_j\ \neq\ -\Psi_j\ \Psi_i, \quad i \neq j \tag{5}$$

Octonions do not commute (unless their imaginary parts are all zero); nor anti-commute (unless their real parts are all zero). And due to the nonassociativity of octonions one has

$$(\Psi_i\Psi_1^{-1})\ (\Psi_1\Psi_j)\ \neq\ \Psi_i\ (\Psi_1^{-1}\Psi_1)\ \Psi_j\ =\ \Psi_i\ \Psi_j, \quad i \neq j \tag{6}$$

The correct procedure requires now finding the expressions $\mathbf{A}_i, i = 2,3,4$ such that

$$\mathbf{A}_i\ (\Psi_1\ \Psi_j)\ =\ \Psi_i\ \Psi_j,\ i = 2,3,4; \quad j = 1,2,3,4 \tag{7}$$

By invoking the key Moufang identity $(\mathbf{x}\mathbf{y})\mathbf{y}^{-1} = \mathbf{x}(\mathbf{y}\mathbf{y}^{-1}) = \mathbf{x}$, one finds that the solution for $\mathbf{A}_i$ obeying eq-(7) is given by

$$\mathbf{A}_i \;=\; (\Psi_i\,\Psi_j)\,(\Psi_1\,\Psi_j)^{-1} \;=\; (\Psi_i\,\Psi_j)\,(\Psi_j^{-1}\,\Psi_1^{-1}) \;\neq\; \Psi_i\,(\Psi_j\,\Psi_j^{-1})\,\Psi_1^{-1} \;=\; \Psi_i\,\Psi_1^{-1} \tag{8}$$

There is *no* sum over j in eq-(8). Thus after multiplying the first row on the left by $\mathbf{A}_i$ given by the solution (8), and by subtracting the second, third and fourth rows, respectively, one can perform the required row operations leading to three *zero* rows and a non-vanishing first row whose entries are

$$\Psi_1\;\Psi_1,\quad \Psi_1\;\Psi_2,\quad \Psi_1\;\Psi_3,\quad \Psi_1\;\Psi_4 \tag{9}$$

In this fashion one obtains a rank-one 4×4 matrix. By equating it to the 4×4 matrix $\sum_{A=1}^{16}\Phi^A\;\Gamma^A_{ij}$ in the right hand side of (3), one will have again three rows of zeros leading to $3\times4=12$ null linear conditions imposed on the 16 complex octonionic variables $\Phi^A=\{\Phi^1,\Phi^2,\cdots,\Phi^{16}\}$. Consequently, out of the 16 complex-octonionic variables Φ^A, only $16-12=4$ of them are truly linearly independent, and such that there is a match with the number of 4 complex octonionic spinor entries $\Psi_i, i=1,2,3,4$ in four-dimensions associated with a complex-octonionic spinor Ψ.

To sum up, setting aside the trivial zero elements of the complex octonions, and by following the procedure described above to obtain a rank-one matrix with a non-vanishing first row, the above relations establish a one-to-one correspondence among the entries of $\mathbf{J}_{3,\alpha\beta}$ in eq-(1) with the entries of the spinors in eqs-(2-3). These relations are just a manifestation of the fact that spinors have a *mixed* Clifford grade : spinors are made of a Clifford scalar, vectors, bivectors, trivectors, $\cdots$.

The linear independence of the 16 gamma matrices Γ^A ensures that the 16 components of λ^A and Φ^A are given by independent linear combinations of the 16 fermionic bilinears $\xi_i\xi_j$ and $\Psi_i\Psi_j$, respectively, with $i,j=1,2,3,4$. And vice versa. If $\xi_i=0$, and $\Psi_i=0$, for all $i=1,2,3,4$, $\Rightarrow\lambda^A=0$ and $\Phi^A=0$, for all $A=1,2,\cdots,16$. And vice versa, if $\lambda^A=0$ and $\Phi^A=0$ for all $A\Rightarrow\xi_i=0$ and $\Psi_i=0$ for all values of i.

Each spinor $\Psi_{j,\alpha}$, where $j=1,2,3$ labels the three generations, has $8\times2\times4=64$ real components that match the number of spinorial degrees of freedom of each fermion generation in $4D$. Their antiparticles have also 64 spinorial degrees of freedom bringing the total to 128. Namely, the 16 fermions of the first generation are $\nu_e, e, u^r, u^b, u^g, d^r, d^b, d^g$, plus their *antiparticles*, given by the electron neutrino, electron, up red/blue/green quark, down red/blue/green quark, and their antiparticles. The 4 complex entries of the Dirac spinors $\Psi^{(f)}_\alpha$ in $4D$ corresponding to each fermion $f=1,2,\cdots,8$ leads to $8\times4\times2=64$ real degrees of freedom. By including their *antiparticles* yields a total of 128 real spinorial degrees of freedom for each generation in $4D$.

Given the algebra $\mathbf{C}\otimes\mathbf{O}\;\cong\;\mathbf{O}\oplus\mathbf{O}$ one then has two copies of $\mathbf{O}$. The Hilbert space of the states of the leptons and quarks with three colors red, blue and green is $\mathbf{C}\oplus\mathbf{C^3}$ [10]. From the correspondence described in [17] $\mathbf{C}\oplus\mathbf{C^3}\leftrightarrow\mathbf{O}$ we then have that one copy of $\mathbf{O}$ corresponds to the electron neutrino ν_e and the up quarks with three colors u^r, u^b, u^g. And the second copy of $\mathbf{O}$ corresponds to the electron e and the down quarks with three colors d^r, d^b, d^g. This will allow us to find the strict correspondence among the complex-octonionic-valued spinors $\boldsymbol{\Psi}$ and the Standard Model fermions. In the Clifford algebra based model by [28] octonions are also used to describe the fermion generations.

Denoting the right/left handed components of the Dirac spinors by R, L, and spin up/down by $1, 2 = \uparrow, \downarrow$, the one-to-one correspondence among the 8 fermions (particles) of the first generation with the complex-octonionic entries $\boldsymbol{\Psi}$ of (3) is given by

$$\Psi^{(1)} \leftrightarrow \begin{pmatrix} \nu_{e,R1} & u^r_{R1} & u^b_{R1} & u^g_{R1} \\ e_{R1} & d^r_{R1} & d^b_{R1} & d^g_{R1} \end{pmatrix}; \;\; \Psi^{(2)} \leftrightarrow \begin{pmatrix} \nu_{e,R2} & u^r_{R2} & u^b_{R2} & u^g_{R2} \\ e_{R2} & d^r_{R2} & d^b_{R2} & d^g_{R2} \end{pmatrix} \tag{10}$$

$$\Psi^{(3)} \leftrightarrow \begin{pmatrix} \nu_{e,L1} & u^r_{L1} & u^b_{L1} & u^g_{L1} \\ e_{L1} & d^r_{L1} & d^b_{L1} & d^g_{L1} \end{pmatrix}; \;\; \Psi^{(4)} \leftrightarrow \begin{pmatrix} \nu_{e,L2} & u^r_{L2} & u^b_{L2} & u^g_{L2} \\ e_{L2} & d^r_{L2} & d^b_{L2} & d^g_{L2} \end{pmatrix} \tag{11}$$

The correspondence with the *antiparticles* of the first generation requires *swapping* the up $\leftrightarrow$ down entries of the $SU(2)$ doublets, the Left $\leftrightarrow$ Right chirality of the spinors [10], and taking the complex conjugation $\Psi \leftrightarrow \Psi^*$:

$$\Psi^{*(1)} \leftrightarrow \begin{pmatrix} e^+_{L1} & \bar{d^r}_{L1} & \bar{d^b}_{L1} & \bar{d^g}_{L1} \\ \bar{\nu}_{e,L1} & \bar{u^r}_{L1} & \bar{u^b}_{L1} & \bar{u^g}_{L1} \end{pmatrix}; \;\; \Psi^{*(2)} \leftrightarrow \begin{pmatrix} e^+_{L2} & \bar{d^r}_{L2} & \bar{d^b}_{L2} & \bar{d^g}_{L2} \\ \bar{\nu}_{e,L2} & \bar{u^r}_{L2} & \bar{u^b}_{L2} & \bar{u^g}_{L2} \end{pmatrix} \tag{12}$$

$$\Psi^{*(3)} \leftrightarrow \begin{pmatrix} e^+_{R1} & \bar{d^r}_{R1} & \bar{d^b}_{R1} & \bar{d^g}_{R1} \\ \bar{\nu}_{e,R1} & \bar{u^r}_{R1} & \bar{u^b}_{R1} & \bar{u^g}_{R1} \end{pmatrix}; \;\; \Psi^{*(4)} \leftrightarrow \begin{pmatrix} e^+_{R2} & \bar{d^r}_{R2} & \bar{d^b}_{R2} & \bar{d^g}_{R2} \\ \bar{\nu}_{e,R2} & \bar{u^r}_{R2} & \bar{u^b}_{R2} & \bar{u^g}_{R2} \end{pmatrix} \tag{13}$$

Since $\Psi^{(1)}, \cdots$, and $\Psi^{*(1)}, \cdots$ are complex-octonionic valued, they have 16 real components each, and which match the number of $8 \times 2 = 16$ real components associated with each single one of the 8 sets of 8 complex-valued entries appearing in the right-hand side of eqs-(10-13).

Repeating this assignment with the other complex-octonionic entries of $\mathbf{J}_{3,ij}$, and $\bar{\mathbf{J}}_{3,ij}$ leads to the correspondence with the 8 fermions, and their antiparticles, of the second and third generation. Therefore, the basis of quantum states for the fermions and their antiparticles of the Standard Model, including *all* their spinorial degrees of freedom, can be described in terms of the off-diagonal complex-octonionic entries of the Jordan *pair* of $(\mathbf{J}_{3,ij}, \bar{\mathbf{J}}_{3,ij})$ block matrices.

Furthermore, we also can incorporate the two complex scalar Higgs doublets $\mathbf{H}_L, \mathbf{H}_R$, stemming from the *diagonal* elements of the Jordan algebra, as explained earlier. Since there are 3 diagonal elements we will have three sets of pairs of complex scalar Higgs doublets $\{\mathbf{H}^i_L, \mathbf{H}^i_R\}$, and their conjugates $\{\bar{\mathbf{H}}^i_R, \bar{\mathbf{H}}^i_L\}$, with $i = 1, 2, 3$, corresponding to each fermion family. The most salient features of this construction is that it extends the work of [17] by including the antiparticles, the explicit spinorial spacetime degrees of freedom of the fermions, and the complex Higgs scalars.

This Jordan-Clifford algebraic approach can be contrasted with the $Cl(6, \mathbf{C})$ algebraic assignment of the fermions of one generation described by [1], [2], [16], and which was succinctly summarized by [13] via the left and right action of $Cl(6, \mathbf{C})$ on the primitive idempotent $\mathbf{p} \equiv \gamma_1\gamma_2\gamma_3\gamma_3^\dagger\gamma_2^\dagger\gamma_1^\dagger$, $\mathbf{p}^2 = \mathbf{p}$:

$$Cl(6, C) \ \rightarrow \ \mathbf{p} \ \leftarrow \ Cl(6, C) \tag{14}$$

The left action of $Cl(6, C)$ on $\mathbf{p}$ generates the 8 states of the first ideal $\mathbf{P}_1$ and which can be assembled into the 8 entries of the first column. While the right action on the first column generates the remaining 7 ideals $\mathbf{P}_2, \mathbf{P}_3, \cdots \mathbf{P}_8$. The full process will lead to $8 \times 8 = 64$ entries associated with the 64 states represented by the 64 complex entries in eqs-(10-13). The latter 64 complex valued entries amount to the 128 real spinorial degrees of freedom of one fermion generation, and which match the $dim_R \ Cl(6, C) = 2 \times 2^6 = 128$. More recently, the authors [13] extended this procedure to the $Cl(8, \mathbf{C})$ case to account for the three fermion generations via triality.

We continue this discussion by focusing on the Jordan products. The commutative but non-associative Jordan product $X \circ Y$ of two Jordan matrices is given by the following anticommutator

$$X \circ Y \ \equiv \ \frac{1}{2} \ (X \ Y \ + \ Y \ X) \tag{15}$$

and obeying the Jordan identity $(X \circ Y) \circ X^2 = X \circ (Y \circ X^2)$.

If one were to define the Jordan product of matrices given by the tensor products $X^A \otimes \Gamma_A$ and $Y^B \otimes \Gamma_B$ as

$$\mathbf{X} \circ \mathbf{Y} \ = \ \frac{1}{2} \ \{\mathbf{X}, \ \mathbf{Y}\} \ = \ \frac{1}{2} \ \{ \ X^A \otimes \Gamma_A, \ Y^B \otimes \Gamma_B \ \} =$$

$$\frac{1}{4} \ \{X^A, \ Y^B\} \otimes \{\Gamma_A, \ \Gamma_B\} \ + \ \frac{1}{4} \ [X^A, \ Y^B] \otimes [\Gamma_A, \ \Gamma_B] \ =$$

$$\mathbf{Z} \ = \ Z^C \otimes \Gamma_C \tag{16}$$

this would lead to a 3×3 matrix

$$Z^C \ = \ \frac{1}{4} \ d_{AB}^{\ \ C} \ \{X^A, \ Y^B\} \ + \ \frac{1}{4} \ f_{AB}^{\ \ C} \ [X^A, \ Y^B] \tag{17}$$

which is *no* longer Hermitian because when $d_{AB}^{\ \ C}, f_{AB}^{\ \ C}$ are the real-valued structure constants of the Clifford algebra

$$\{\Gamma_A, \Gamma_B\} \ = \ d_{AB}^{\ \ C} \ \Gamma_C, \quad [\Gamma_A, \Gamma_B] \ = \ f_{AB}^{\ \ C} \ \Gamma_C \tag{18}$$

the 3×3 matrix Z^C in eq-(12) will be comprised of a Hermitian *plus* an anti-Hermitian matrix, respectively, when X^A, Y^B (for $A, B = 1, 2, \cdots, 16$) are 3×3

Hermitian Jordan matrices. The anti-commutator of two Hermitian matrices is Hermitian, while the commutator is anti-Hermitian. Therefore, the product in eq-(10) does *not* lead to a matrix of the form $J_3^C \otimes \Gamma_C$ because the 3×3 matrix Z^C is *not* Hermitian. Furthermore, a careful inspection reveals also that the product (16) is *not* consistent with the Jordan identity $(\mathbf{X} \circ \mathbf{Y}) \circ \mathbf{X}^2 = \mathbf{X} \circ (\mathbf{Y} \circ \mathbf{X^2})$.

For this reason one must *modify* the product (16) so that the 3×3 matrix Z^C is Hermitian. Given $\mathbf{X} = X^A \otimes \Gamma_A$, and $\mathbf{Y} = Y^A \otimes \Gamma_A$ described by eq-(1), the modified Jordan product $\bullet$ is now defined as

$$\mathbf{X} \bullet \mathbf{Y} \ \equiv \ \frac{1}{4} \, d_{AB}{}^{C} \ \{X^A, \ Y^B\} \otimes \Gamma_C \ = \ \mathbf{Z} \ = \ Z^C \otimes \Gamma_C \tag{19}$$

where X^A, Y^B and Z^C (for $A, B, C = 1, 2, \cdots, 16$) are 3×3 Hermitian Jordan matrices. Despite that one has attained *closure* in the product $\mathbf{X} \bullet \mathbf{Y} = \mathbf{Z}$, with $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in J_3^A \otimes \Gamma_A$, one should point out that this modified product $\bullet$ does *not* obey the Jordan identity

$$(\mathbf{X} \bullet \mathbf{Y}) \bullet \mathbf{X}^2 \ \neq \ \mathbf{X} \bullet (\mathbf{Y} \bullet \mathbf{X^2}) \tag{20}$$

Exceptional Jordan F_4, E_6 (Chern-Simons-like) matrix models involving similar tensor products of Jordan matrices $J_3[\mathbf{O}], J_3[\mathbf{C} \otimes \mathbf{O}]$ with $u(N)$-valued matrix generators were constructed by [20], [21]. F_4 and E_6 are the automorphism groups of the Jordan algebras $J_3[\mathbf{O}], J_3[\mathbf{C} \otimes \mathbf{O}]$, respectively. The large N limit of these Exceptional Jordan matrix models and its relation to a nonperturbative bosonic formulation of M-theory in $D = 27$ was analysed by [22].

Following these findings we shall repeat their procedure and construct an action by replacing the $u(N)$ algebra with the Clifford $Cl(4, C)$ one. Given the structure constants d_{ABC}, f_{ABC} of the Clifford algebra, one candidate action is given by the Jordan trace of the cubic form

$$S_1 \ = \ (X^A, \ X^B, \ X^C) \ d_{ABC} \ = \ tr_3(X^A \circ (X^B \times_F X^C)) \ d_{ABC} \tag{21}$$

involving the Jordan product $\circ$, and the symmetric Freudenthal product

$$Y \times_F Z \ = \ Y \circ Z \ - \frac{1}{2} \ tr_3(Y) Z \ - \frac{1}{2} tr_3(Z) Y \ + \frac{1}{2} \ tr_3(Y) \ tr_3(Z) \ - \frac{1}{2} tr_3(Y \circ Z) \ \mathbf{1} \tag{22}$$

Another candidate action is

$$S_2 \ = \ (\rho^2(X^{[A}), \ \rho(X^B), \ X^{C]}) \ f_{ABC} \tag{23}$$

where $\rho, \rho^3 = 1$ is the cycle mapping based on the triality symmetry of $SO(8)$ that takes the index $I \to I + 1$ modulo 3. This cycle mapping is essential otherwise the action (18) would be identically zero due to the fact that the cubic form is symmetric in its entries while f_{ABC} is antisymmetric. A more general action is given by the sum $S_1 + S_2$ of eqs-(21,23).

Ohwashi [21] has shown that the cubic action (23) (when the structure constants f_{ABC} correspond to the $u(N)$ matrix algebra) is invariant under global rigid E_6 transformations which are encoded as automorphisms of the $J_3[\mathbf{C}\otimes\mathbf{O}]$ algebra under the transformations $\mathbf{J}\to\alpha\mathbf{J}$, where α is a 3×3 matrix whose entries are numerical constants. The action also posseses the $u(N)$ gauge symmetry resulting from $f_{ABC} = 2Tr_N(\mathbf{T_A}[\mathbf{T_B},\mathbf{T_C}])$; the cycle symmetry with respect to the fields, and the matrix translation symmetry with respect to the diagonal part of the fields. The explicit components of the complex-valued action (18) can be found in [21]. The real-valued Smolin action was based on the trilinear form $tr_3(X\circ(Y\circ Z)$ instead of the cubic form associated with the Freudenthal product $tr_3(X\circ(Y\times Z)$.

It remains to explore further the physical role of the algebra $J_3[\mathbf{C}\otimes\mathbf{O}]\otimes Cl(4,\mathbf{C})$ beyond the mere counting of degrees of freedom associated to a vector space. Having described how the degrees of freedom of the three fermion generations, with the inclusion of three pairs of complex scalar Higgs doublets, plus their conjugates, can be incorporated within the entries of the tensor product of the algebras $J_3[\mathbf{C}\otimes\mathbf{O}]\otimes Cl(4,\mathbf{C})$, in the next section we shall explain how the symmetry groups $SO(10); SU(3)\times SU(3)\times SU(3); SU(3)_C\times SU(2)_L\times SU(2)_R\times U(1)$ emerge from the internal symmetries of the Jordan algebra and which also account for the standard model gauge symmetries.

3 Emergence of $SU(3)\times SU(2)_L\times SU(2)_R\times U(1)$

A close inspection of the fermion structure displayed in eqs-(10-13) reveals that it fits naturally with the **16** complex-dimensional representation of the internal left/right symmetric gauge group $G_{LR} = SU(3)_C\times SU(2)_L\times SU(2)_R\times U(1)$ [6]. The left-handed/right-handed leptons can be assigned to the $\{\mathbf{1},\mathbf{2},\mathbf{1},-\frac{\mathbf{1}}{\mathbf{2}}\}$, $\{\mathbf{1},\mathbf{1},\mathbf{2},-\frac{\mathbf{1}}{\mathbf{2}}\}$ representations, respectively. The left-handed/right-handed quarks can be assigned to the $\{\mathbf{3},\mathbf{2},\mathbf{1},\frac{\mathbf{1}}{\mathbf{6}}\}$, $\{\mathbf{3},\mathbf{1},\mathbf{2},\frac{\mathbf{1}}{\mathbf{6}}\}$ representations, respectively. The assignment of their corresponding anti-particles is obtained by flipping Left for Right, $\mathbf{3}$ for $\bar{\mathbf{3}}$, and changing the sign of the $U(1)$ charges. The latter $U(1)$ charge (not to be confused with the electromagnetic charge) is defined by $\frac{B-L}{2}$, where B is the baryon number (quarks have $\frac{1}{3}$, anti-quarks have $-\frac{1}{3}$), and L is the lepton number (leptons have 1, anti-leptons have -1). In addition, there are 3 sets of pairs (one set for each family) of complex Higgs-doublets $\mathbf{H}^i_L, \mathbf{H}^i_R; i = 1,2,3$, and their conjugates.

Next we shall review [5] the geometrical background showing how $SU(3)\times SU(2)\times SU(2)\times U(1)$ is the intersection of $SO(10)$ with $SU(3)\times SU(3)\times SU(3)$ in E_6 . The automorphism group of the complex exceptional Jordan algebra $\mathbf{J}_3(\mathbf{C}\otimes\mathbf{O})$ that preserves the inner product $<\mathbf{X},\mathbf{Y}>=(\mathbf{X}^*,\mathbf{Y})$, and the cubic form (16) is E_6. There are three maximal-rank-6 subgroups of E_6 given by

$$SO(10)\times U(1),\quad SU(3)\times SU(3)\times SU(3),\quad SU(2)\times SU(6) \tag{24}$$

The significance of the first two subgroups (24) goes as follows. Firstly, as shown by Yokota [5], $Spin(10)$ is the subgroup of E_6 that preserves a rank-one idempotent of the Jordan algebra, and which can be taken to be

$$\mathbf{P} \ = \ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{25}$$

The rank-one idempotents of the Jordan algebra $\mathbf{J}_3(\mathbf{C} \otimes \mathbf{O})$ correspond to the points of the complex-octonionic projective plane $(\mathbf{C} \otimes \mathbf{O})\mathbf{P^2}$ such that the isometries of $(\mathbf{C} \otimes \mathbf{O})\mathbf{P^2}$ correspond to the automorphisms of the Jordan algebra $\mathbf{J}_3(\mathbf{C} \otimes \mathbf{O})$. Furhermore, the complex-octonionic projective plane can be represented as the coset space $(\mathbf{C} \otimes \mathbf{O})\mathbf{P^2} = \frac{\mathbf{G}}{\mathbf{H}} = \frac{E_6}{(Spin(10) \times U(1))/Z_4}$ where E_6 is the symmetry group $\mathbf{G}$, and $\mathbf{H} = (Spin(10) \times U(1))/Z_4$ is the subgroup that stabilizes a point (fixes a rank-one idempotent up to a phase).

Secondly, the subgroup of E_6 that preserves the embedding of $\mathbf{C}$ into $\mathbf{O} = \mathbf{C} \oplus \mathbf{C}^3$ is $SU(3) \times SU(3) \times SU(3)$ [5]. Such embedding permits the following decomposition of the complex exceptional Jordan algebra $\mathbf{J}_3(\mathbf{C} \otimes \mathbf{O}) = \mathbf{J}_3^C(\mathbf{C}) \oplus \mathbf{M}_3^C(\mathbf{C})$, and given in terms of the direct sum of the complexification of the Jordan algebra $\mathbf{J}_3(\mathbf{C})$ of complex hermitian 3×3 matrices $\mathbf{h}_3(\mathbf{C})$, and the complexification of the 3×3 matrix algebra $\mathbf{M}_3(\mathbf{C})$ over the complex numbers.

The embedding of the complex algebra into the octonion-algebra $\mathbf{C} \subset \mathbf{O} = \mathbf{C} \oplus \mathbf{C}^3$ can be chosen such that $\mathbf{C} = x_o + x_4 e_4$ is based on the e_4 imaginary unit of the seven imaginary units of the octonions, and the octonion $\mathbf{x} = x_o e_o + x_a e_a$ can be rewritten as

$$\mathbf{x} \ = \ (x_o + x_4 e_4) \ + \ e_1 \ (x_1 + x_5 e_4) \ + \ e_2 \ (x_2 + x_6 e_4) \ + \ e_3 \ (x_3 + x_7 e_4) \tag{26}$$

The complexification of the octonion algebra, and the exceptional Jordan algebra, is based on the extra imaginary unit $i = \sqrt{-1}$. One may then construct the idempotents [5]

$$\iota = \frac{1}{2}(1 + i e_4), \quad \bar{\iota} = \frac{1}{2}(1 - i e_4), \quad \iota^2 = \iota, \quad \bar{\iota}^2 = \bar{\iota}, \quad \iota + \bar{\iota} = 1, \quad \iota \, \bar{\iota} = 0 \tag{27}$$

which allows to decompose a 3×3 matrix $V = V_R \iota + V_L \bar{\iota}$ into a "right" and "left" component. The bar conjugation $\bar{V}$ denotes complex conjugation $i \rightarrow -i$. Whereas $V^\dagger$ denotes the matrix transpose followed by $e_4 \rightarrow -e_4$ resulting from the octonion conjugation of the 7 imaginary octonion units $e_a \rightarrow -e_a, a = 1, 2, \cdots, 7$. Thus, given $V = V_R \iota + V_L \bar{\iota}$ one has $\bar{V}^\dagger = V_R^\dagger \iota + V_L^\dagger \bar{\iota}$.

Consequently, given $\mathbf{X} = y + \mathbf{m} \in h_3(\mathbf{C} \otimes \mathbf{O}) = h_3^C(\mathbf{C}) \oplus \mathbf{M}_3^C(\mathbf{C})$ with $y \in \mathbf{h}_3^C(\mathbf{C})$, and $\mathbf{m} \in \mathbf{M}_3^C(\mathbf{C})$, the automorphisms that preserve such embedding are of the form [5]

$$y' \ = \ (V_R \ \iota + V_L \ \bar{\iota}) \ y \ (V_R^\dagger \ \iota + V_L^\dagger \ \bar{\iota}), \quad \mathbf{m}' \ = \ U \ \mathbf{m} \ (V_R^\dagger \ \iota + V_L^\dagger \ \bar{\iota}) \tag{28}$$

and are implemented by the three 3×3 matrices U, V_R, V_L corresponding to the group $SU(3)\times SU(3)_R\times SU(3)_L$.

The author [11] has recently pointed out that the automorphisms that simultaneously preserve the embedding of $\mathbf{C}$ into $\mathbf{O}$, and also preserve the idempotent $\mathbf{P}$ (25) are of the form indicated by eq-(28) with the provision that V_R, V_L have the block-diagonal form

$$V_R \;=\; \begin{pmatrix} \varphi\; v_R & 0 \\ 0 & \varphi^{-2} \end{pmatrix}, \quad V_L \;=\; \begin{pmatrix} \varphi\; v_L & 0 \\ 0 & \varphi^{-2} \end{pmatrix} \tag{29}$$

with $v_R, v_L \in SU(2)$ and $\varphi \in U(1)$ such that the determinant of V_R, V_L is one. Thus an element $\{U, v_L, v_R, \varphi\}$ belonging to $SU(3)\times SU(2)_L\times SU(2)_R\times U(1)$ determines a transformation in the intersection of $SO(10)$ with $SU(3)\times SU(3)\times SU(3)$ in E_6. Furthermore, because these transformations are unaffected by the replacements [11] $\{U, v_L, v_R, \varphi\}\to\{U, -v_L, -v_R, -\varphi\}$, or the replacements $\{U, v_L, v_R, \varphi\}\to\{\omega\; U, v_L, v_R, \omega\; \varphi\}$, and $\{U, v_L, v_R, \varphi\}\to\{\omega^2\; U, v_L, v_R, \omega^2\; \varphi\}$, with ω being the cube-root of unity $\omega = -\frac{1}{2}+\frac{\sqrt{3}}{2}e_4$, the actual intersection is the left-right symmetric gauge group given by the minimal extension of the standard model group [6]

$$[SU(3)\times SU(2)_L\times SU(2)_R\times U(1)]/Z_6 \tag{30}$$

The intersection of $SO(10)$ with $SU(3)\times SU(3)\times SU(3)$ in E_6 given by $SU(3)\times SU(2)\times SU(2)\times U(1)$ can also be inferred from the branching rules of $SO(10)$ [8]

$$SO(10)\;\supset\; SU(4)\times SU(2)\times SU(2)\;\supset\; SU(3)\times SU(2)\times SU(2)\times U(1) \tag{31}$$

and

$$SU(3)\times SU(3)\times SU(3)\;\supset\; SU(3)\times SU(2)\times SU(2)\times U(1)\times U(1) \tag{32}$$

From the last terms of eqs-(32) one can simply read-off the sought-after intersection of the two groups.

To sum up, we have seen how the first two subgroups of E_6 in eq-(24) play an important role in the exceptional complex Jordan algebra $J_3(\mathbf{C}\otimes\mathbf{O})$ and their intersection leads to the $G_{LR}\equiv SU(3)_C\times SU(2)_L\times SU(2)_R\times U(1)$ left/right symmetric group G_{LR}. The $Cl(4,\mathbf{C})$ algebra in the tensor product $J_3(\mathbf{C}\otimes\mathbf{O})\otimes Cl(4,\mathbf{C})$ represents an *external* four-dim spinorial spacetime symmetry compared to the *internal* space symmetry underlying the automorphisms of the complex Jordan algebra $J_3(\mathbf{C}\otimes\mathbf{O})$.

It so happens that the **16** complex-dimension of the $Cl(4,\mathbf{C})$ algebra equals also the **16** complex dimension of the space $(\mathbf{C}\otimes\mathbf{O})^{\mathbf{2}}$, and which in turn, coincides with the tangent space of $(\mathbf{C}\otimes\mathbf{O})\mathbf{P}^{\mathbf{2}}$. As pointed out by [11] this tangent space also appears in the Barton-Sudbery decomposition of the $\mathbf{e}_6$ Lie algebra [9]

$$\mathbf{e}_6 \;=\; \mathbf{u(1)} \oplus \mathbf{so(8)} \oplus (\mathbf{C}\otimes\mathbf{O}) \oplus (\mathbf{C}\otimes\mathbf{O})^{\mathbf{2}} \;= \\ \mathbf{u(1)} \oplus \mathbf{so(10)} \oplus (\mathbf{C}\otimes\mathbf{O})^{\mathbf{2}} \tag{33}$$

and which is consistent with the fact that the complex-octonionic projective plane is the coset space $(\mathbf{C}\otimes\mathbf{O})\mathbf{P^2} = \frac{E_6}{(Spin(10)\times U(1))/Z_4}$. Furthermore, the generators of the tangent space $(\mathbf{C}\otimes\mathbf{O})^{\mathbf{2}}$ transform as the **16** complex-dimensional representation of $Spin(10)$, which also matches the **16** complex-dimensional representation of the group G_{LR} involving the leptons, quarks, and their antiparticles, as described at the beginning of this section.

The author [11] has argued that the Barton-Sudbery decomposition of the $\mathbf{e}_6$ Lie algebra displayed in eq-(33) requires choosing one copy $(\mathbf{C}\otimes\mathbf{O})$ (of the three copies $(\mathbf{C}\otimes\mathbf{O})^{\mathbf{3}}$) to include inside $\mathbf{so(10)}$. Because these three copies are permuted by the $\mathbf{so(8)}$ triality symmetry, the author [11] postulated that the standard model fermions may inevitably arise in three triality-related ways when constructing the complex-octonionic projective plane $(\mathbf{C}\otimes\mathbf{O})\mathbf{P^2}$ as the coset space of E_6. This argument should be contrasted with the emergence of three fermion generations based on the triality symmetry of the $Cl(8)$ algebra given by [13].

Our construction of the three fermion generations is very different. From the very start it uses the external four-dim spacetime algebra $Cl(4,\mathbf{C})$ in the tensor product $J_3(\mathbf{C}\otimes\mathbf{O})\otimes Cl(4,\mathbf{C})$ leading explictly to three fermion generations after invoking the construction displayed by eqs-(10-13). The triality symmetry is just a cyclic symmetry, rotating the already-existing three fermion generations among each other, rather than *generating* three fermion generations by *choosing* one copy $(\mathbf{C}\otimes\mathbf{O})$ (out of the three copies $(\mathbf{C}\otimes\mathbf{O})^{\mathbf{3}}$) in three different ways.

One should note that if the $Cl(4,\mathbf{C})$ algebra were interpreted as an internal symmetry, rather than a four-dim spinorial spacetime symmetry, due to the fact that $Cl(4,\mathbf{C}) \simeq M(4,\mathbf{C})$, where $M(4,\mathbf{C})$ is the 4×4 matrix algebra over the complex numbers and it is also the complexification of $u(4)$ ($sl(4,\mathbf{C})$ is the complexification of $su(4)$), then the $Cl(4,\mathbf{C})$ algebra can be decomposed into two copies of $u(4)$: $Cl(4,\mathbf{C}) \simeq u(4)\oplus u(4)$. And the latter $u(4)\oplus u(4)$ algebra is large enough to accomodate the Standard Model and Pati-Salam algebras, respectively, $su(3)\oplus su(2)\oplus u(1)$ and $su(4)\oplus su(2)\oplus su(2) \supset su(3)\oplus su(2)\oplus su(2)\oplus u(1)$. Once again, one recovers the algebra associated with the G_{LR} group.

Therefore, it is essential that the $Cl(4,\mathbf{C})$ algebra is interpreted as an external four-dim spacetime symmetry and that there is *no* mixing of external with internal symmetries are required by the Coleman-Mandula theorem. However it would be interesting to propose a *reciprocal* picture where $Cl(4,\mathbf{C})$ is now an internal space symmetry algebra, and the 27 complex-dimensional Jordan algebra $J_3(\mathbf{C}\otimes\mathbf{O})$ corresponds to an external space symmetry algebra.

We finalize with a series of concluding remarks. In [22] we showed how the decomposition $27 = 16 + 8 + 3 = 16 + 11$ associated with a compactification of the 27-dim bosonic M-theory down to 11-dim along an internal 16-dim space was connected to the large $N \rightarrow \infty$ limit of a matrix model based on the algebra $J_3(\mathbf{O}) \otimes SU(N)$. It incorporated the 3-dim world volume of a bosonic membrane moving in 27-dimensions. One of the octonionic variables of the 3×3 matrix belonging to the $J_3(\mathbf{O})$ algebra represents the 8 transverse dimensions to a membrane in 11-dimensions. The 3 diagonal entries represent the three-dim world volume of the membrane. And the remaining two octonionic variables represent the internal 16 dimensions.

Another possibility is to construct a Yang-Mills-like theory based on a Clifford $Cl(4, \mathbf{C})$-bundle over $(\mathbf{C} \otimes \mathbf{O})\mathbf{P^2}$. The isometries of the external space $(\mathbf{C} \otimes \mathbf{O})\mathbf{P^2}$ are given by E_6, and the internal gauge symmetry is provided by the $Cl(4, \mathbf{C})$ algebra. Clifford algebras were instrumental in the Chamseddine-Connes grand-unified model based on the spectral action in Noncommutative geometry [14]

One may also ask what is the role of the other maximal-rank 6 subgroup of E_6 given by $SU(2) \times SU(6)$. Given the complexified octonions $\mathbf{C} \otimes \mathbf{O} \simeq \mathbf{O} \oplus \mathbf{O}$ one can have a $2 + 6$ split of the form $\mathbf{C}^2 \oplus \mathbf{C}^6$ representing 2 longitudinal directions and 6 transverse directions in $\mathbf{C}^8$. The $SU(2)$ acts naturally on $\mathbf{C}^2$ and $SU(6)$ acts on $\mathbf{C}^6$. It is similar to having 2 light-cone and 6 transverse directions in a complexified $8D$-spacetime. Focusing now on the 8-fermion set decomposition in eqs-(10-13) one can split the 8 fermions into 2 leptons ν_e, e, and 6 quarks $u^r, u^b, u^g, d^r, d^b, d^g$. The $SU(2)$ symmetry acts on the the ν_e, e and u, d doublets, whereas the $SU(6)$ rotates the 6 quarks, which is reminiscent of hexaquarks.

Finally, it remains to explore the matrix Chern-Simons-like cubic actions displayed in eqs-(16,18). Smolin [20] has already provided important clues how to provide explicit dynamics by introducing additional dimensions leading to integrals involving derivatives of the variables inside the entries of the matrices belonging to the Jordan algebra. All these questions and the possibility that the extra pairs of complex scalar Higgs-doublets could be dark matter candidates warrants further investigation.

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References

[1] G. M. Dixon, *"Division Algebras, Octonions, Quaternions, Complex Numbers, and the Algebraic Design of Physics"* (Kluwer, Dordrecht, 1994).

G. M. Dixon, *Division Algebras, Lattices, Physics and Windmill Tilting* (ICG 2010)

G. M. Dixon, "Derivation of the Standard Model", Il Nuovo Cimento **B 105** issue 3 (1990) 349.

G.M. Dixon, "Division Algebras, Family Replication", J. Math. Phys **45** , no 10 (2004) 3678.

[2] C. Furey, "Standard Model from an Algebra ? " (Ph.D thesis) arXiv : 1611.09182.

C. Furey. Generations: three prints, in colour. Journal of High Energy Physics, 2014(10):1-11, 2014.

C. Furey, "$SU(3)_C \times SU(2)_L \times U(1)_Y (\times U(1)_X)$ as a symmetry of division algebraic ladder operators" Eur. Phys. J. C (2018) 78:375.

C. Furey, "Three generations, two unbroken gauge symmetries, and one eight-dimensional algebra" Phys. Lett. **B 785** (2018) 84.

[3] C. Castro, "A Note on Jordan Algebras, Three Generations and Exceptional Periodicity" to appear in the IJGMMP, (2020).

[4] M. Gunaydin and F. Gursey, " Quark structure and Octonions", Journal of Mathematical Physics, **14** issue 11 (1973) 1651.

M Gunaydin and F Gursey, "Quark statistics and Octonions", Physical Review **D 9** issue 12 (1974) 3387.

[5] I. Yokota, "Exceptional Lie Groups", arXiv : 0902.0431.

[6] R. Mohapatra and J. Pati, Phys. Rev. **D 11** (1975) 566.

[7] L. Hall and K. Harigaya, "Implications of Higgs Discovery for the Strong CP Problem and Unification," JHEP **10**, 130 (2018).

L. J. Hall and K. Harigaya, "Higgs Parity Grand Unification," JHEP **11**, 033 (2019).

J. Dror, D. Dunsky, L. Hall and K. Harigaya, "Sterile Neutrino Dark Matter in Left-Right Theories," [arXiv:2004.09511 [hep-ph]].

[8] N. Yamatsu, "Finite-Dimensional Lie Algebras and Their Representations for Unified Model Building " arXiv : 1511.08771.

[9] J. Baez, " The Octonions", Bulletin of the American Mathematical Society, **39** issue. 2 (2002) 145.

C. Barton, and A. Sudbery, "Magic squares and matrix models of Lie Algebras" Advances in Mathematics **180** (2003) 596.

[10] J. Baez and J. Huerta, "The Algebra of Grand Unified Theories". arXiv : 0904.1556.

[11] L. Boyle, "The Standard Model. The Exceptional Jordan Algebra, and Triality" arXiv : 2006.16265.

[12] S. Catto, Y. Gurcan, A. Khalfan and L. Kurt, "Quantum Symmetries: From Clifford and Hurwitz Algebras to M-Theory and Leech Lattices" Adv. Appl. Clifford Algebras **28** (2018) 81.

[13] A. Gillard and N. Gresnigt, "The $Cl(8)$ algebra of three fermion generations with spin and full internal symmetries" arXiv : 1906.05102.

A. Gillard , N. Gresnigt , "Three fermion generations with two unbroken gauge symmetries from the complex Sedenions", arXiv : 1904.03186.

N. Gresnigt, "The Standard Model gauge symmetries and particle content from the minimal ideals of two Clifford algebras" arXiv : 2003.08814.

[14] A.H. Chamseddine and A. Connes, "Noncommutative geometry as a framework for unification of fundamental interactions including gravity", Fortsch. Phys. **58** (2010) 553/

A.H. Chamseddine, A. Connes, W.D. van Suijlekom, "Beyond the spectral standard model: emergence of the Pati-Salam unification, JHEP **1311** (2013) 132.

[15] G. Trayling and W. Baylis, "The $Cl(7)$ approach to the standard model", In Clifford Algebras, pages 547-558. Springer, 2004.

"A geometric basis for the standard-model gauge group", Journal of Physics **A**: Mathematical and General, **34** issue 15 (2001).

[16] O. Stoica, "Leptons, quarks, and gauge from the complex clifford algebra $Cl(6)$", Advances in Applied Clifford Algebras, **28** (3) (2018) 52.

[17] M. Dubois-Violette, " Exceptional quantum geometry and particle physics. Nuclear Physics **B 912** (2016) 426.

I. Todorov and M. Dubois-Violette, " Deducing the symmetry of the standard model from the automorphism and structure groups of the exceptional Jordan algebra", International Journal of Modern Physics **A 33** , issue. 20 (2018) 1850118,

M. Dubois-Violette and I. Todorov, "Exceptional Quantum Geometry and Particle Physics **II**", Nuclear Physics **B 938** (2019) 751.

[18] C. Castro, "R $\otimes$ C $\otimes$ H $\otimes$ O-valued Gravity as a Grand Unified Field Theory" Adv. in Appl. Clifford Algebras, (online January 2019).

C. Castro, "On C $\otimes$ H $\otimes$ O-valued Gravity, Sedenions, Hermitian Matrix Geometry and Nonsymmetric Kaluza-Klein Theory", to appear in Adv. in Appl. Clifford Algebras, 2019.

[19] S. Marques and C. Oliveira, "An extension of quaternionic metrics to octonions" J. Math. Phys **26** (1985) 3131.

S. Marques and C. Oliveira, "Geometrical properties of an internal local octonionic space in curved space-time", Phys. Rev **D 36** (1987) 1716.

[20] L. Smolin, "The exceptional Jordan algebra and the matrix string" arXiv : hep-th/0104050

L. Smolin, "Quantum reference frames and triality" arXiv : 2007.05957.

[21] Y. Ohwashi "E_6 Matrix Model", arXiv : hep-th/0110106.

Y. Ohwashi, "$Sp(4, H)/Z_2$ pair universes in E_6 matrix models" arXiv : arXiv : hep-th/0510252.

[22] C. Castro, "The large N limit of Exceptional Jordan Matrix Models and M, F theory", Journal of Geometry and Physics **57** (2007) 1941.

[23] C. Castro, "A Clifford Algebra Based Grand Unification Program of Gravity and the Standard Model : A Review Study", Canadian Journal of Physics

C. Castro, "A Clifford $Cl(5, C)$ Unified Gauge Field Theory of Conformal Gravity, Maxwell and $U(4) \times U(4)$ Yang-Mills in $4D$" Advances in Applied Clifford Algebras **22**, no. 1 (2012), page 1-21

C. Castro, "The Noncommutative and Nonassociative Geometry of Octonionic Spacetime, Modified Dispersion Relations and Grand Unification" J. Math. Phys, **48**, no. 7 (2007) 073517.

[24] S. Okubo and N. Kamiya, "Jordan-Lie Super Algebra and Jordan-Lie Triple System" Journal of Algebra **198** (1997) 388.

M. Gunaydin, "Generalized Conformal and Superconformal Group Actions and Jordan Algebra " hep-th/9301050

[25] P. Truini, M. Rios, A. Marrani, "The Magic Star of Exceptional Periodicity" arXiv : 1711.07881.

P.Truini, A. Marrani, M. Rios, "Magic Star and Exceptional Periodicity: an approach to Quantum Gravity" arXiv: 1811.11202.

P.Truini, A. Marrani, M. Rios, "Exceptional Periodicity and Magic Star Algebras. I : Foundations" , arXiv : 1909.00357.

P.Truini, A. Marrani, M. Rios, "Exceptional Periodicity and Magic Star Algebras. II : Gradings and HT-Algebras" , arXiv : 1910.07914.

[26] M. Rios, A. Marrani, D. Chester, "The Geometry of Exceptional Super Yang-Mills Theories", Phys. Rev. **D 99**, 046004 (2019)

M. Rios, A. Marrani, D. Chester, "Exceptional Super Yang-Mills in D=27+3 and Worldvolume M-Theory" arXiv : 1906.10709.

[27] E.B Vinberg, "The theory of Convex Homogeneous Cones", in Transaction of the Moscow Mathematical Society 1963, 340-403, American Mathematical Society, Providence RI 1965.

[28] T. Smith, "Geisteswissenschaft and Cl(16) Physics", vixra.org/pdf/1810.03651.

$Cl(16)$ Physics: E_8 Lagrangian and $Fr_3(O)$ String Theory, vixra.org/pdf/1807.0166v1.pdf